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SOME COMBINATORIAL ASPECTS OF THE SCHUBERT CALCULUS

Richard P. Stanley[1]

Schubert calculus is a branch of algebraic geometry essentially founded in 1874 by H. Schubert. Schubert developed his calculus to answer questions in enumerative geometry, i.e., to find the number of points, lines, planes, etc., satisfying certain geometric conditions . In subsequent years algebraic geometers developed a rigorous foundation to the Schubert calculus, perhaps spurred by Hilbert's Fifteenth Problem (see [Kl]). This foundation is intimately connected with the branch of combinatorics which deals with symmetric functions, Young tableaux, plane partitions, etc. Our aim in this paper is to explain this connection as straightforwardly as possible. We will be addressing ourselves mainly to combi-

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natorialists unfamiliar with the Schubert calculus. It is hoped, however, that algebraic geometers will also find something of interest here. Most of the results which we state are well-known to researchers in the Schubert calculus in at least an implicit form, but we can perhaps give some additional insight by our explicit combinatorial approach. The only result which appears to be really new is the concept of "skew Schubert varieties." I am grateful to S. Kleiman, B. Kostant, D. Laksov, and J. Wahl for helpful discussions.

1. Schubert calculus. An excellent account of this topic may be found in the survey article of Kleiman and Laksov [K-L]. A detailed history of Schubert calculus appears in [Kℓ]. We shall briefly outline those facts relevant to us here, referring the reader to [K-L] for further details.

Let $\underset{\sim}{P}^n = \underset{\sim}{P}^n(\underset{\sim}{C})$ denote n-dimensional complex projective space. Thus the points of $\underset{\sim}{P}^n$ are defined by (n+1)-tuples $(p(0), p(1),\ldots,p(n))$ of complex numbers not all zero. Two (n+1)-tuples define the same point if they are scalar multiples of each other. Let G_{dn} be the set of all d-dimensional subspaces (called d-planes, for short) of $\underset{\sim}{P}^n$. Thus $G_{0n} = \underset{\sim}{P}^n$ and $G_{nn} = \{\underset{\sim}{P}^n\}$. Let $N = \binom{n+1}{d+1} - 1$. Thus points in $\underset{\sim}{P}^N$ have $\binom{n+1}{d+1}$ coordinates. We will index these coordinates by integer sequences $(c_0,c_1,\ldots,c_d)$, where $0 \leq c_0<c_1<\ldots<c_d\leq n$. (Clearly there are $\binom{n+1}{d+1}$ such sequences.) There is a natural way of associating with a d-plane $L\in G_{dn}$ a point of $\underset{\sim}{P}^N$, with the indexing of coordinates defined above. The coordinates of L (regarded as an element of $\underset{\sim}{P}^N$) are called the Plücker coordinates of L. (For the precise definition, see [K-L].)

Distinct d-planes L define distinct elements of $\mathbf{P}^N$, so we have an embedding of G_{dn} into $\mathbf{P}^N$. It turns out that this embedding makes G_{dn} into a manifold, the <u>Grassmann manifold</u>. One of the basic problems in setting up the foundations of the Schubert calculus is to describe the cohomology ring of G_{dn} (say with integer coefficients). To this end, given an integer sequence $0 \le a_0 < a_1 < \ldots < a_d \le n$, let $A_0 \subset A_1 \subset \ldots \subset A_d$ be a chain (or <u>flag</u>) of subspaces of $\mathbf{P}^n$ satisfying $\dim A_i = a_i$. Define $\Omega(A_0 \ldots A_d)$ to be the

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1.1 __Theorem.__ (The basis theorem, quoted from [K-L].) Each even dimensional integral cohomology group $H^{2p}(G_{dn};\underset{\sim}{Z})$ is a free abelian group and the Schubert cycles

$$\Omega(a_0 \ldots a_d) \text{ with } [(d+1)(n-d) - \sum_{i=0}^{d} (a_i - i)] = p \text{ form a basis.}$$

Each odd dimensional group is zero. (Here $\underset{\sim}{Z}$ denotes the integers.)

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mann varieties were one of the original pieces of evidence for the Weil conjectures.) We require the fact that G_{dn} is actually a (projective) <u>variety</u>, i.e., the points of $G_{dn} \subset \underset{\sim}{P}^N$ are the set of zeros of a system of homogeneous polynomials in the variables $X(j_0 j_1 \ldots j_d)$ (the coordinates of $\underset{\sim}{P}^N$). These equations are given explicitly by

$$\sum_{i=0}^{d+1} (-1)^i X(j_0 \ldots j_{d-1} k_i) X(k_0 \ldots \check{k}_i \ldots k_{d+1}) = 0 ,$$

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$\prod_{j=1}^{\beta_i}(1-\alpha_{ij}x)$, $|\alpha_{ij}| = q^{i/2}$. Thus β_i = deg $P_i(x)$. But $N(k) = \left[{n+1 \atop d+1}\right]_{\lambda=q^k}$, and $\left[{n+1 \atop d+1}\right]$ is a polynomial in λ, say $\Sigma\gamma_i\lambda^i$. Therefore

$$\exp \sum_{k=1}^{\infty} \frac{N(k)x^k}{k} = \exp \sum_{k=1}^{\infty} (\Sigma\gamma_i q^{ik}) \frac{x^k}{k}$$

$$= \Pi(1-q^i x)^{-\gamma_i},$$

whence from (2) we have $\beta_{2i} = \gamma_i$, as desired.

The above argument generalizes immediately to finding the Betti numbers of the so-called <u>partial flag manifolds</u> $G_n(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_r)$, where $0\leq\varepsilon_1<\varepsilon_2<\ldots<\varepsilon_r<n$. The elements of $G_n(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_r)$ consist of all flags $L_1\subset L_2\subset\ldots\subset L_r\subset \underset{\sim}{P}^n$, where dim $L_i = \varepsilon_i$. We can make $G_n(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_r)$ into a manifold (and also a variety) in a manner analogous to G_{dn} (the special case r=1, ε_1=d). We then obtain as above that if β_i is the i-th Betti number of $G_n(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_r)$, then

$$\sum_p \beta_{2p}\lambda^p = \left[{n+1 \atop \varepsilon_1+1,\ \varepsilon_2-\varepsilon_1,\varepsilon_3-\varepsilon_2,\ldots,n-\varepsilon_r}\right]$$

where

$$\left[{m \atop k_1,k_2,\ldots,k_s}\right] = \frac{[m]!}{[k_1]!\ldots[k_s]!} \qquad (3)$$

$$[k]! = (1-\lambda^k)(1-\lambda^{k-1})\ldots(1-\lambda).$$

The expression (3) is the <u>q-multinomial coefficient</u> and when $\lambda=q$ is equal to the number of flags $L_1 \subset L_2 \subset \ldots \subset L_r \subset \underset{\sim}{P}^n(q)$, where $\dim L_i = \varepsilon_i$.

We have mentioned that G_{dn} is a variety. As suggested by the terminology, the Schubert varieties $\Omega(\underset{\sim}{A})$ are also varieties (in general they are singular). If we take $\underset{\sim}{A}$ to be a standard flag, then the equations for $\Omega(\underset{\sim}{A})$ are obtained by adjoining to (1) the linear relations

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<u>Schubert cycle</u>. Note that if $\Omega(\underline{A},\underline{B})$ and $\Omega(\underline{A}^*,\underline{B}^*)$ represent the same skew Schubert cycle $\Omega(\underline{a},\underline{b})$, then there is an invertible linear transformation of $\underset{\sim}{P}^N$ which carries G_{dn} into G_{dn} and $\Omega(\underline{A},\underline{B})$ into $\Omega(\underline{A}^*,\underline{B}^*)$.

If we pick $\underline{A}$ to be a standard flag and $\underline{B}$ to be the flag $B_0 \subset B_1 \subset \ldots \subset B_d$ such that B_i consists of all points in $\underset{\sim}{P}^n$ of the form $(0,0,\ldots,0,p(b_{d-i}), p(b_{d-i}+1),\ldots,p(n))$, then it is easy to see that the equations defining $\Omega(\underline{A},\underline{B})$ are given by (1), (4), and

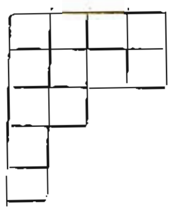
$$X(j_0 j_1 \cdots j_d) = 0 , \qquad (5)$$

over all sequences $0 \le j_0 < j_1 < \ldots < j_d \le n$ for which $j_i < b_i$ for some i. We call such a pair $(\underline{A},\underline{B})$ a <u>standard pair</u>. We also remark that if $(\underline{A},\underline{B})$ is standard and if $a_i = b_i$ for all i, then $\Omega(\underline{A},\underline{B})$ (which we know consists of a single d-plane) consists of that d-plane whose Plücker coordinates are all zero except for $X(\underline{a})$.

2. <u>Schur functions</u>. We now review some basic results about symmetric functions. A more detailed survey of this subject appears in $[S_1]$. For convenience we take our symmetric functions to be in infinitely many variables $x_1, x_2, \ldots$ (so we are really speaking of symmetric formal power series, not "functions"), although it would suffice to take sufficiently many variables, namely, $(d+1)(n-d)$ of them.

Let λ be a partition of a non-negative integer m (written $\lambda \vdash m$). By this we mean $\lambda = (\lambda_1, \lambda_2, \ldots)$, where the λ_i's are integers satisfying $\lambda_1 \ge \lambda_2 \ge \ldots \ge 0$ and $\Sigma\lambda_i = m$. If $\lambda_{r+1} = 0$ we also write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$.

If $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$, then the <u>diagram</u> $Y(\lambda)$ of λ is a left-justified array of squares with λ_i squares in the i-th row. For instance $Y(4,4,2,1,1)$ is given by



Given a partition λ, define the <u>monomial symmetric function</u> k_λ in the variables $x_1, x_2, \ldots$ to be the sum of all distinct monomials whose exponents are $\lambda_1, \lambda_2, \ldots$ in some order. We also write symbolically $k_\lambda = \Sigma x_1^{\lambda_1} x_2^{\lambda_2} \ldots$. Thus for instance $\Sigma x_1 = x_1+x_2+x_3+\cdots = \sum_i x_i$, $\Sigma x_1x_2 = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + \cdots = \sum_{i<j} x_ix_j$, etc. Let $\mathcal{A}$ denote the ring of all symmetric functions (formal power series) in the variables $x_1, x_2, \ldots$, which as an abelian group is free with basis k_λ, where λ ranges over all partitions of all non-negative integers m . (By convention, 0 has the unique partition $\emptyset$, and $k_\emptyset = 1$.) We also define the <u>complete homogeneous symmetric functions</u> h_m and the <u>elementary symmetric functions</u> a_m by

$$h_m = \sum_{\lambda \vdash m} k_\lambda \ , \ a_m = \Sigma\, x_1x_2\ldots x_m \ .$$

The "fundamental theorem of symmetric functions" states that the a_m are algebraically independent and $\mathcal{A} = \mathbb{Z}[a_1, a_2, \ldots]$. It is also easy to see that the h_m are algebraically independent and $\mathcal{A} = \mathbb{Z}[h_1, h_2, \ldots]$. We now wish to define a new $\mathbb{Z}$- basis for $\mathcal{A}$, the <u>Schur functions</u> e_λ, where $\lambda \vdash m$. Define

$$e_\lambda = \Sigma\, M(\pi) \ ,$$

where the sum is over all ways π of filling the squares of $Y(\lambda)$ with positive integers such that the integers along any row are non-increasing and along any column are strictly decreasing. Here $M(\pi) = x_1^{a_1} x_2^{a_2} \ldots$, where exactly a_i entries of π one equal to i . It follows that e_λ is homogeneous of degree m . It is by no means evident from this definition that e_λ is a symmetric function, but such is indeed the case.

<u>Example</u>. Let $\lambda=(3,1)$. Then the coefficient of $x_1x_2x_3x_4$

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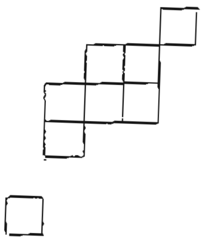
2.2 Theorem (Aitken) Let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)$ be a partition, and let $\lambda'=(\lambda_1',\lambda_2',\ldots,\lambda_s')$ be the conjugate partition. (The diagram $Y(\lambda')$ is obtained from $Y(\lambda)$ by a reflection in the main diagonal.) Then

$$e_\lambda = |a_{\lambda_i'-i+j}| \qquad (1\le i,j\le s).$$

The above results on Schur functions can be generalized to the so-called skew Schur functions. Suppose $\lambda=(\lambda_1,\lambda_2,\ldots)$ and $\mu=(\mu_1,\mu_2,\ldots)$ are partitions with $\mu_i\le\lambda_i$ for all i. We then write $\mu\le\lambda$. The skew diagram $Y(\lambda/\mu)$ is obtained from $Y(\lambda)$ by removing $Y(\mu)$. For example, if $\lambda=(5,4,4,2,1,1)$ and $\mu=(4,2,1,1,1)$, then $Y(\lambda/\mu)$ looks like



We now define the skew Schur function $e_{\lambda/\mu}$ exactly analogously to our definition of e_λ. It turns out that $e_{\lambda/\mu}$ is a symmetric function, and generalizing Theorems 2.1 and 2.2 we have

$$\begin{aligned} e_{\lambda/\mu} &= \left|h_{\lambda_i-\mu_j-i+j}\right| \\ &= \left|a_{\lambda_i'-\mu_j'-i+j}\right| . \end{aligned}$$

Since $e_{\lambda/\mu}$ is a symmetric function, it can be written as a $\mathbb{Z}$-linear combination of the e_ν's. The next result relates the coefficients in this linear combination to the multiplication of Schur functions.

2.3 Theorem (E.D. Roe, D.E. Littlewood) The coefficient of e_ν in $e_{\lambda/\mu}$ (when written as a linear combination of Schur functions) is equal to the coefficient of e_λ in $e_\nu e_\mu$.

Theorem 2.3 gives a method for computing $e_\nu e_\mu$, but in general it is not much better than brute force since $e_{\lambda/\mu}$ is not easy to express in terms of the e_ν's. What is desired is a direct rule for finding the coefficient of e_λ in $e_\nu e_\mu$. This is the content of the next result. An

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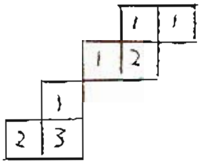
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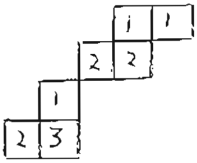
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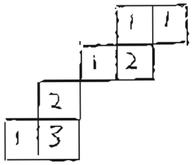
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Hence the coefficient of e_λ in $e_\mu e_\nu$ is 3. (Another example appears in [Li, pp. 96-98].)

Now given $0 \leq d < n$, let λ be the partition with d+1 parts equal to n-d. Hence $Y(\lambda)$ is a $(d+1)\times(n-d)$ rectangle. Let I_{dn} be the $\mathbb{Z}$-submodule of $\mathcal{A}$ generated by all e_μ for which $\mu \nleq \lambda$. In other words, either $\mu_1 > d+1$ or μ has more than n-d parts. Equivalently, μ is not of the form $(a_d - d, a_{d-1} - d+1, \ldots, a_0)$, where $0 \leq a_0 < a_1 < \ldots < a_d \leq n$. It follows from Theorem 2.4 that if $e_\mu \in I_{dn}$ and if e_ρ appears with a positive coefficient in $e_\mu e_\nu$, then $\mu \leq \rho$. Thus $e_\rho \in I_{dn}$, so I_{dn} is an ideal of $\mathcal{A}$. The quotient $\mathcal{A}/I_{dn}$ is generated as a $\mathbb{Z}$-module freely by those e_ρ for which $\rho \leq \lambda$. When multiplying Schur functions modulo I_{dn}, simply use the Littlewood-Richardson rule and ignore all $e_\mu \in I_{dn}$.

There is another rule for multiplying Schur functions in $\mathcal{A}/I_{dn}$ discovered essentially by Jacobi and rediscovered by I.R. Porteous (see [La_1, pp. 174-175] for a sketched proof).

2.5 Theorem. Let e_μ, e_ν be Schur functions, so by Theorem 2.1, $e_\mu = |h_{\mu_i - i+j}|$ and $e_\nu = |h_{\nu_i - i+j}|$. Then in the ring $\mathcal{A}/(x_{r+1}, x_{r+2}, \ldots)$,

$$e_\mu e_\nu = |h_{\mu_i + \nu_{r+1-j} - i+j}|,$$

where the determinants are all $r \times r$.

3. <u>The cohomology ring</u> $H^*(G_{dn};\ \mathbb{Z})$. We are now able to describe the multiplicative structure of $H^*(G_{dn};\ \mathbb{Z})$.

3.1 <u>Theorem</u> (the multiplication rule). There is an isomorphism $\theta: \mathcal{Q}/I_{dn} \longrightarrow H^*(G_{dn};\ \mathbb{Z})$ given by $\theta(e_\lambda)=\Omega(\underline{a})$, where $\lambda=(n-a_0-d,\ n-a_1-d+1,\dots,n-a_d)$.

The connection between multiplication of Schur functions and the cohomology of G_{dn} was first observed explicitly by Lesieur [Le], though the result was capable of

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it is evident that $\mathcal{Q}/I_{dn}$ is generated as a ring by $e_0=1$, $e_1,\ldots,e_{n-d}$, where e_i denotes e_λ when λ has a single part i. (Note that $e_i=h_i$, the complete homogeneous symmetric function.) Hence $H^*(G_{dn};\, \underset{\sim}{Z})$ is generated as a ring by the <u>special Schubert cycles</u> $\sigma(i)=\theta(e_{n-d-i})=\Omega(i,n-d+1,\ldots,n)$, for $i=0,1,\ldots, n-d$. As an immediate consequence of Theorem 2.1 we obtain:

3.2 <u>Theorem</u> (the determinantal formula). For all sequences of integers $0\leq a_0<a_1<\ldots<a_d\leq n$, the following formula holds in the cohomology ring $H^*(G_{dn};\, \underset{\sim}{Z})$:

$$\Omega(\underline{a}) \quad = \quad |\sigma(a_i-j)|, \quad 0\leq i,j\leq d,$$

where $\sigma(i)=0$ if $i<0$ or $i>n-d$.

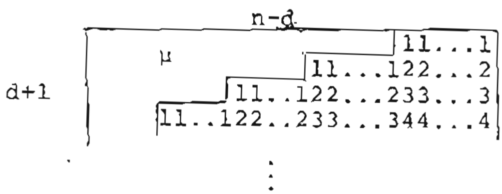
Now consider the product $\Omega(\underline{a})\cdot\sigma(h)$. This corresponds to the multiplication $e_\mu e_{n-d-h}$, where $\mu=(n-a_0-d,\, n-a_1-d+1,\ldots,n-a_d)$ By the Littlewood-Richardson rule, the coefficient of e_λ in $e_\mu e_{n-d-h}$ (modulo I_{dn}) is one if $\mu\leq\lambda$, $\lambda_1\leq n-d$, and $Y(\lambda/\mu)$ has exactly $n-d-h$ squares and no columns with more than one square. Otherwise the coefficient is zero. If $Y(\lambda/\mu)$ has c_i squares in the i-th row (counting the top row as the 0-th row), then define $b_i=a_i-c_i$. It follows that $0\leq b_0\leq a_0<b_1\leq a_1<\ldots<b_d\leq a_d$ and $\Sigma b_i=\Sigma a_i-(n-d-h)$. Conversely such a sequence of b_i's contributes a term to $e_\mu e_{n-d-h}$. Thus we have proved:

3.3 <u>Theorem</u> (Pieri's formula). For all sequences of integers $0\leq a_0<a_1<\ldots<a_d\leq n$ and for all $h=0,1,\ldots,n-d$, the following formula holds in the cohomology ring $H^*(G_{dn};\underset{\sim}{Z})$:

$$\Omega(\underline{a})\cdot\sigma(h)=\Sigma\Omega(\underline{b})$$

where the sum ranges over all sequences of integers $b_0,\ldots,b_d$ satisfying $0\leq b_0\leq a_0<b_1\leq a_1<\ldots<b_d\leq a_d$ and $\Sigma b_i=\Sigma a_i-(n-d-h)$.

A further classical result in the Schubert calculus is the <u>duality theorem</u>. In order to state this theorem, first note that by Theorem 1.1, the highest non-vanishing cohomology group of $H^*(G_{dn};\mathbb{Z})$ occurs in dimension $2(d+1)(n-d)$, and is generated by the cycle $\Omega(0,1,\ldots,d)$. This cycle corresponds to the Schur function e_λ, where $Y(\lambda)$ is a $(d+1)\times(n-d)$ rectangle. Hence if $\mu\vdash s$ and $\nu\vdash t$ where $s+t=(d+1)(n-d)$, then in $\mathcal{A}/I_{dn}$ we have $e_\mu e_\nu = c_{\mu\nu}e_\lambda$ for some $c_{\mu\nu}\in\mathbb{Z}$. We can calculate $c_{\mu\nu}$ by the Littlewood-Richardson rule. If $\nu=(\nu_1,\nu_2,\ldots)$ we wish to insert ν_1 1's, ν_2 2's,... into the squares of $Y(\lambda/\mu)$. It is easy to see that if conditions (i) and (ii) of Theorem 2.4 are to be satisfied, the ν_i's must be arranged as follows:



Hence if $\mu=(\mu_1,\ldots,\mu_{d+1})$, there must be exactly $n-d-\mu_{d+1}$ 1's, $n-d-\mu_d$ 2's,..., $n-d-\mu_1$ d+1's. Therefore we have:

3.4 <u>Theorem.</u> Let λ be the partition whose diagram $Y(\lambda)$ is a $(d+1)\times(n-d)$ rectangle, and let $\mu\vdash s$, $\nu\vdash t$, where $s+t=(d+1)(n-d)$. Then in the ring $\mathcal{A}/I_{dn}$, we have

$$e_\mu e_\nu=\begin{cases} e_\lambda, & \text{if } \mu_i+\nu_{d+2-i} = n-d \text{ for } i=1,2,\ldots,d+1 \\ 0, & \text{otherwise.}\end{cases}$$

If Theorem 3.4 is translated into a statement about Schubert cycles, it becomes:

3.4' <u>Theorem</u> (the duality theorem). The basis $\{\ldots,\Omega(a_0,a_1,\ldots,a_d),\ldots\}$ of the group $H^{2p}(G_{dn};\mathbb{Z})$ and the basis $\{\ldots,\ \Omega(n-a_d,\ldots,n-a_1,\ n-a_0),\ldots\}$ of the group $H^{2[(d+1)(n-d)-p]}(G_{dn};\ \mathbb{Z})$ are dual under the pairing $v,w \mapsto \deg(v\cdot w)$ of Poincaré duality.

the basis $\{\ldots,\ \Omega(n-a_d,\ldots,n-a_1,\ n-a_0),\ldots\}$ of the group $H^{2[(d+1)(n-d)-p]}(G_{dn};\ \mathbb{Z})$ are dual under the pairing $v,w \mapsto \deg(v\cdot w)$ of Poincaré duality.

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3.7 Corollary. Let $\lambda=(n-a_0-d, n-a_1-d+1,\ldots,n-a_d)$ and $\mu=(n-b_0-d, n-b_1-d+1,\ldots,n-b_d)$ be partitions with $\mu\leq\lambda$. In the isomorphism $\Theta: \mathcal{Q}/I_{dn}\to H^*(G_{dn};\underset{\sim}{Z})$, we have $\Theta(e_{\lambda/\mu})=\Omega(\underline{a},\underline{b})$.

Proof. This is an immediate consequence of Theorem 2.3, Corollary 3.6, and the fact (Theorem 3.1) that $\Theta(e_\lambda)=\Omega(\underline{a})$ and $\Theta(e_\mu)=\Omega(\underline{b})$.

In other words, skew Schubert cycles correspond to skew Schur functions in the same way that Schubert cycles correspond to Schur functions. It now follows from the determinantal expression for $e_{\lambda/\mu}$ (preceding Theorem 2.3) that

$$\Omega(\underline{a},\underline{b}) = |\sigma(a_i-b_j)|,\quad 0\leq i,j\leq d,$$

thereby generalizing Theorem 3.2.

4. Homogeneous coordinate rings. Recall that if $A_0\subset A_1\subset\ldots\subset A_d$ is a flag in $\underset{\sim}{P}^n$, then we have defined an explicit embedding $\Omega(\underline{A})\subset \underset{\sim}{P}^N$, where $N = \binom{n+1}{d+1} - 1$. More generally, if $(\underline{A},\underline{B})$ is a pair of flags defining a skew Schubert variety, then $\Omega(\underline{A},\underline{B})\subset \underset{\sim}{P}^N$ since $\Omega(\underline{A},\underline{B})\subseteq\Omega(\underline{A})$. Let $R = \underset{\sim}{C}[\ldots, X(i_0\ldots i_d),\ldots]$, the polynomial ring in the $\binom{n+1}{d+1}$ variables $X(i_0\ldots i_d)$, $0\leq i_0<\ldots<i_d\leq n$, which coordinatize $\underset{\sim}{P}^N$. Let $J(\underline{A},\underline{B})$ be the ideal of R consisting of all polynomials which vanish at every point of $\Omega(\underline{A},\underline{B})$. The quotient ring $R/J(\underline{A},\underline{B})$ is called the homogeneous coordinate ring of $\Omega(\underline{A},\underline{B})$. Let T_m denote the $\underset{\sim}{C}$-subspace of $R/J(\underline{A},\underline{B})$ generated by all homogeneous polynomials of degree m. The function $H: \underset{\sim}{N}\to\underset{\sim}{N}$ defined by $H(m) = \dim_{\underset{\sim}{C}} T_m$ is the Hilbert function of $R/J(\underline{A},\underline{B})$. Since any two skew Schubert varieties representing $\Omega(\underline{a},\underline{b})$ are projectively equivalent, it follows that H depends only on $(\underline{a},\underline{b})$. If

the numbers a_i and b_i are not clear from context, then we write $H(\underline{a},\underline{b};m)$. We are interested in computing $H(\underline{a},\underline{b};m)$. To do so, we may assume that $(\underline{A},\underline{B})$ is a standard pair of flags, so that the equations defining $\Omega(\underline{A},\underline{B})$ are given by (1), (4), and (5). We shall write $J(\underline{a},\underline{b})$ for $J(\underline{A},\underline{B})$ when $(\underline{A},\underline{B})$ is standard. We also write $J(\underline{a})$ for $J(\underline{a},\underline{b})$ when $\underline{b} = (0,1,\ldots,d)$ (so that $\Omega(\underline{A},\underline{B}) = \Omega(\underline{A})$).

In [H-P, Ch.XIV,§9] it is shown that when $b_0=0,\ldots,b_d=d$ (the case of "ordinary" Schubert varieties), a vector space

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In [H-P, Ch.XIV,§9] it is shown that when $b_0=0,\ldots,b_d=d$

$$
\begin{array}{c}
\qquad\qquad\qquad \geq \\
\vee \quad
\begin{array}{|c|cccc|c|}
\hline
a_d & i_d & i'_d & \cdots & i''_d & b_d \\
a_{d-1} & i_{d-1} & i'_{d-1} & \cdots & i''_{d-1} & b_{d-1} \\
\vdots & \vdots & \vdots & & & \vdots \\
a_0 & i_0 & i'_0 & \cdots & i''_0 & b_0 \\
\hline
\end{array}
\end{array}
\qquad (7)
$$

Thus $H(\underline{a},\underline{b};m)$ is equal to the number of arrays (7) with m columns (excluding the columns of a_j's and b_j's at the ends). We now transform the array (7), with the first and last column excluded, into a form which has been previously considered by combinatorialists. Number the rows $0,1,\ldots,d$ from the bottom up. Now do the following steps: (i) subtract j from the j-th row, (ii) replace each row by its conjugate partition (in the sense of Theorem 2.2) (iii) remove entries (which will always equal zero) from the ends of each row so that the i-th row has $a_i - i$ entries, (iv) remove the first $b_i - i$ entries from the i-th row. If we let $\lambda=(a_d-d, a_{d-1}-d+1,\ldots,a_0)$ and $\mu=(b_d-d, b_d-d+1,\ldots,b_0)$, then we have obtained a set of integers in the squares of $Y(\lambda/\mu)$ which are non-increasing in every row and column, and the largest integer is at most m (the number of columns of (7), excluding the first and last). Such an array is called a plane partition of shape λ/μ and largest part at most m. The above process is reversible, i.e., given a plane partition π of shape λ/μ and an integer m no smaller than the largest part of π, we can uniquely recover (7). Thus we obtain:

4.1 Theorem. $H(\underline{a},\underline{b};m)$ is equal to the number of plane partitions of shape λ/μ and largest part at most m.

As an explicit illustration of steps (i)-(iv) above, let $\underline{a}=(1,4,5,7)$, $\underline{b}=(0,1,3,4)$. Then a typical array (7) and its transformation (i)-(iv) is given by

$$\begin{array}{c|c|c} 7 & 755 & 4 \\ 5 & 443 & 3 \\ 4 & 221 & 1 \\ 1 & 110 & 0 \end{array} \longrightarrow \begin{array}{l} 422 \\ 221 \\ 110 \\ 110 \end{array} \longrightarrow \begin{array}{l} 3311 \\ 3200 \\ 2000 \\ 2000 \end{array}$$

$$\longrightarrow \begin{array}{l} 3311 \\ 320 \\ 200 \\ 2 \end{array} \longrightarrow \begin{array}{r@{}l} & 311 \\ & 20 \\ 2 & 00 \\ 2 & \end{array}$$

We remark that it is easily proved from Theorem 4.1 that $H(m)$ is a polynomial for __all__ $m \geq 0$. (__A priori__ one only knows that $H(m)$ is a polynomial for m sufficiently large.)

Plane partitions have been extensively studied since P.A. MacMahon. MacMahon derived (implicitly) a determinantal formula [M,§495] which in the special case $x=1$ gives the number of plane partitions of shape λ and largest part at most m. An equivalent formula was found by Hodge and Littlewood (see [HP, Ch. XIV,§9]) and is called the __postulational formula__. Thus the postulational formula gives an expression for $H(m)$ when $\mu=\emptyset$. The proof of the postulational formula generalizes to any λ/μ. A result equivalent to this generalization was proved by G. Kreweras [Kr, §2.5.3].

4.2 __Theorem__ (the postulational formula for skew Schubert varieties). Let $0 \leq a_0 < \ldots < a_d \leq n$ and $0 \leq b_0 < \ldots < b_d \leq n$, with $b_i \leq a_i$ for $i=0,1,\ldots,d$. Let $H(\underline{a},\underline{b};m)$ be the Hilbert function of the homogeneous coordinate ring $R/J(\underline{a},\underline{b})$ of $\Omega(\underline{A},\underline{B})$. Then $H(\underline{a},\underline{b};m)$ is given by the $(d+1)\times(d+1)$ determinant

$$H(\underline{a},\underline{b};m) = \left| \binom{m+a_i-b_j-i+j}{a_i-b_j} \right| \qquad (0 \leq i,j \leq d), \quad (8)$$

with the convention $\binom{\alpha}{\beta}=0$ if $\beta<0$.

In certain cases the formula (8) for H(m) can be greatly simplified, essentially by evaluating the determinant of (8). In the following theorem, part (a) is equivalent to [S_1,Thm. 18.1], while part (b) seems to be new.

4.3 <u>Theorem.</u> (a) Let H(m) be the Hilbert function of R/J(n-d,n-d+1,...,n), the homogeneous coordinate ring of the Grassmann variety G_{dn}. Then

n ...: min(i,n-i+1, d+1, n-d)

new.

4.3 Theorem. (a) Let H(m) be the Hilbert function of R/J(n-d,n-d+1,...,n), the homogeneous coordinate ring of the Grassmann variety G_{dn}. Then

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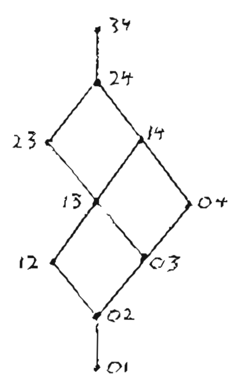
4.3 Theorem. (a) Let H(m) be the Hilbert function of R/J(n-d,n-d+1,...,n), the homogeneous coordinate ring of the Grassmann variety G_{dn}. Then

Theorem 4.3(a). Presumably part (b) of Theorem 4.3 has a similar explanation, but at present this remains open.

We now wish to point out a remarkable relationship between the Hilbert functions $H(\underline{a},\underline{b};m)$ and the inclusion relationships among Schubert varieties. Define a partially ordered set L_{dn} (actually a distributive lattice) as follows. The elements of L_{dn} consist of all integer sequences $\underline{a}=(a_0,a_1,\ldots,a_d)$ with $0\le a_0<a_1<\ldots<a_d\le n$. Thus L_{dn} has $\binom{n+1}{d+1}$ elements. The elements of L_{dn} are ordered component-wise, that is, $(b_0,b_1,\ldots,b_d)\le(a_0,a_1,\ldots,a_d)$ if $b_i\le a_i$ for $i=0,1,\ldots,d$. Thus L_{dn} is a distributive lattice under the operations of component-wise max and min. If we identify the sequence $\underline{a}$ with the standard Schubert variety $\Omega(\underline{A})$, then L_{dn} may be identified with the set of standard Schubert varieties in G_{dn}, ordered by inclusion. Note that each interval $[\underline{b},\underline{a}] = \{\underline{c} \in L_{dn} | \underline{b} \le \underline{c} \le \underline{a}\}$ of L_{dn} corresponds to a unique skew Schubert cycle $\Omega(\underline{a},\underline{b})$. As an illustration, the Hasse diagram of L_{14} is shown below.



Now consider the array (7). Each column represents an element of L_{dn}, and the columns occur from left to right in descending order as elements of L_{dn}. Since $H(\underline{a},\underline{b};m)$ is the number of arrays (7) with m columns (excluding the first and last), there follows:

4.4 Theorem. $H(\underline{a},\underline{b};m)$ is equal to the number of chains $\underline{b}=x_0 \le x_1 \le \ldots \le x_{m+1}=\underline{a}$ in L_{dn}.

In the language of $[S_3, \S 3]$, $H(\underline{a},\underline{b};m+1)$ is the zeta polynomial of the interval $[\underline{b},\underline{a}]$. It is easy to see that the degree ℓ of $H(\underline{a},\underline{b};m)$ (i.e., the dimension of $\Omega(\underline{A},\underline{B})$) is equal to the length of the longest chain between $\underline{a}$ and $\underline{b}$, viz., $\Sigma(a_i-b_i)$. Moreover, $\ell!$ times the leading coefficient of $H(\underline{a},\underline{b};m)$ (called the degree of

polynomial of the interval $[\underline{b},\underline{a}]$. It is easy to see that the degree ℓ of $H(\underline{a},\underline{b};m)$ (i.e., the dimension of $\Omega(\underline{A},\underline{B})$) is equal to the length of the longest chain between $\underline{a}$ and $\underline{b}$, viz., $\Sigma(a_i-b_i)$. Moreover, $\ell!$ times the leading coefficient of $H(\underline{a},\underline{b};m)$ (called the degree of

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where δ_i is the number of elements $\underline{c}$ in the interval $[\underline{b},\underline{a}]$ of L_{dn} such that the maximal chains of $[\underline{b},\underline{c}]$ have length i.

For instance, if $\underline{a} = (2,3)$ and $\underline{b} = (0,2)$, then from the above diagram of L_{14} we see that $\emptyset(q) = 1 + 2q + q^2 + q^3$.

We now turn to consideration of the generating function

$$F(\underline{a},\underline{b};x) = \sum_{m=0}^{\infty} H(\underline{a},\underline{b};m)x^m ,$$

which is sometimes called the Poincaré series of the coordinate ring $R/J(\underline{a},\underline{b})$. Since $H(m)$ is a polynomial, say of degree $\ell = \Sigma(a_i - b_i)$, it follows that there exist integers $w_0, w_1, \ldots, w_\ell$ such that

$$F(\underline{a},\underline{b};x) = \frac{w_0 + w_1 x + \ldots + w_\ell x^\ell}{(1-x)^{\ell+1}} \qquad (9)$$

Note $w_0 = H(0) = 1$. Equivalently,

$$H(\underline{a},\underline{b};m) = \sum_{i=0}^{\ell} w_i \binom{m+\ell-i}{\ell} .$$

Now it can be shown that the coordinate ring $R/J(\underline{a},\underline{b})$ is Cohen-Macaulay. This is well-known for $\underline{b} = (0,1,\ldots,d)$ and can be proved in general by induction, repeatedly using [Hc, p. 329, lines 14-17]. It follows from a theorem of Macaulay (see [S_4, Cor.3.2]) that the w_i's have the following algebraic interpretation. Let $\theta_1, \theta_2, \ldots, \theta_{\ell+1}$ be a homogeneous system of parameters for $R/J(\underline{a},\underline{b})$, all of degree one. (Such a sequence of parameters exists, e.g., by [A-M, p.69, Ex.16].) Thus the quotient ring $U = R/(J(\underline{a},\underline{b}) + (\theta_1, \ldots, \theta_{\ell+1}))$ is a graded (Artinian)

$\underset{\sim}{C}$-algebra with the grading induced from R, say $U = U_0 + U_1 + \ldots$. Then w_i is simply the Hilbert function of U , i.e., $w_i = \dim_{\underset{\sim}{C}} U_i$.

We will now give a combinatorial (rather than algebraic) interpretation of the integers w_i. This interpretation is essentially a special case of [S_2, Prop.13.3(i)], once Theorem 4.1 is known.

4.6 Theorem. Let w_i be as in (9), and let

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4.6 Theorem. Let w_i be as in (9), and let

more convenient to use is the following. Let $\lambda=(a_d-d, a_{d-1}-d+1,\ldots,a_0)$ and $\mu=(b_d-d,\ b_{d-1}-d+1,\ldots,b_0)$. Then w_i is equal to the number of ways of placing the integers $1,2,\ldots,\ell$ in the squares of the skew diagram $Y(\lambda/\mu)$ such that (a) the integers are increasing in every row and column, and (b) for exactly i integers $j \in \{1,2,\ldots,\ell-1\}$ is the row containing j below the row containing $j+1$.

5. Affine coordinate rings. Let $R=\underset{\sim}{C}[\ldots,X(i_0\ldots i_d),\ldots]$ as in the previous section. If $A_0 \subset A_1 \subset \ldots \subset A_d$ is a standard flag in $\underset{\sim}{P}^n$, define the ideal $K(\underline{a}) = K(a_0\ldots a_d)$ of R by

$$K(\underline{a}) = J(\underline{a}) + (X(01\cdots d)-1),$$

where $J(\underline{a})$ is the ideal defined in the previous section. We call $R/K(\underline{a})$ the affine coordinate ring of $\Omega(\underline{A})$, since it is the ideal of polynomials which vanish on those points of $\Omega(\underline{A})$ which lie in the affine space of all points $X \in \underset{\sim}{P}^n$ satisfying $X(01\ldots d)=1$. (We could have defined more generally the affine coordinate ring $R/K(\underline{a},\underline{b})$ of $\Omega(\underline{A},\underline{B})$, but our results below on $R/K(\underline{a})$ do not extend to the more general case.)

The main interest in the rings $R/K(\underline{a})$ stems from the fact that they can be obtained by dividing out by the determinants of certain minors in the affine space of $(d+1)\times(n-d)$ matrices. For instance, the following result is obtained in [K-L, p.1077]. Suppose that for some $s\leq d$ we have $a_i=d-s+i$ for $i=0,\ldots,s$. Let $R' = \underset{\sim}{C}[Y_{ij}\,|\,0\leq i\leq d,\ 1\leq j\leq n-d]$. The indeterminates Y_{ij} are regarded as the entries of a generic $(d+1)\times(n-d)$ matrix M over $\underset{\sim}{C}$. Let $K'(\underline{a})$ be the ideal of R' generated by the determinants of all the $(d-i+1)\times(d-i+1)$ minors from the last $n-a_i$ columns of M, for all $i\geq s$. Then $R/K(\underline{a}) \cong R'/K'(\underline{a})$. In particular, suppose that for certain integers α,β,γ with $1\leq\gamma\leq \min(\alpha,\beta)$ and $n=\alpha+\beta-1$, we have

$$a_i = \gamma-1+i \quad \text{if } 0\leq i\leq \alpha-\gamma$$

$$a_i = \beta+i \quad \text{if } \alpha-\gamma+1\leq i\leq \alpha-1 = d\ .$$

Then $s=\alpha-\gamma$ and $K'(\underline{a})$ is generated by two types of minors:

(i) $(\alpha-i)\times(\alpha-i)$ minors from the last $n-a_i = \alpha-i-1$ columns of M if $\alpha-\gamma+1\leq i\leq\alpha-1$, and (ii) $\gamma\times\gamma$ minors from the last

$$a_i = \beta+i \quad \text{if } \alpha-\gamma+1\leq i\leq \alpha-1 = d\ .$$

Then $s=\alpha-\gamma$ and $K'(\underline{a})$ is generated by two types of minors:

(i) $(\alpha-i)\times(\alpha-i)$ minors from the last $n-a_i = \alpha-i-1$ columns of M if $\alpha-\gamma+1\leq i\leq\alpha-1$, and (ii) $\gamma\times\gamma$ minors from the last

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Note in particular that a basis for the first homogeneous part (m=1) consists of all variables $X(i_0 \ldots i_d)$ for which $i_{d-1} \leq d$ and $i_d > d$. There are clearly $(d+1)(n-d)$ such variables, corresponding to the $(d+1)(n-d)$ indeterminates Y_{ij}.

We now associate the monomial (10) with the array (6) as we did for the homogeneous coordinate ring. If we transform (6) into a plane partition π of shape $\lambda = (a_d - d, a_{d-1} - d+1, \ldots, a_0)$ as in the previous section, then the number of entries of (6) (excluding the first column of a_i's) greater than d becomes the sum of the main diagonal elements of π, which we call the trace of π (even though π is not a square array). We therefore conclude from Theorem 5.1 the following result.

5.2 Theorem. Let $H'(\underline{a},m)$ denote the Hilbert function of the affine coordinate ring $R'/K'(\underline{a})$, with the grading defined by $\deg Y_{ij} = 1$. Then $H'(\underline{a},m)$ is equal to the number of plane partitions whose shape is $\lambda = (a_d - d, a_{d-1} - d+1, \ldots, a_0)$ and whose trace is m.

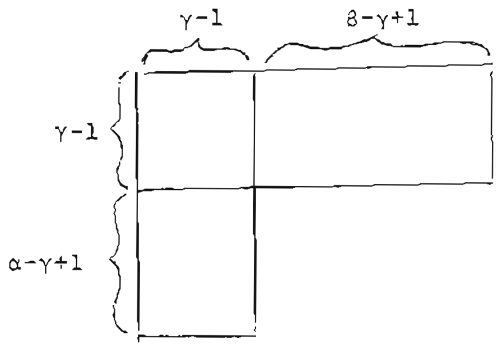
Note that it is by no means apparent from the description of $H'(\underline{a},m)$ in Theorem 5.2 that $H'(\underline{a},m)$ is a polynomial in m for m sufficiently large, though in fact $H'(\underline{a},m)$ is a polynomial for all $m \geq 0$. Although we do not know a formula for $H'(\underline{a},m)$ as explicit as Theorem 4.2, the following result can be proved. For simplicity, we treat only the special case $K'(\alpha,\beta,\gamma)$.

5.3 Theorem. Let $H(m)$ be the Hilbert function of $R'/K'(\alpha,\beta,\gamma)$, the polynomial ring over $\underset{\sim}{C}$ in the entries of a generic $\alpha \times \beta$ matrix, modulo all $\gamma \times \gamma$ minors. Then

$$H(m) = \sum_{i \geq 1} f_i \binom{m-1}{i-1},$$

where f_i is the number of sets of i entries of an $\alpha\times\beta$ matrix which do not contain the main diagonal of any $\gamma\times\gamma$ minor.

In the special case $R'/K'(\alpha,\beta,\gamma)$, the partition λ of Theorem 5.2 is a "hook" of horizontal length β, vertical length α, and width $\gamma-1$:



Note that when $\alpha=\gamma-1$ or $\beta=\gamma-1$, we obtain an $\alpha\times\beta$ rectangle. Now in this case there are no $\gamma\times\gamma$ minors of an $\alpha\times\beta$ matrix. Hence $K'(\alpha,\beta,\gamma)=(0)$, and the Hilbert function $H(m)$ of $R'/K'(\alpha,\beta,\gamma)$ is just $\binom{\alpha\beta+m-1}{\alpha\beta-1}$. We therefore obtain from Theorem 5.2 the following interesting purely combinatorial result.

5.4 <u>Theorem</u>. Let $H(m)$ be the number of $\alpha\times\beta$ matrices of non-negative integers whose entries are non-increasing in every row and column, and whose trace (= sum of entries on main diagonal, even if $\alpha\neq\beta$) is m. Then $H(m) = \binom{\alpha\beta+m-1}{\alpha\beta-1}$.

It is possible to give a purely combinatorial proof of Theorem 5.4. We have that $\binom{\alpha\beta+m-1}{\alpha\beta-1}$ is equal to the number of $\alpha\times\beta$ matrices M of non-negative integers whose entries sum to m . Knuth [Kn] (see also [S_1,§6]) shows how M can be associated with what may be called a pair of "column-strict plane partitions of the same shape." Bender and Knuth [B-K] (see also [S_1,§19]) show how such a pair can be "merged" into a single plane partition π . One sees that π fits into an $\alpha\times\beta$ matrix and has trace m . This establishes a one-to-one correspondence which proves Theorem 5.4.

The correspondence mentioned above between pairs of column-strict plane partitions and ordinary plane partitions shows that one could index the $\underset{\sim}{C}$-basis elements of $R'/K'(\underline{a})$ (for any $\underline{a}$) by pairs of column-strict plane partitions, rather than by ordinary plane partitions as we have done. If one examines in this context how elements of $R'/K'(\underline{a})$ multiply, one obtains the "straightening formula" of [D-R-S].

6. The bracket ring. There is a generalization of the homogeneous coordinate ring $R/J(\underline{a},\underline{b})$ due to G.-C. Rota [R] and studied by N. White [W_1][W_2]. This generalization is known as the bracket ring of a pregeometry (or "matroid"). A finite pregeometry G is a finite set S and a collection of subsets of S known as independent sets. The independent sets satisfy the two axioms: (i) if T is any subset of S , then the maximal independent sets in T all have the same cardinality, and (ii) a subset of an independent set is independent. The archetypal example is to take S to be any finite set of points in affine space, and to let "independent" mean "linearly independent." The bracket ring B(G) of G (over $\underset{\sim}{C}$) may be defined as follows. Let the elements of S be the integers 0,1,...,n. Suppose the largest independent set of S has d+1 elements. Let

R/J_{dn} be the homogeneous coordinate ring of the Grassmann variety G_{dn}. Then $B(G) = R/(J_{dn}+Q)$, where Q is the ideal generated by all variables $X(j_0 j_1 \cdots j_d)$ for which $\{j_0, j_1, \ldots, j_d\}$ is not an independent set.

It is easy to see that the homogeneous coordinate rings $R/J(\underline{a},\underline{b})$ are bracket rings. In other words, the sets $\{j_0, j_1, \ldots, j_d\}$ for which $X(j_0 j_1 \cdots j_d)=0$ in $R/J(\underline{a},\underline{b})$ have the property that they are the dependent (=not independent) sets of some pregeometry. Thus it is of interest to ask which of the known results about $R/J(\underline{a},\underline{b})$ can be generalized to bracket rings. For instance, what bracket rings are domains (as are $R/J(\underline{a})$ and presumably $R/J(\underline{a},\underline{b})$)? White has given many examples of bracket rings which are not domains, and he conjectures that a certain class of pregeometries, known as "unimodular", yield domains. What bracket rings are Cohen-Macaulay (as are $R/J(\underline{a},\underline{b})$)? No examples of bracket rings are known which are not Cohen-Macaulay. What is the Krull dimension of $B(G)$? Can it be expressed in a simple way in terms of the structure of G? Finally, are there analogues of Theorems 4.1 and/or Theorem 4.2 for computing the Hilbert function of $B(G)$?

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Richard P. Stanley
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Mass. 02139
U. S. A.