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Stirling Polynomials

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An investigation is made of the polynomials $f_k(n) = S(n + k, n)$ and $g_k(n) = (-1)^k s(n, n - k)$, where S and s denote the Stirling numbers of the second and first kind, respectively. The main result gives a combinatorial interpretation of the coefficients of the polynomial $(1 - x)^{2k+1} \sum_{n=0}^{\infty} f_k(n)x^n$ analogous to the well-known combinatorial interpretation of the Eulerian numbers in terms of descents of permutations.

1. ELEMENTARY PROPERTIES

If k and n are positive integers, let $S(n, k)$ denote the number of partitions of an n-element set into k blocks. Also let $c(n, k)$ denote the number of permutations of an n-element set with k cycles. Thus $S(n, k) = c(n, k) = 0$ if $k > n$. $S(n, k)$ is a Stirling number of the second kind, and $c(n, k)$ is related to the Stirling numbers $s(n, k)$ of the first kind by $c(n, k) = (-1)^{n-k} s(n, k)$. The $c(n, k)$'s are sometimes called the "signless Stirling numbers of the first kind." For further information, see [9, pp. 32–34, 70–72; 3, Chap. V], or [6, Chap. IV]. Throughout this paper we will use the following notation:

$\mathbb{N}$ set of nonnegative integers,
$\mathbb{P}$ set of positive integers,
$\mathbb{Z}$ set of all integers,
$\mathbb{C}$ set of complex numbers,
$[n]$ the set $\{1, 2, \ldots, n\}$, where $n \in \mathbb{P}$.

We are interested in studying $S(n, k)$ and $c(n, k)$ as functions of the difference $n - k$. Hence if $k \in \mathbb{N}$ and $n \in \mathbb{P}$, we define $f_k(n) = S(n + k, n)$ and $g_k(n) = c(n, n - k)$. For instance, $f_0(n) = g_0(n) = 1$ and $f_1(n) = \binom{n+1}{2}$,

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$g_1(n) = \binom{n}{2}$. The next two propositions are not new, but for the sake of completeness we have included proofs.

PROPOSITION 1.1 (see, e.g., [6, Sect. 52]). *Fix a nonnegative integer k. Then $f_k(n)$ and $g_k(n)$ are polynomial functions of the positive integer n. Both these polynomials have degree 2k and leading coefficient* $(1 \cdot 3 \cdot 5 \cdots (2k-1))/(2k)!$.

Proof. It is well known that $S(n, k)$ satisfies the recursion $S(n+1, k+1) = S(n, k) + (k+1)\, S(n, k+1)$ for $n, k \in \mathbb{P}$. This is equivalent to

$$\Delta f_k(n) = (n+1) f_{k-1}(n+1), \qquad \text{for all } n, k \in \mathbb{P}, \tag{1}$$

where Δ is the usual difference operator (defined by $\Delta h(n) = h(n+1) - h(n)$).

We now prove by induction on k that $f_k(n)$ is a polynomial of degree $d_k = 2k$ and leading coefficient $\alpha_k = (1 \cdot 3 \cdot 5 \cdots (2k-1))/(2k)!$ This statement is clearly true for $k = 0$, since $f_0(n) = 1$. Now assume it for k. Recall that a function $h\colon \mathbb{P} \to \mathbb{C}$ is a polynomial of degree d and leading coefficient α if and only if $\Delta h(n)$ is a polynomial of degree $d - 1$ and leading coefficient αd. Hence by (1) and the induction hypothesis, $f_{k+1}(n)$ is a polynomial of degree $d_k + 2 = 2(k+1)$ and leading coefficient $\alpha_k/(2k+2) = (1 \cdot 3 \cdot 5 \cdots (2k+1))/(2k+2)!$.

The proof for $g_k(n)$ is analogous, using the recursion $c(n+1, k+1) = c(n, k) + n \cdot c(n, k+1)$ in the form $\Delta g_k(n) = n \cdot g_{k-1}(n)$. This completes the proof.

We call the polynimials $f_k(n)$ and $g_k(n)$ *Stirling polynomials*. (These are not to be confused with the related but not identical "Stirling polynomials" of [6, Sect. 77].) Although $f_k(n)$ and $g_k(n)$ were originally defined only for $n \in \mathbb{P}$, as polynomials they can now be evaluated at any complex number. In particular, in the spirit of [11] we can ask for a combinatorial interpretation of $f_k(n)$ and $g_k(n)$ at *negative* integers n. A brief history of this problem appears in [5]. It may be regarded as a reformulation of the well-known orthogonality between $S(n, k)$ and $s(n, k)$, although we will give a direct proof.

PROPOSITION 1.2. *We have*

$$f_k(0) = f_k(-1) = \cdots = f_k(-k) = 0, \qquad \textit{for all } k \in \mathbb{P}, \tag{2}$$

and

$$f_k(-n) = g_k(n), \qquad \textit{for all } n \in \mathbb{Z},\ k \in \mathbb{P}. \tag{3}$$

Proof. There are several ways to prove this result, of which the following is perhaps the most straightforward. The polynomials $f_k(n)$ are uniquely determined by the conditions $f_0(n) = 1$, $f_k(1) = 1$ if $k \in \mathbb{P}$, and $\Delta f_k(n) = (n+1) f_{k-1}(n+1)$ if $k \in \mathbb{P}$. This latter identity we originally observed for $n \in \mathbb{P}$, but by Proposition 1.1 must be a polynomial identity and therefore valid for all $n \in \mathbb{Z}$ (or even $n \in \mathbb{C}$).

We prove (2) by induction on k. It is true for $k = 1$ since $f_1(n) = \binom{n+1}{2}$. Assuming its validity for $k-1$ we have $\Delta f_k(0) = f_{k-1}(1) = 1$ and $\Delta f_k(-i) = (-i+1) f_{k-1}(-i+1) = 0$ if $i \in [k]$. These $k+1$ equations, together with $f_k(1) = 1$, imply (2).

Define $h_k(n) = f_k(-n)$. Then $h_0(n) = 1$. Moreover, since $-f_k(-k-1) = \Delta f_k(-k-1) = -k \cdot f_{k-1}(-k)$ for $k \geqslant 1$ and since $f_0(-1) = 1$, we get $h_k(k+1) = k!$. Finally, putting $-n-1$ for n in (1), we get $\Delta h_k(n) = n \cdot h_{k-1}(n)$. But these conditions on $h_k(n)$ are precisely the conditions which uniquely determine $g_k(n)$. Hence $h_k(n) = g_k(n)$, and the proof is complete.

2. Stirling Permutations

Our main result on Stirling polynomials can best be motivated by recalling some properties of the Eulerian numbers $A_{k,i}$. They are defined by the equation

$$\sum_{n=0}^{\infty} n^k x^n = \left(\sum_{i=1}^{k} A_{k,i} x^i \right) / (1-x)^{k+1}.$$

The Eulerian number $A_{k,i}$ has the following combinatorial interpretation: $A_{k,i}$ is the number of permutations $a_1 a_2 \cdots a_k$ of the set $[k]$ having exactly i *descents* (or *falls*); i.e., for exactly i values of $j \in [k]$ do we have $a_j > a_{j+1}$ or $j = k$. See, e.g., [8; 3, pp. 240–246; 12, Example 4.10; 7, Vol. 3, 5.1.3; 4].

Now Propositions 1.1 and 1.2 imply that for each $k \in \mathbb{P}$, there are integers $B_{k,i}$, $i \in [k]$, such that

$$\sum_{n=0}^{\infty} f_k(n)\, x^n = \left(\sum_{i=1}^{k} B_{k,i} x^i \right) / (1-x)^{2k+1} \tag{4}$$

and

$$\sum_{n=0}^{\infty} g_k(n)\, x^n = \left(\sum_{i=k+1}^{2k} B_{2k-i+1,i} x^i \right) / (1-x)^{2k+1}, \tag{5}$$

where $\sum_{i=1}^{k} B_{k,i} = 1 \cdot 3 \cdot 5 \cdots (2k-1)$. The deduction of (4) and (5) from Propositions 1.1 and 1.2 follows from basic results in the theory of generating functions, see e.g., [12, Corollaries 4.5 and 4.6]. Thus we can ask, in analogy to the Eulerian numbers, whether there is a combinatorial interpretation of the numbers $B_{k,i}$. (It is not even evident a priori that they are nonnegative.) Ideally $B_{k,i}$ will count the number of permutations, from some class of $1 \cdot 3 \cdot 5 \cdots (2k-1)$ permutations, with exactly i descents. This is precisely the content of the next theorem. We shall give three different proofs of this theorem, reflecting three different combinatorial features of Stirling polynomials.

First let us remark that the numbers $B_{k,i}$ have been previously investigated by Carlitz [1, 2] and Riordan [10, Sect. 4] using a different notation. Carlitz derives their formal properties and asks for a combinatorial interpretation. Riordan states such an interpretation, but not in terms of descents of permutations.

THEOREM 2.1. *Let Q_k be the set of all permutations $a_1 a_2 \cdots a_{2k}$ of the multiset $M_k = \{1, 1, 2, 2, \ldots, k, k\}$ such that if $u < v < w$ and $a_u = a_w$, then $a_v > a_u$. (We call the elements of Q_k Stirling permutations.) Then $B_{k,i}$ is equal to the number of permutations $a_1 a_2 \cdots a_{2k} \in Q_k$ with exactly i descents, i.e., such that $a_j > a_{j+1}$ or $j = 2k$ for exactly i values of $j \in [2k]$.*

First proof. Let $C_{k,i}$ be the number of permutations in Q_k with exactly i descents. Thus $C_{1,1} = 1$, $C_{1,i} = 0$ if $i \neq 1$, $C_{k,0} = 0$. We claim that

$$C_{k,i} = i \cdot C_{k-1,i} + (2k - i)\, C_{k-1,i-1}, \qquad \text{for all } k, i \geqslant 2. \tag{6}$$

Every permutation π in Q_k can be obtained uniquely by choosing a permutation σ in Q_{k-1} and inserting two consecutive k's somewhere in σ (there are $2k - 1$ ways to insert the two k's). In order that π have i descents, there are two possibilities: (a) σ has i descents and the two k's are inserted between a descent or at the end, (b) σ has $i - 1$ descents and the two k's are not inserted between a descent or at the end. In (a), there are $C_{k-1,i}$ choices for σ and i ways to insert the k's. In (b), there are $C_{k-1,i-1}$ choices for σ and $2k - i$ ways to insert the k's. This proves (6).

It is clear that $B_{k,i}$ satisfies the initial conditions $B_{1,1} = 1$, $B_{1,i} = 0$ if $i \neq 1$, $B_{k,0} = 0$. Hence the proof will be complete if we can show $B_{k,i}$ satisfies the same recursion (6) which holds for $C_{k,i}$. Let

$$F_k(x) = \sum_{n=0}^{\infty} f_k(n)\, x^n = \Big(\sum_{i=1}^{k} B_{k,i} x^i\Big)/(1 - x)^{2k+1}.$$

Then $(1 - x) F_k(x) = \sum_{n=0}^{\infty} [\Delta f_k(n - 1)]\, x^n = \sum_{n=0}^{\infty} n \cdot f_{k-1}(n)\, x^n = x\, d/dx\, F_{k-1}(x)$. Thus

$$\frac{\sum_{i=1}^{k} B_{k,i} x^i}{(1 - x)^{2k}} = \frac{\sum_{i=1}^{k-1} i \cdot B_{k-1,i} x^i}{(1 - x)^{2k-1}} + \frac{(2k - 1) \sum_{i=1}^{k-1} B_{k-1,i} x^{i+1}}{(1 - x)^{2k}} \tag{7}$$

Multiplying (7) by $(1 - x)^{2k}$ and equating coefficients of x^i gives $B_{k,i} = i \cdot B_{k-1,i} - (i - 1)\, B_{k-1,i-1} + (2k - 1)\, B_{k-1,i-1} = i \cdot B_{k-1,i} + (2k - i) \times B_{k-1,i-1}$. This completes the proof.

Note that Theorem 2.1 implies that $|Q_k| = 1 \cdot 3 \cdot 5 \cdots (2k - 1)$, since we already observed that $\sum_{i=1}^{k} B_{k,i} = 1 \cdot 3 \cdot 5 \cdots (2k - 1)$. A direct combinatorial proof of this is implicit in the above proof, since we noted that there are $2k - 1$ ways of choosing $\pi \in Q_k$ from $\sigma \in Q_{k-1}$ by inserting two consecutive k's, and each π is obtained exactly once in this way. We leave to the reader the easy problem of giving a direct combinatorial proof that $B_{k,k} = k!$. Table I gives the values of $B_{k,i}$ for $1 \leqslant k \leqslant 8$.

TABLE I

The Numbers $B_{k,i}$

	k							
i	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2		2	8	22	52	114	240	494
3			6	58	328	1452	5610	19950
4				24	444	4400	32120	195800
5					120	3708	58140	644020
6						720	33984	785304
7							5040	341136
8								40320

Second proof of Theorem 2.1. We now give a direct combinatorial proof that $C_{k,i} = B_{k,i}$, where $C_{k,i}$ is the number of permutations in Q_k with exactly i descents and $B_{k,i}$ is defined by (4). We show that

$$\sum_{n=0}^{\infty} f_k(n)\, x^n = \left(\sum_{i=1}^{k} C_{k,i} x^i \right) / (1-x)^{2k+1} \tag{8}$$

by constructing objects whose generating function is the right side of (8) and exhibiting a bijection between these objects and partitions of sets.

Let $\pi = a_1 a_2 \cdots a_{2k}$ be a permutation in Q_k for $k > 0$. We define the "spaces" of π to be the integers $0, 1, \ldots, 2k$, and we think of space i as lying between a_i and a_{i+1} for $1 \leqslant i \leqslant k$, with space 0 before a_1 and space $2k$ after a_{2k}. We say that a space i is a *descent* of π if $a_i > a_{i+1}$ or if $i = 2k$.

Now for any permutation π, we define a *barred permutation* on π to be a sequence of integers and bars (/) formed from π by inserting bars in some of the spaces of π. For example, /1/2//21 is a barred permutation on 1221. We now define P_k to be the set of barred permutations on elements of Q_k with at least one bar in each descent. For example, //123//3/2/144/ is an element of P_4. Let $P_{k,n}$ be the set of elements of P_k with n bars.

LEMMA 2.2. $\sum_{n=0}^{\infty} | P_{k,n} | x^n = (\sum_{i=1}^{k} C_{k,i} x^i)/(1-x)^{2k+1}$, *where* $C_{k,i}$ *is the number of permutations in* Q_k *with i descents.*

Proof. For $\pi \in Q_k$, let $d(\pi)$ be the number of descents of π. Observe that every element of $P_{k,n}$ can be obtained uniquely from some permutation

π in Q_k by first putting a bar in each of the $d(\pi)$ descents of π and then putting any number of bars in each of the $2k+1$ spaces of π. Thus

$$\sum_{n=0}^{\infty} |P_{k,n}| x^n = \left(\sum_{\pi \in Q_k} x^{d(\pi)}\right)(1 + x + x^2 + \cdots)^{2k+1}$$

$$= \left(\sum_{i=1}^{k} C_{k,i} x^i\right) / (1 - x)^{2k+1}.$$

THEOREM 2.3. $|P_{k,n}| = S(n+k, n) = f_k(n)$.

Proof. We exhibit a bijection between $P_{k,n}$ and the set of partitions of $[n+k]$ into n blocks. First note that if π is an element of P_k for $k > 0$ then π can be obtained uniquely from some $\pi' \in P_{k-1}$ by inserting to the left of some bar in π' a pair of k's separated by some number of bars. For example, 112/3//3/2/ is obtained from 112//2 by inserting 3//3 to the left of the second bar. Let us call a pair of integers separated by some number of bars an *adjunct*. Then by iteration of the above procedure, any element of P_k can be built up in one way from an element of P_0 (which is a sequence of bars) by successively inserting adjuncts.

As we construct a barred permutation π in $P_{k,n}$, we simultaneously label its bars and the left occurrence of each of its numbers with the lable set $\{1, 2,..., n+k\}$ and construct a partition of this label set into n blocks, according to the following procedure:

The labels are assigned in the natural order: $1, 2,..., n+k$. We start with a sequence of bars which we label from left to right. We then insert the adjunct $1// \cdots 1$ and label the left 1, then the bars from left to right. We then insert $2// \cdots 2$ and label the left 2, then the bars from left to right, and so on.

As we label, we construct a partition of the set of labels. When we label a bar we put the label in a new block, and when we label a number we put the label in the same block as the label of the bar to the left of which it is inserted.

For example, let $\pi = 11/2/344/3/2//$. Then the above construction yields

/// 123	{1}{2}{3}
11/// 4 123	{1, 4}{2}{3}
11/2//2// 4 15 67 23	{1, 4}{2, 5}{3}{6}{7}
11/2/3/3/2// 4 15 68 9 7 23	{1, 4}{2, 5}{3}{6}{7, 8}{9}
11/2/344/3/2// 4 15 68 10 9 7 23	{1, 4}{2, 5}{3}{6}{7, 8}{9, 10}.

Now suppose we are given a partition of $[n+k]$ into n blocks for which we wish to find the corresponding element of $P_{k,n}$. Let us associate to each block $\{a_1, \ldots, a_j\}$ of the partition, where $a_1 < a_2 < \cdots < a_j$, the permutation $a_2a_2a_3a_3 \cdots a_ja_j\underline{a_1}$, where we underline a_1. Let the permutations obtained in this manner be $\pi_1, \pi_2, \ldots, \pi_n$, in increasing order of underlined number, and let c_i be the underlined number of π_i. Now let $\sigma_1 = \pi_1$, and for $1 \leqslant i < n$ construct σ_{i+1} by inserting π_{i+1} into σ_i immediately to the right of the leftmost occurrence of $c_i - 1$ in σ_i. Then to obtain the desired element of $P_{k,n}$, in σ_n change all underlined numbers to bars, and renumber the remaining numbers 1, 1, 2, 2,..., k, k, keeping the same order.

For example, if our partition is {1, 4}{2, 5}{3}{6}{7, 8}{9, 10} then the above procedure yields

$$
\begin{array}{ll}
\pi_1:\ 44\underline{1} & \sigma_1:\ 44\underline{1} \\
\pi_2:\ 55\underline{2} & \sigma_2:\ 44\underline{1}55\underline{2} \\
\pi_3:\ \underline{3} & \sigma_3:\ 44\underline{1}55\underline{2}\underline{3} \\
\pi_4:\ \underline{6} & \sigma_4:\ 44\underline{1}5\underline{6}5\underline{2}\underline{3} \\
\pi_5:\ 88\underline{7} & \sigma_5:\ 44\underline{1}5\underline{6}88\underline{7}5\underline{2}\underline{3} \\
\pi_6:\ 10\ 10\ \underline{9} & \sigma_6:\ 44\underline{1}5\underline{6}8, 10, 10, \underline{9}8\underline{7}5\underline{2}\underline{3}
\end{array}
$$

Changing underlined numbers to bars gives 44/5/8 10 10/8/5// and renumbering gives 11/2/344/3/2//. We leave it to the reader to show that these correspondences are in fact inverse to each other. This completes the second proof of Theorem 2.1.

If π is a permutation in Q_k then we may define the "nondescents" of π to be those spaces of π which are not descents. If π has i descents, then π has $2k + 1 - i$ nondescents. If we now define $\bar{P}_k$ to be the set of barred permutations on elements of Q_k with at least one bar in each nondescent, and $\bar{P}_{k,n}$ to the set of elements of $P_{k,n}$ with n bars, then by the same reasoning as in Lemma 2.2, we see that

$$\sum_{n=0}^{\infty} |\bar{P}_{k,n}|\, x^n = \left(\sum_{i=k+1}^{2k} C_{k,2k+1-i}x^i\right)/(1-x)^{2k+1}.$$

Then one can show that $|\bar{P}_{k,n}| = c(n, n-k) = g_k(n)$ by a correspondence between the elements of $\bar{P}_{k,n}$ and the set of permutations of $[n]$ with $n - k$ cycles, in analogy to Theorem 2.3.

Third proof of Theorem 2.1. We give an alternate proof that $|P_{k,n}| = S(n+k, n)$ by showing that the generating functions for $|P_{k,n}|$ and $S(n+k, n)$ satisfy the same differential equation.

We must first clarify our definition of $P_{k,n}$ for $k = 0$: By definition Q_0 has one element (the "empty permutation"), which has one space and no descents. Thus $| P_{0,n} | = 1$ for all $n \in \mathbb{N}$.

Now let $G(t) = \sum_{k,n=0}^{\infty} | P_{k,n} | x^n(t^k/k!)$.

THEOREM 2.4. *$G(t)$ satisfies the differential equation*

$$G' = G^2(G - 1), \tag{9}$$

where $G' = dG/dt$, together with the initial condition $G(0) = 1/(1 - x)$.

Proof. Equation (9) is equivalent to the recursion

$$| P_{k+1,n} | = \sum \binom{k}{k_1, k_2, k_3} | P_{k_1,n_1} | \cdot | P_{k_2,n_2} | \cdot | P_{k_3,n_3} |, \tag{10}$$

where the sum is over all 6-tuples $(k_1, k_2, k_3, n_1, n_2, n_3) \in \mathbb{N}^6$ satisfying $k_1 + k_2 + k_3 = k$, $n_1 + n_2 + n_3 = n$, and $n_3 \neq 0$. (We have used the fact that $P_{k_3,0} = 0$ for $k_3 > 0$.) The combinatorial interpretation of (10) is best illustrated by an example. Consider the barred permutation 244/2/13355/1/66/ in $P_{6,5}$. If we "remove" the 1's, the barred permutation splits into three parts, 244/2/, 3355/, and /66/. If we now "reduce" these barred permutations, i.e., renumber each with 1, 1, 2, 2, etc., keeping the same original order, then we get 122/1/, 1122/, and /11/, elements of $P_{2,2}$, $P_{2,1}$, and $P_{1,2}$. In general, from any $\pi \in P_{k+1,n}$ (with $k \geqslant 0$) we get by this procedure three barred permutations $\pi_1 \in P_{k_1,n_1}$, $\pi_2 \in P_{k_2,n_2}$, and $\pi_3 \in P_{k_3,n}$ with $k_1 + k_2 + k_3 = k$ and $n_1 + n_2 + n_3 = n$. Moreover, $n_3 \neq 0$ since π must end with a bar.

Moreover given three barred permutations $\pi_1 \in P_{k_1,n_1}$, $\pi_2 \in P_{k_2,n_2}$, and $\pi_3 \in P_{k_3,n_3}$, with $n_3 \neq 0$, we can construct elements of $P_{k+1,n}$, where $k = k_1 + k_2 + k_3$ and $n = n_1 + n_2 + n_3$, as follows: Pick pairwise disjoint sets S_1, S_2, and S_3 such that $S_1 \cup S_2 \cup S_3 = \{2, 3, 4, \ldots, k_1 + k_2 + k_3 + 1\}$, $| S_1 | = k_1$, $| S_2 | = k_2$, and $| S_3 | = k_3$. This can be done in $\binom{k_1+k_2+k_3}{k_1,k_2,k_3}$ ways. Then for $i = 1, 2, 3$, renumber π_i with the elements of S_i, keeping the same order. Call the barred permutations so obtained π_1', π_2', and π_3'. Then the barred permutation $\pi_1' 1 \pi_2' 1 \pi_3'$ is an element of $P_{k+1,n}$, and every element of $P_{k+1,n}$ is obtained uniquely in this manner. This proves (10), and hence (9); moreover, the initial condition $G(0) = 1/(1 - x)$ is equivalent to the statement that $| P_{0,n} | = 1$ for all $n \in \mathbb{N}$.

To solve Eq. (9) we make the substitution $G = (1 - xe^H)^{-1}$, which yields

$$H' = (1 - xe^H)^{-1} = G \tag{11}$$

and the initial condition $H(0) = 0$. Thus, integrating (11) we get

$$H - x(e^H - 1) = t. \tag{12}$$

Equation (12) can be solved by Lagrange's inversion formula [3, p. 148; 12, Theorem 5.6], which gives

$$H = \sum_{k,n=0}^{\infty} S(n+k-1, n)\, x^n \frac{t^k}{k!}. \tag{13}$$

Alternatively, we can show that if $H = \sum_{k,n=0}^{\infty} T_{k,n} x^n (t^k/k!)$, then Eq. (12) implies that $T_{k,n}$ is the number of rooted trees with k labeled nodes and n unlabeled nodes such that the labeled nodes are the terminal nodes (or leaves). This number is known to be $S(n+k-1, n)$ [7, Vol. 1, p. 397, Exercise 19, Solution p. 585].

From (13) and (11) we get $G = H' = \sum_{k,n=0}^{\infty} S(n+k, n)\, x^n (t^k/k!)$, thus $|P_{k,n}| = S(n+k, n)$. This completes the third proof of Theorem 2.1.

Similarly if we set $\bar{G}(t) = \sum_{k,n=0}^{\infty} |\bar{P}_{k,n}|\, x^n (t^k/k!)$, then one can show that $\bar{G}' = (\bar{G}+1)\,\bar{G}^2$, with the initial condition $\bar{G}(0) = x/(1-x)$. This equation has the solution $\bar{G} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} g_k(n)\, x^n (t^k/k!)$.

3. An Open Problem

Can Theorem 2.1 be extended in some way to more general multisets than M_k? Suppose, for instance, we let $r, k \in \mathbb{P}$ and let M be the multiset consisting of r copies of each integer $i \in [k]$. Define $Q_k^{(r)}$ to be the set of all permutations $a_1 a_2 \cdots a_{rk}$ of M such that if $u < v < w$ and $a_u = a_w$, then $a_v \geqslant a_u$. Define $B_{k,i}^{(r)}$ to be the number of permutations in $Q_k^{(r)}$ with exactly i descents. Thus $B_{k,i}^{(1)} = A_{k,i}$ and $B_{k,i}^{(2)} = B_{k,i}$. Define

$$F_k^{(r)}(x) = \left(\sum_{i=1}^{k} B_{k,i}^{(r)} x^i \right) / (1-x)^{rk+1} = \sum_{n=0}^{\infty} f_k^{(r)}(n)\, x^n.$$

Can any combinatorial significance be attached to $f_k^{(r)}(n)$? For instance, $f_k^{(1)}(n) = n^k$, the number of functions $[k] \to [n]$; while $f_k^{(2)}(n) = f_k(n) = S(n+k, n)$, the number of partitions of an $(n+k)$-element set into n blocks. It can be shown, by generalizing the first proof of Theorem 2.1, that $f_k^{(r)}(n)$ satisfies the recursion

$$\Delta^{r-1} f_k^{(r)}(n) = (n+r-1)\, f_{k-1}^{(r)}(n+r-1),$$

for all $n \in \mathbb{N}$, $k \geqslant 2$. However, a "nice" combinatorial interpretation of $f_k^{(r)}(n)$ is desired.

It is also easy to see, generalizing Theorem 2.4, that if we define

$$G_r = G_r(t) = \sum_{k=0}^{\infty} F_k^{(r)}(x)\, t^k/k!$$

(so that $G_2(t) = G(t)$), then $G_r(t)$ satisfies the differential equation

$$G_r' = G_r^r(G_r - 1), \qquad G_r(0) = 1/(1-x).$$

From this it is easy to deduce the following functional equation which generalizes (12):

$$\log\left(1 - \frac{1}{G_r}\right) + \sum_{i=1}^{r-1} \frac{1}{iG_r^i} = t + \log x + \sum_{i=1}^{r-1} \frac{1}{i}(1-x)^i.$$

Similarly the equation for $\bar{G}(t)$ can be generalized to

$$\bar{G}_r' = \bar{G}_r^r(\bar{G}_r + 1), \qquad \bar{G}_r(0) = 1/(1-x).$$

These considerations do not seem to shed much light on the problem of interpreting $f_k^{(r)}(n)$.

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