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The Upper Bound Conjecture and Cohen-Macaulay Rings*

By Richard P. Stanley

Let Δ be a triangulation of a $(d-1)$-dimensional sphere with n vertices. The Upper Bound Conjecture states that the number of i-dimensional faces of Δ is less than or equal to a certain explicit number $c_i(n, d)$. A proof is given of a more general result. The proof uses the result, proved by G. Reisner, that a certain commutative ring associated with Δ is a Cohen-Macaulay ring.

1. Introduction

In this paper we shall show how commutative algebra can be used to obtain information on the number of faces of a simplicial complex. Our treatment will be rather sketchy; in a subsequent paper a more comprehensive account will be given. We begin with the basic terminology and notation and discuss the historical background of the subject. Next we introduce a commutative ring A_Δ associated with a simplicial complex Δ. We show that Δ satisfies the so-called "upper bound conjecture" if A_Δ is a Cohen-Macaulay ring. A recent result of G. Reisner gives necessary and sufficient conditions for A_Δ to be Cohen-Macaulay. From Reisner's result we deduce the previously open "upper bound conjecture for spheres".

2. Background

The following notation is fixed once and for all. Δ denotes a simplicial complex on a finite set $V = \{v_1, v_2, \ldots, v_n\}$ of n elements. Hence, Δ is a collection of subsets of V such that (a) if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$, and (b) $\{v\} \in \Delta$ for all $v \in V$. The elements of Δ are called the *faces* of Δ. If the largest face of Δ has d elements, then we say that Δ has *dimension* $d-1$ and write

$$d = 1 + \dim \Delta = \max_{F \in \Delta} |F|.$$

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The *f-vector* of Δ is $f = (f_0, f_1, \ldots, f_{d-1})$, where exactly f_i faces of Δ have $i + 1$ elements. Thus $f_0 = n$. Let $\mathbf{N}$ denote the non-negative integers, and define a function $H : \mathbf{N} \to \mathbf{N}$ as follows:

$$H(m) = \begin{cases} 1 & \text{if } m = 0 \\ \sum_{i=0}^{d-1} f_i \binom{m-1}{i} & \text{if } m > 0. \end{cases} \tag{1}$$

Hence for $m > 0$, $H(m)$ is a polynomial function of m of degree $d - 1$. It follows from elementary facts concerning generating functions that there exist integers $h_0, h_1, \ldots, h_d$ such that

$$(1 - x)^d \sum_{m=0}^{\infty} H(m)x^m = h_0 + h_1 x + \cdots + h_d x^d, \tag{2}$$

regarded as an identity in the formal power series ring $\mathbf{C}[[x]]$ (where $\mathbf{C}$ denotes the complex numbers). It is easily seen that $h_0 = 1$ and $h_d = (-1)^d(1 - \chi(\Delta))$, where $\chi(\Delta) = f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1}$ is the *Euler characteristic* of Δ. We call $(h_0, h_1, \ldots, h_d)$ the *h-vector* of Δ.

The primary question to concern us here is the following. Given information about Δ, what can be said about the f-vector of Δ? Frequently the information about Δ will be topological in character; we will specify the topological type of the underlying space $X = |\Delta|$ of Δ (as defined, e.g., in [18, pp. 110–111]), or equivalently will say that Δ is a *triangulation* of X. (Note that we make no assumptions about the links of faces of Δ, as is frequently done in combinatorial topology.)

The upper bound conjecture

Let $n > d > 1$. Let $C(n, d)$ be the convex hull of any n distinct points on the "moment curve" $(\tau, \tau^2, \ldots, \tau^d)$, $-\infty < \tau < \infty$, in d-space. $C(n, d)$ is called a *cyclic polytope* and was first investigated by Carathéodory in 1907. Gale and Motzkin rediscovered the concept. (See [5, p. 127] for further historical details.) Let $c_i(n, d)$ be the number of i-dimensional faces of $C(n, d)$. Motzkin [16] conjectured (implicitly) that if P is any d-dimensional convex polytope with n vertices and f_i i-dimensional faces, then $f_i \leqslant c_i(n, d)$. The reason for this conjecture is the following. First, it can be shown by the process of "pulling the vertices" [5, p. 80; 15, Sec. 2.5] that it suffices to assume that P is a simplicial convex polytope, i.e., every face of P except P itself is a simplex. (Thus the boundary complex of P is a simplicial complex.) Secondly, it can be shown [5, Sec. 4.7; 10, Sec. 2.3 (vi)] that $C(n, d)$ is a simplicial convex polytope with $c_i(n, d) = \binom{n}{i+1}$ if $0 \leqslant i \leqslant m - 1$, where $m = [d/2]$. Since each i-dimensional face $(0 \leqslant i < d)$ of a d-dimensional simplicial convex polytope P has $i + 1$ vertices, clearly $f_i(P) \leqslant \binom{n}{i+1}$. Hence the cyclic polytopes maximize

f_i when $1 \leqslant i \leqslant m-1$. Now the Dehn-Sommerville equations [5, Sec. 9.2; 15, Sec. 2.4] allow us to solve for $f_m, f_{m+1}, \ldots, f_{d-1}$ in terms of $f_0, f_1, \ldots, f_{m-1}$, when $(f_0, f_1, \ldots, f_{d-1})$ is the f-vector of a simplicial convex polytope of dimension d. Thus it is natural to expect that maximizing $f_1, f_2, \ldots, f_{m-1}$ simultaneously maximizes $f_m, f_{m+1}, \ldots, f_{d-1}$. This is the content of Motzkin's conjecture, known as the *upper bound conjecture* (UBC) *for convex polytopes*. After some special cases of the UBC for convex polytopes were settled by Fieldhouse, Gale, and Klee, finally in 1970 McMullen [13; 15, Chapter 5] proved Motzkin's conjecture. McMullen's proof used a result of Brugesser and Mani [2] that the boundary complex of a simplicial convex polytope (or any convex polytope) is "shellable".

Let Δ be the boundary complex of a d-dimensional simplicial convex polytope with n vertices. It is convenient (and indeed necessary for proofs) to formulate the UBC for convex polytopes in terms of the h-vector defined by (2) rather than in terms of the f-vector. It can be shown that the UBC is equivalent to

$$h_i \leqslant \binom{n-d+i-1}{i}, \qquad 0 \leqslant i \leqslant d. \tag{3}$$

See for example [15, Lemma 14, p. 173], where $g_k^{(d)}(P)$ is used for our h_{k+1}. In 1964 Klee [9] extended the UBC to arbitrary manifolds. His conjecture is equivalent to the following. Let Δ be a triangulation of a $(d-1)$-dimensional manifold, with $f_0 = n$. Then Klee in effect conjectured that (3) holds for Δ. Thus for *any* simplicial complex Δ of dimension $d-1$ with $f_0 = n$, we say that *the UBC holds for* Δ if (3) holds for Δ. The special case of Klee's conjecture when $|\Delta|$ is a sphere is known as the *UBC for spheres* (sometimes called the "UBC for simplicial spheres" or the "UBC for spherical polytopes"). Note that it is not *a priori* evident that the UBC for spheres is stronger than the UBC for convex polytopes, i.e., that there is a triangulation Δ of a sphere which is not the boundary complex of some simplicial convex polytope. Such triangulations do, however, exist [15, Sec. 11.5]. Hence the UBC for convex polytopes does not subsume the UBC for spheres. Moreover, McMullen's proof of the UBC for convex polytopes cannot be extended in an obvious way to spheres, since there are known to be triangulations of spheres which are not shellable. Thus until now the UBC for spheres has remained open. We shall indicate a proof of the UBC for various Δ which include triangulations of spheres, thereby establishing the UBC for spheres. We will in fact give an even stronger condition on the h_i's than (3).

3. Hilbert functions

Let Δ be a simplicial complex on $V = \{v_1, v_2, \ldots, v_n\}$, and let K be any field. Form the polynomial ring $R = K[v_1, v_2, \ldots, v_n]$, where the v_i are regarded as independent (commuting) indeterminates. Let I be the homogeneous ideal of R generated by all square-free monomials $v_{i_1} v_{i_2} \cdots v_{i_r}$ with $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} \not\in \Delta$. Let $A_\Delta = R/I$. We wish to obtain information about the f-vector of Δ by studying algebraic properties of A_Δ. To do this, we first show that the f-vector of

Δ determines the Hilbert function of A_Δ. Recall that if I is any homogeneous ideal of R, then $A = R/I$ is a *graded ring*,

$$A = A_0 + A_1 + A_2 + \cdots, \qquad A_iA_j \subseteq A_{i+j},$$

where A_m is the vector space (over K) of all homogeneous polynomials in A of degree m (see, e.g., [20, p. 149; 1, p. 106; 12, Chapter 4]). The *Hilbert function* $H : \mathbf{N} \to \mathbf{N}$ of A is defined by

$$H(m) = \dim_K A_m.$$

For the sake of completeness we recall two standard results about Hilbert functions (see, e.g., [1, Chapter 11; 20, p. 247]).

PROPOSITION 3.1. *Let I be a homogeneous ideal of $R = K[v_1, \ldots, v_n]$, and let $A = R/I$. The Hilbert function $H(m)$ of A is a polynomial function of m for m sufficiently large.* [*This polynomial is called the Hilbert polynomial of A. It agrees with $H(m)$ for all m sufficiently large.*] *The Krull dimension* $\dim A$ *of A (i.e., the length of a longest chain of prime ideals of A, or equivalently the maximum number of elements of A which are algebraically independent over K) is one more than the degree of the Hilbert polynomial of A.* ■

We now are ready to compute the Hilbert function of A_Δ.

PROPOSITION 3.2. *The Hilbert function of A_Δ is the function $H(m)$ of* (1).

Proof: Let $\bar{v}_i$ be the image of v_i under the natural homomorphism $R \to A_\Delta$. A K-basis for $(A_\Delta)_m$, the mth homogeneous part of A_Δ, consists of all monomials $w = \bar{v}_1^{a_1}\bar{v}_2^{a_2} \cdots \bar{v}_n^{a_n}$ such that $\deg w = m$ (i.e., $a_1 + a_2 + \cdots + a_n = m$) and $\operatorname{supp} w \in \Delta$, where $\operatorname{supp} w$ is the *support* of w, defined by $\operatorname{supp} w = \{v_j : a_j \geq 0\}$. The only monomial of degree 0 is 1, which has support $\emptyset$. Hence the Hilbert function of A_Δ at $m = 0$ is 1. Moreover, 1 is the only monomial with support $\emptyset$. If $F \in \Delta$ has $i+1$ elements, $i \geq 0$, then the number of monomials of degree $m > 0$ with support F is $\binom{m-1}{i}$. Hence the Hilbert function of A_Δ agrees with $H(m)$ when $m > 0$, so the proof follows. ■

As an immediate corollary of Propositions 3.1 and 3.2, we get that the Krull dimension $\dim A_\Delta$ of A_Δ is $d = 1 + \dim \Delta$.

We now require the concept of an "order ideal of monomials." We say that a set M of monomials in variables $y_1, y_2, \ldots$ is an *order ideal of monomials* if whenever $w \in M$ and w' divides w, then $w' \in M$. For instance, if $y_1^2y_2y_3 \in M$ and if M is an order ideal of monomials, then $y_1^2y_2, y_1^2y_3, y_1y_2y_3, y_1^2, y_1y_2, y_1 y_3, y_2y_3, y_1, y_2, y_3, 1 \in M$. The next proposition is essentially due to Macaulay [11].

PROPOSITION 3.2. *Let $R = K[v_1, \ldots, v_n]$ as before, let I be any homogeneous ideal of R, and let $A = R/I$. Also let $\bar{v}_i$ be the image of v_i under the natural homomorphism $R \to A$. Then one can find a K-basis for A consisting of an order ideal of monomials in the variables $\bar{v}_1, \ldots, \bar{v}_n$.*

Sketch of proof: Order all monomials in the variables $\bar{v}_1, \ldots, \bar{v}_n$ lexicographically. This means that $w < w'$ if either (a) $\deg w < \deg w'$ or (b) $\deg w = \deg w'$ (say $w = \bar{v}_1^{a_1} \cdots \bar{v}_n^{a_n}$, $w' = \bar{v}_1^{a'_1} \cdots \bar{v}_n^{a'_n}$), and for sufficiently large A (specifically $A > \deg w$) we have $\Sigma a_i A^{i-1} < \Sigma a'_i A^{i-1}$. Define a K-basis $b_1, b_2, \ldots$ for A as follows: (a) $b_1 = 1$, (b) once $b_1, \ldots, b_i$ are chosen, choose b_{i+1} to be the least monomial (in the lexicographic ordering we have defined) linearly independent of $b_1, \ldots, b_i$. (If no such b_{i+1} exists, then the process terminates with b_i.) Then it is straightforward to check that $b_1, b_2, \ldots$ is an order ideal of monomials. ∎

We say that a sequence $k = (k_0, k_1, \ldots)$, finite or infinite, of integers is an *O-sequence* if there is an order ideal M of monomials such that $k_i = |\{w \in M : \deg w = i\}|$. Proposition 3.2 implies the following corollary.

COROLLARY 3.3. *Let I be a homogeneous ideal of $R = K[v_1, \ldots, v_n]$, and let $H(m)$ be the Hilbert function of R/I. Then $(H(0), H(1), \ldots)$ is an O-sequence.* ∎

It is natural to ask for a purely combinatorial criterion for a sequence $(k_0, k_1, \ldots)$ to be an O-sequence. This question was essentially answered by Macaulay [11], with later proofs by Sperner, by Whipple, and most recently by Clements and Lindström [3]. (See also [4, Sec. 7].) We now state this result, in a more explicit form than given by the above persons. Recall that if h and i are positive integers, then h can be written uniquely in the form

$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geqslant j \geqslant 1$. Following McMullen [14], define

$$h^{\langle i \rangle} = \binom{n_i + 1}{i + 1} + \binom{n_{i-1} + 1}{i} + \cdots + \binom{n_j + 1}{j + 1}.$$

Also define $0^{\langle i \rangle} = 0$.

PROPOSITION 3.4. *A sequence $(k_0, k_1, \ldots)$ of integers is an O-sequence if and only if $k_0 = 1$, $k_1 \geqslant 0$, and for all $i \geqslant 1$ for which k_{i+1} is defined we have*

$$0 \leqslant k_{i+1} \leqslant k_i^{\langle i \rangle}. \quad ∎$$

4. Cohen-Macaulay rings

We now come to the question of how the Hilbert function of A is affected by additional algebraic conditions on A. The condition to concern us here will be that A is a *Cohen-Macaulay ring*. (Some other conditions are mentioned in [19] and will be discussed in a subsequent paper.) We refer the reader to the literature for the definition of Cohen-Macaulay rings (e.g., [20, p. 396; 8, Sec. 3.1; 12, p. 103]). We will state, however, a characterization (sometimes known as "Hironaka's criterion") of Cohen-Macaulay rings of the type we have been considering. For further information, see [7, p. 1036].

PROPOSITION 4.1. *Let I be a homogeneous ideal of $R = K[v_1, \ldots, v_n]$, and let $A = R/I$ have Krull dimension d. Then A is Cohen-Macaulay if and only if there exist d homogeneous elements $\theta_1, \ldots, \theta_d$ of A which may be chosen to be of degree 1, if K is infinite, and finitely many homogeneous elements $\eta_1, \ldots, \eta_t$ of A, such that every $y \in A$ can be written uniquely in the form $y = \sum_{i=1}^{t} \eta_i p_i(\theta_1, \ldots, \theta_d)$, where $p_i(x_1, \ldots, x_d) \in K[x_1, \ldots, x_d]$.* ∎

Remark: Even if A isn't Cohen-Macaulay in Proposition 4.1, we can always find algebraically independent elements $\theta_1, \ldots, \theta_d$ (which may be chosen to be homogeneous of degree 1 if K is infinite) and finitely many homogeneous elements $\eta_1, \ldots, \eta_t$ such that every $y \in A$ can be written in the form $y = \Sigma \eta_i p_i(\theta_1, \ldots, \theta_d)$. The θ_i's are called a *sequence of parameters*, and their existence is guaranteed by the Noether normalization lemma. The crucial property which makes A Cohen-Macaulay is that the η_i's can be chosen so that the above representation is *unique* for all $y \in A$. For the remainder of this section we assume that K is infinite.

Suppose that $A = R/I$ (as in Proposition 4.1) is Cohen-Macaulay, and let $\theta_1, \ldots, \theta_d, \eta_1, \ldots, \eta_t$ be the elements of A whose existence is guaranteed by Proposition 4.1. Then a K-basis for A obviously can be taken to be all elements of A of the form $\eta_i w$, where w is a monomial in the θ_i's. For fixed η_i, the number of elements $\eta_i w$ of degree m is clearly the coefficient of x^m in $x^{\deg \eta_i}/(1-x)^d$. Summing over all i, we get

$$\sum_{m=0}^{\infty} H(m)x^m = \frac{\sum_{i=1}^{t} x^{\deg \eta_i}}{(1-x)^d}. \tag{4}$$

We are now ready to state the main result of this section (which is given implicitly in [11]).

THEOREM 4.2. *Let I be a homogeneous ideal of $R = K[v_1, \ldots, v_n]$, and suppose $A = R/I$ is Cohen-Macaulay of Krull dimension d. Let $H(m)$ be the Hilbert function of A, and define integers h_i by*

$$(1-x)^d \sum_{m=0}^{\infty} H(m)x^m = h_0 + h_1 x + \cdots \tag{5}$$

Then $(h_0, h_1, \ldots)$ is an O-sequence. Moreover, $h_i = 0$ for i sufficiently large. Finally, if I contains no elements of degree 1, *then $h_1 = n - d$.*

Proof: Choose the θ_i's and η_j's as in Proposition 4.1. Define $B = A/(\theta_1, \ldots, \theta_d)$. It follows from Proposition 4.1 that the images of the η_j's in B are a K-basis for B. Thus by (4) and (5) the Hilbert function for B is just h_m. Hence by Corollary 3.3 we have that $(h_0, h_1, \ldots)$ is an O-sequence. Also, $h_i = 0$ for i large, since the η_i are finite in number. Finally, the space B_1 of forms of degree 1 in B will be a quotient space of A_1 by the K-space spanned by $\theta_1, \ldots, \theta_d$. Since $\dim A_1 = n$ if I contains no elements of degree 1, and since $\theta_1, \ldots, \theta_d$ are necessarily linearly independent (even algebraically independent), we have $h_1 = \dim B_1 = \dim A_1 - d = n - d$. ■

COROLLARY 4.3. *Let Δ be as in Section* 2 *and suppose that A_Δ is Cohen-Macaulay. Then the h-vector $(h_0, h_1, \ldots, h_d)$ of Δ is an O-sequence with $h_1 = n - d$.* ■

COROLLARY 4.4. *If A_Δ is Cohen-Macaulay, then the UBC holds for Δ.*

Proof: By Corollary 4.3, $(h_0, h_1, \ldots, h_d)$ is an O-sequence with $h_1 = n - d$. Hence there is an order ideal M of monomials in $n - d$ variables $y_1, \ldots, y_{n-d}$ such that $h_i = |\{w \in M : \deg w = i\}|$. Thus h_i can be no more than the *total* number of monomials of degree i in $n - d$ variables, this number being $\binom{n-d+i-1}{i}$. The proof follows from our definition of the UBC. ∎

We remark that Corollary 4.3 is the best possible, in the sense that given any O-sequence $(h_0, h_1, \ldots, h_d)$, there exists a $(d-1)$-dimensional simplicial complex Δ with h-vector $(h_0, h_1, \ldots, h_d)$ such that A_Δ is Cohen-Macaulay. One may even choose Δ to be semi-shellable in the sense of [10]; it can be shown that if Δ is semi-shellable, then A_Δ is Cohen-Macaulay over any field K.

5. The upper bound conjecture

The following question now arises: For what Δ is A_Δ Cohen-Macaulay? This question was recently answered by G. Reisner [17] (who was unaware of the relevance of his work to the UBC). In order to state Reisner's theorem, recall that if $F \in \Delta$, then the *link* of F, denoted lk F, is defined by

$$\text{lk}\, F = \{G \in \Delta : G \cap F = \emptyset \text{ and } G \cup F \in \Delta\}.$$

Thus lk F is a subcomplex of Δ. In particular, $\text{lk}\,\emptyset = \Delta$.

THEOREM 5.1 (G. Reisner). *The following two conditions are equivalent*:

(i) A_Δ *is Cohen-Macaulay*,

(ii) *for each* $F \in \Delta$ (*including* $F = \emptyset$), *the reduced homology of* lk F *with coefficients in* K *vanishes in all dimensions except possibly the dimension of* lk F. ∎

Using Theorem 5.1, a routine application of the universal coefficient theorem shows that if A_Δ is Cohen-Macaulay over some field K, then A_Δ is Cohen-Macaulay over the rationals $\mathbf{Q}$. Hence we lose nothing in the following corollary when we assume $K = \mathbf{Q}$.

COROLLARY 5.2. *If* Δ *satisfies condition* (ii) *of Theorem* 5.1 *with* $K = \mathbf{Q}$, *then* $(h_0, h_1, \ldots, h_d)$ *is an O-sequence and* Δ *satisfies the UBC.* ∎

If $|\Delta|$ is a manifold (with boundary allowed), then for any $F \neq \emptyset$ it is known that lk F is either a homology sphere or homology cell (the latter never occurring if $|\Delta|$ is without boundary). Hence we deduce:

COROLLARY 5.3. *If* $|\Delta|$ *is a manifold or manifold with boundary whose reduced homology with coefficients in* $\mathbf{Q}$ *vanishes outside of dimension* $d - 1 = \dim|\Delta|$, *then* $(h_0, h_1, \ldots, h_d)$ *is an O-sequence and the UBC holds for* Δ. *In particular, the UBC holds for spheres and cells.* ∎

References

1. M. F. ATIYAH and I. G. MACDONALD, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Mass., 1969.

2. H. BRUGGESSER and P. MANI, Shellable decompositions of cells and spheres, *Math. Scand.* **29**, 197–205 (1971).

3. G. CLEMENTS and B. LINDSTRÖM, A generalization of a combinatorial theorem of Macaulay, *J. Comb. Theory* **7**, 230–238 (1969).

4. C. Greene and D. J. Kleitman, Methods in extremal set theory, *MAA Stud. Comb.*, to be published.
5. B. Grünbaum, *Convex Polytopes*, Wiley-Interscience, London, New York, Sydney, 1967.
6. P. J. Hilton and S. Wylie, *Homology Theory*, Cambridge U. P., 1960.
7. M. Hochster and J. A. Eagon, Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci, *Am. J. Math.* **93**, 1020–1058 (1971).
8. I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.
9. V. Klee, The number of vertices of a convex polytope, *Can. J. Math.* **16**, 701–720 (1964).
10. V. Klee and G. Danaraj, Shellings of spheres and polytopes, *Duke Math. J.* **41**, 443–451 (1974).
11. F. S. Macaulay, Some properties of enumeration in the theory of modular systems, *Proc. Lond. Math. Soc.* **26**, 531–555 (1927).
12. H. Matsumura, *Commutative Algebra*, Benjamin, New York, 1970.
13. P. McMullen, The maximum numbers of faces of a convex polytope, *Mathematika* **17**, 179–184 (1970).
14. P. McMullen, The numbers of faces of simplicial polytopes, *Isr. J. Math.* **9**, 559–570 (1971).
15. P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, Lond. Math. Soc. Lect. Note Ser. 3, Cambridge U. P., 1971.
16. T. S. Motzkin, Comonotone curves and polyhedra, Abstr. 111, *Bull. Am. Math. Soc.* **63**, 35 (1957).
17. G. A. Reisner, Cohen-Macaulay quotients of polynomial rings, Ph.D. thesis, Univ. of Minn., 1974.
18. E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.
19. R. Stanley, Cohen-Macaulay rings and constructible polytopes, Bull. Am. Math. Soc. **81**, 133–135 (1975).
20. O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, van Nostrand, Princeton, N. J., 1960.

Massachusetts Institute of Technology

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