THE FIBONACCI LATTICE*

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1. DISTRIBUTIVE LATTICES

Our object is to investigate a certain distributive lattice F_1 closely related to the Fibonacci numbers. First we will review some basic properties of distributive lattices and discuss some general combinatorial problems associated with them. Thus this paper can be regarded as a semi-expository survey of some combinatorial aspects of distributive lattices.

In order that the combinatorial invariants we will be considering are finite, we need to restrict ourselves to distributive lattices L satisfying the following property:

(W) L is locally finite with a unique minimal element 0, and only finitely many elements of any given rank (or height).

By *locally finite,* we mean that every segment $[x,y] = \{z \mid x \leq z \leq y\}$ of L is finite. The *rank k* of an element $x \in L$ is the length of the longest chain between 0 and x. In any distributive lattice, if the length k of the longest chain between two elements x and y is finite, then the length of any saturated (or unrefinable) chain between x and y is also k. A distributive lattice satisfying property *(W)* will be called a *W-distributive lattice.*

Recall that an *order ideal* of a partially ordered set P is a subset $I \subseteq P$ such that if $x \in I$ and $y \leq x$, then $y \in I$. By a fundamental theorem of Garrett Birkhoff [2, Ch. III, §3], corresponding to every W-distributive lattice L is a partially ordered set P, uniquely determined up to isomorphism, satisfying the following three properties:

(i) Every element of P is contained in a finite order ideal of P,

(ii) P has only finitely many order ideals of any given finite cardinality k,

(iii) L is isomorphic to the set of finite order ideals of P, ordered by inclusion.

Conversely, given any partially ordered set P satisfying (i) and (ii), the lattice of finite order ideals of P (ordered by inclusion) is a W-distributive lattice. A partially ordered set satisfying (i) and (ii) is called a *W-ordered set.* The correspondence between W-ordered sets P and W-distributive lattices L is denoted $L = J(P)$. P is isomorphic to the sub-ordered set of L consisting of all the join-irreducible elements of L. If I is a finite order ideal of P, then the cardinality $|I|$ of I is equal to the rank of I in $J(P)$.

If P is a W-ordered set, then we define a *P-partition of n* [18] to be an order-reversing map $\sigma : P \to \{0, 1, 2, \cdots\}$ satisfying

$$\sum_{x \in P} \sigma(x) = n.$$

(In particular, only finitely many elements x of P satisfy $\sigma(x) > 0$.) The statement that σ is *order-reversing* means that if $x \leq y$ in P, then $\sigma(x) \geq \sigma(y)$. The *parts* of σ are the non-zero values $\sigma(x)$ (counting multiplicities). Let $a(m,n)$ denote the number of P-partitions of n with largest part $\leq m$. Since P is a W-ordered set, it follows easily that $a(m,n)$ is finite. It can be shown that $a(m,n)$ is the number of order ideals of cardinality n in the direct product $P \times \underline{m}$, where $\underline{m}$ denotes an m-element chain,

$$\underline{m} = \{1, 2, \cdots, m\}.$$

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Furthermore, let $a(n)$ denote the total number of P-partitions of n. Hence

$$\lim_{m\to\infty} a(m,n) = a(n),$$

and $a(n)$ is the number of order ideals of cardinality n in the partially ordered set $P\times\underline{N}$, where $\underline{N}$ denotes the natural numbers,

$$\underline{N} = \{1, 2, 3, \cdots\}.$$

In particular, $a(1,n)$ is equal to the number of order ideals of cardinality n in P (equivalently, the number of elements of rank n in $J(P)$), since $P\times\underline{1} = P$. In fact, there is a one-to-one correspondence $\sigma \leftrightarrow I(\sigma)$ between order-reversing maps $\sigma : P \to \{0,1\}$ satisfying

$$\sum_{x\in P} \sigma(x) = n,$$

and order ideals $I(\sigma)$ of P of cardinality n, *viz.*,

$$I(\sigma) = \{x \mid \sigma(x) = 1\}.$$

The number $a(1,n)$ is denoted $j_n(P)$ or simply j_n. If P is finite, then the total number of order ideals of P is denoted $j(P)$, so $j(P) = |J(P)|$.

If $L = J(P)$ is a W-distributive lattice and $I \in L$, then define $e(I)$ to be the number of saturated chains between 0 and I. (This number is obviously finite.) It is not difficult to see that $e(I)$ is equal to the number of order-preserving bijections $\sigma : I \to \underline{k}$, where $|I| = k$. In fact, such a bijection σ corresponds to the saturated chain

$$\phi \subset \sigma^{-1}(\underline{1}) \subset \sigma^{-1}(\underline{2}) \subset \cdots \subset \sigma^{-1}(\underline{k}). \tag{1}$$

Thus a saturated chain between 0 and I corresponds to a permutation $\sigma^{-1}(1), \sigma^{-1}(2), \cdots, \sigma^{-1}(k)$ of the elements of I. This provides a systematic basis for studying relationships between sequences and lattice paths which occur frequently in combinatorial theory and probability theory.

2. EXAMPLES

By now the reader may be overwhelmed by a plethora of definitions and anxious to see the point of them. We will give several examples, some of which will be used later, to illustrate the significance of the above concepts.

Example 1. Let $P = \underline{N}$, the natural numbers with their usual ordering. Then a P-partition of n with largest part $\leqslant m$ is just an *ordinary partition* of n with largest part $\leqslant m$ [8, Ch. 19]. As is well-known,

$$\sum_{n=0}^{\infty} a(m,n)x^n = \prod_{i=1}^{m} (1-x^i)^{-1}.$$

Similarly $a(n)$ is equal to the total number of partitions of n (usually denoted $p(n)$), with the corresponding generating function

$$\sum_{n=0}^{\infty} a(n)x^n = \prod_{i=1}^{\infty} (1-x^i)^{-1}.$$

To tie in with subsequent results, we state the trivial formulas

$$\sum_{|I|=k} e(I) = 1, \qquad \sum_{|I|=k} e(I)^2 = 1, \tag{2}$$

where the sum is over all order ideals I of $\underline{N}$ of cardinality k.

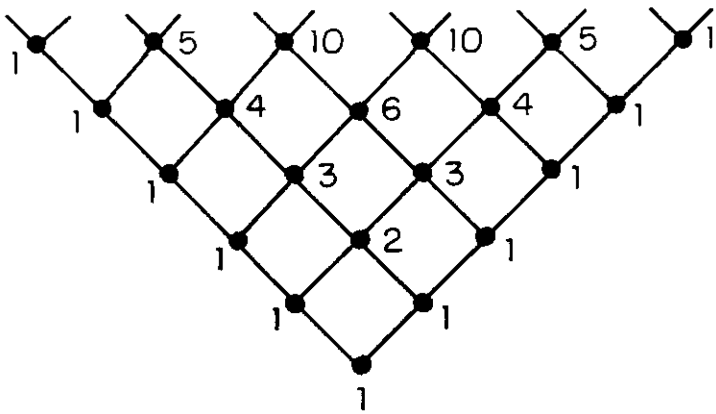
Example 2. Let P be the disjoint union of two copies of $\underline{N}$, denoted $P = \underline{N} + \underline{N} = 2\underline{N}$. Thus $J(P)$ is isomorphic to the direct product $\underline{N}\times\underline{N} = \underline{N}^2$. Here the numbers $a(m,n)$ are not so significant (in particular,

$$\sum_{n=0}^{\infty} a(m,n)x^n = \prod_{i=1}^{m} (1 - x^i)^{-2} .$$

We will rather discuss the numbers $e(I)$, $I \in J(P)$. For any W-ordered set P and $I \in J(P)$, let $I_1, I_2, \cdots, I_r$ be the elements of $J(P)$ which I *covers*, i.e., $I_i < I$ and no $I' \in J(P)$ satisfies $I_i < I' < I$. It follows that

(3) $$e(I) = e(I_1) + e(I_2) + \cdots + e(I_r).$$

For the lattice $\underline{N}^2$ under consideration, (3) is precisely the "addition formula" for constructing Pascal's triangle.



The numbers $e(I)$ are just the binomial coefficients, and in analogy to (2) we have the well-known formulas

$$\sum_{|I|=k} e(I) = 2^k, \qquad \sum_{|I|=k} e(I)^2 = \binom{2k}{k} .$$

More precisely, for any $I \in J(P)$ the segment $[0,I]$ has the form

$$\underline{a+1} \times \underline{b+1}, \quad \text{and} \quad e(I) = \binom{a+b}{b} .$$

Now $\underline{a+1} \times \underline{b+1} = J(\underline{a} + \underline{b})$. Thus from (1), we have that

$$\binom{a+b}{b}$$

is equal to the number of order-preserving bijections $\sigma : \underline{a} + \underline{b} \to \underline{a+b}$. The map σ is determined by the image of $\underline{a}$ (or $\underline{b}$), so we get the usual combinatorial interpretation of

$$\binom{a+b}{b}$$

as the number of combinations of $a + b$ things taken b at a time.

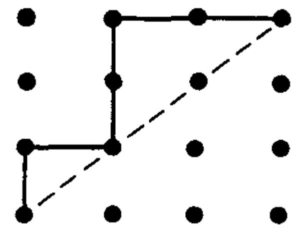
The above discussion motivates defining a *generalized Pascal triangle* to be a W-distributive lattice together with the function e. The entries $e(I)$ of a generalized Pascal triangle have three features in common with the ordinary binomial coefficients:

(a) They can be obtained by an additive recursion,

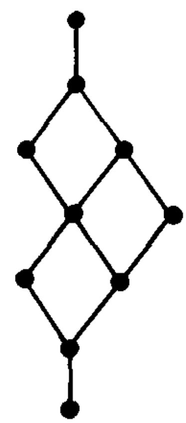
(b) They can be interpreted as counting certain types of permutations or sequences.

(c) They can be interpreted as counting certain types of lattice paths in Euclidean space, since every finite distributive lattice can be "imbedded" in a Cartesian grid of sufficiently high dimension.

To illustrate the lattice path interpretation (c), consider the well-known problem of counting the number of lattice paths in an $(n+1) \times (n+1)$ array of lattice points from a fixed corner to the opposite corner, such that the path never goes below the diagonal. For instance, in the 4×4 case we have as one path the following:



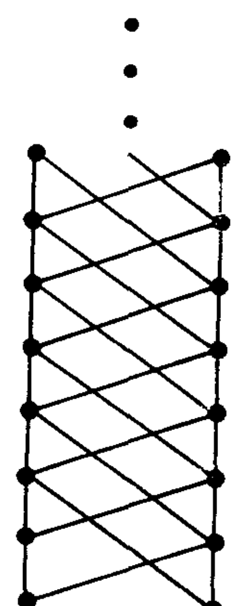
The total number in the 4×4 case is the number of maximal chains in the following distributive lattice L:



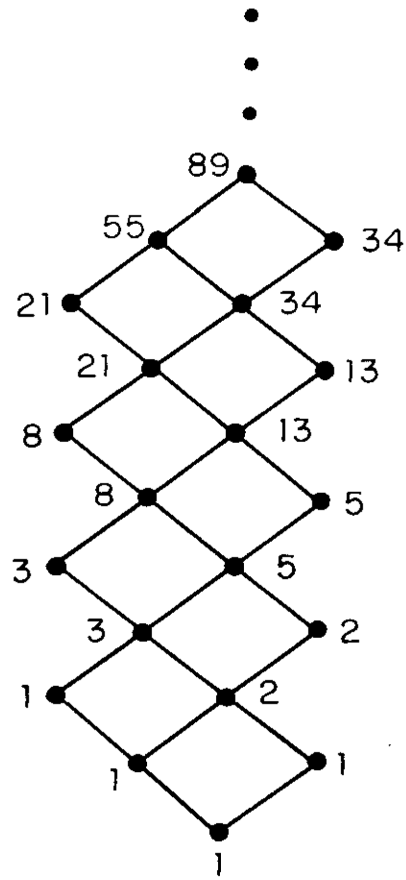
Here $L = J(\underline{2}\times\underline{3})$. In the general $(n+1) \times (n+1)$ case, the appropriate distributive lattice is $L = J(\underline{2}\times\underline{n})$. The number of maximal chains in $J(2\times n)$ is known to be the *Catalan number*

$$\frac{1}{n+1}\binom{2n}{n}.$$

Many other known lattice path problems can be formulated in a similar context. We give a further example, arising from a lattice path problem considered by Frankel [6]. Here if we take P to look like



then the generalized Pascal triangle corresponding to $J(P)$ looks like



The entries $e(I)$ are all Fibonacci numbers.

Example 3. Let $P = \underline{N}^2$. Then the lattice $J(P)$ is denoted $\underline{T}$ and is called *Young's lattice* (cf. Kreweras [11]). $\underline{T}$ can also be regarded as the lattice of all decreasing sequences $\lambda = (\lambda_1, \lambda_2, \cdots)$ (with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$) of non-negative integers λ_i, all but finitely many equal to 0, ordered coordinatewise. Hence λ may be regarded as a partition of $|\lambda| = \Sigma\lambda_i$. Thus if $\lambda = (\lambda_1, \lambda_2, \cdots) \in \underline{T}$ and $\mu = (\mu_1, \mu_2, \cdots) \in \underline{T}$, then $\lambda \leqslant \mu$ if and only if $\lambda_i \leqslant \mu_i$ for all $i = 1, 2, \cdots$. From this it follows that $j_k(\underline{T}) = p(k)$, the number of partitions of k. The lattice $\underline{T}$ is intimately connected with the theory of plane partitions and the representation theory of the symmetric group (cf. Stanley [19], and the references cited there). We will merely state some of the remarkable properties of the lattice $\underline{T}$.

First, we have the beautiful formulas, originally due to MacMahon [13, Sect. 495],

$$\sum_{n=0}^{\infty} a(m,n)x^n = \prod_{i=1}^{\infty} (1-x^i)^{-\min(i,m)}, \qquad \sum_{n=0}^{\infty} a(n)x^n = \prod_{i=1}^{\infty} (1-x^i)^{-i}.$$

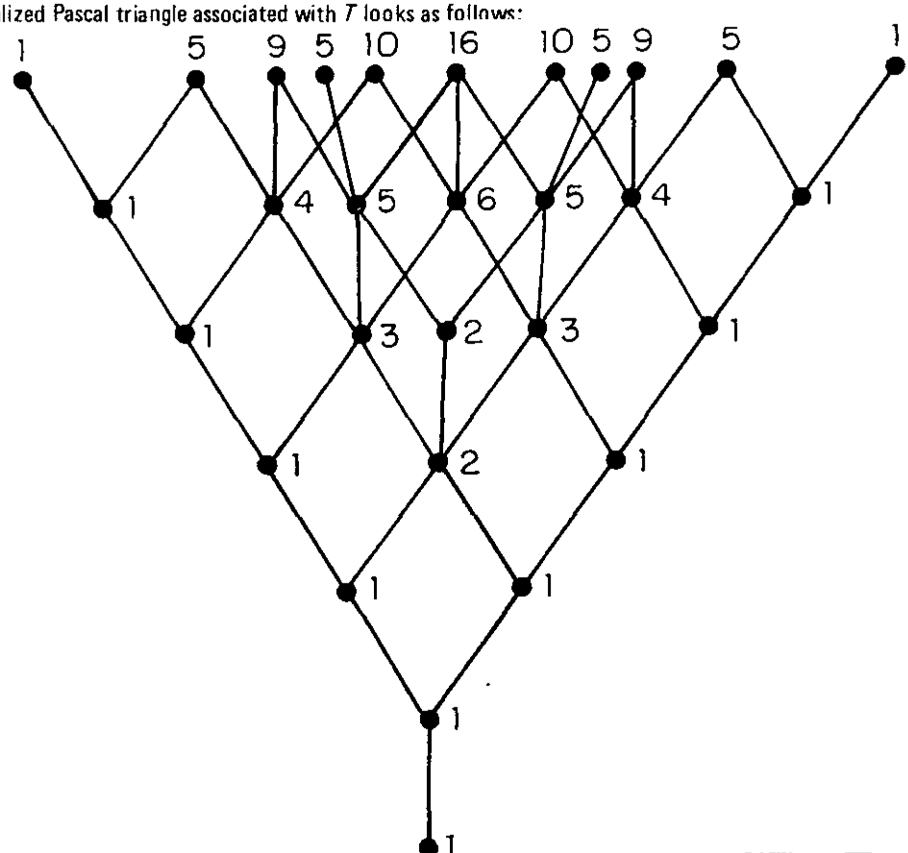
If $\lambda \in \underline{T}$ and $|\lambda| = k$, then the number $e(\lambda)$ is traditionally denoted f^λ and is equal to the degree of the irreducible representation of the symmetric group S_k corresponding to the partition λ. By either group-theoretic or combinatorial means, the following formulas can be proved:

$$\sum_{|\lambda|=k} e(\lambda) = t_k, \qquad \sum_{|\lambda|=k} e(\lambda)^2 = k! \tag{4}$$

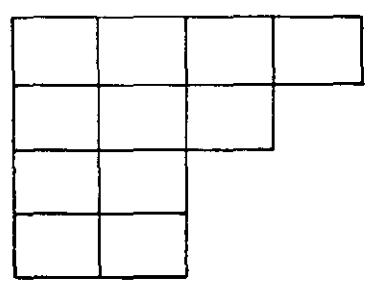
Here t_k is the number of elements $\pi \in S_k$ satisfying $\pi^2 = 1$. It is most easily computed from the recursion

$$t_0 = t_1 = 1, \qquad t_{k+1} = t_k + kt_{k-1}, \qquad k \geq 1.$$

The generalized Pascal triangle associated with T looks as follows:



Let us consider the problem of computing the individual $e(\lambda)$'s, $\lambda \in \underline{T}$. The element $\lambda = (\lambda_1, \lambda_2, \cdots)$ of $\underline{T}$ is represented schematically as an array of left-justified squares, with λ_i squares in the i^{th} row. This array is called the *graph* of λ. For instance, if $\lambda = (4,3,2,2,0,0,\cdots)$, then the graph of λ is



A maximal chain from 0 to λ in $\underline{T}$ corresponds to filling in the squares of the graph of λ with the integers $1, 2, \cdots, |\lambda|$, such that these integers are increasing in every row and column. Such an array is called a *Young tableau of shape* λ. For instance, one of the Young tableaux of shape $(4,3,2,2)$ is

1	3	4	10
2	5	8	
6	9		
7	11		

With each square S of the graph of a partition λ, we associate an integer $h(S)$, defined to be the number of squares directly to the right or directly below S, counting S itself exactly once. This number $h(S)$ is called the *hook length* of S. The hook lengths for $\lambda = (4,3,2,2)$ are given by

7	6	3	1
5	4	1	
3	2		
2	1		

A basic result of Frame, Robinson, and Thrall [5] states that

$$e(\lambda) = k!/h(S_1)h(S_2)\cdots h(S_k),$$

where $|\lambda| = k$ and $S_1, \cdots, S_k$ are the squares in the graph of λ.

Formulas (4) can be stated in terms of Young tableaux as follows:

(i) The number of Young tableaux with k squares is t_k.

(ii) The number of ordered pairs of Young tableaux of the same shape and with k squares is $k!$.

For instance, when $k = 3$, we have the following $t_3 = 4$ Young tableaux:

```
123   12   13   1
      3    2    2
                3
```

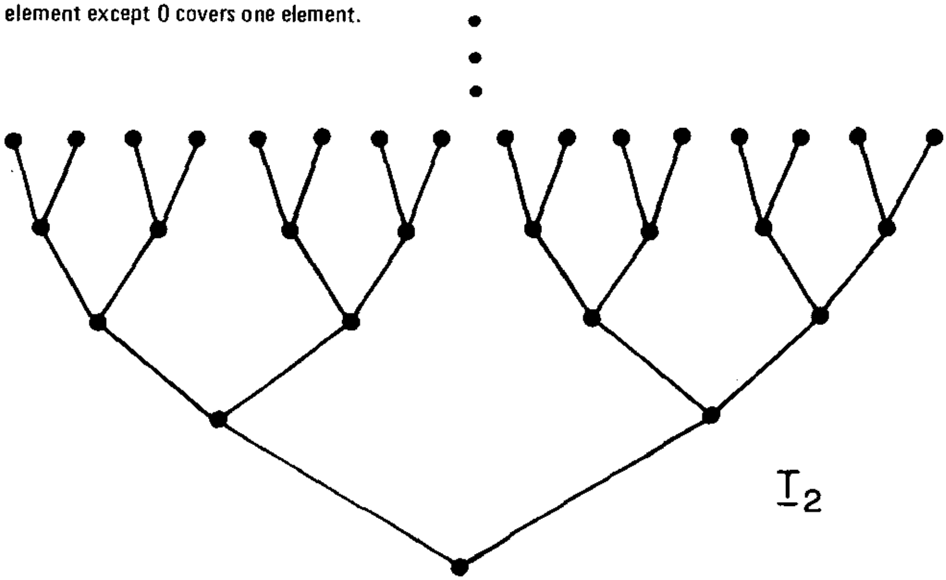
We also have the following $3! = 6$ pairs:

```
123 123   12 12   12 13   1 1   13 12   13 13
          3  3    3  2    2 2   2  3    2  2
                          3 3
```

In view of (i) and (ii), it is natural to ask for an explicit one-to-one correspondence $\pi \to (P,Q)$ between permutations π of $1, 2, \cdots, k$ and ordered pairs (P,Q) of Young tableaux of the same shape and with k squares, such that if $\pi \to (P,Q)$, then $\pi^{-1} \to (Q,P)$ (so that $\pi^2 = 1$ if and only if $\pi \to (P,P)$ for some P). Such a correspondence was discovered in a rather vague form by Robinson [14] and later more explicitly by Schensted [16]. Further aspects of this correspondence were considered by Schützenberger [17] and Knuth [9], [10, §5.2.4]. We refer the reader to these sources for the details.

It is natural to try to extend the results about $\underline{T} = J(\underline{N}^2)$ to the lattices $J(\underline{N}^r)$, $r > 2$. Unfortunately, all the "expected" results turn out to be false, and very little is known about the numbers $a(m,n)$ and $e(l)$.

Example 4. Our final example in this section is when P is the *universal binary tree* $\underline{T}_2$. This partially ordered set is characterized by the property that it is a W-ordered set with 0 such that every element is covered by two elements, and every element except 0 covers one element.



$\underline{T}_2$

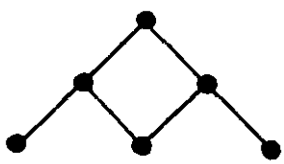
A finite order ideal of $\underline{T}_2$ (or an element of $J(\underline{T}_2)$) is a *plane binary tree*. The number j_k of order ideals of $\underline{T}_2$ of cardinality k is the *Catalan number*

$$\frac{1}{k+1}\binom{2k}{k}.$$

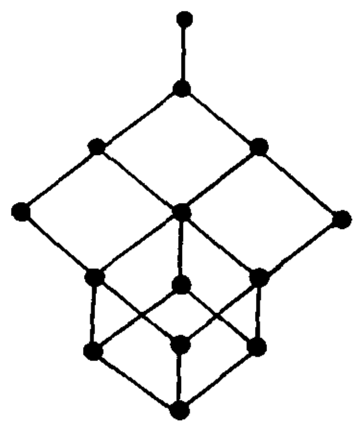
We thus have two order-theoretic interpretations of the Catalan numbers: (a) as the number of maximal chains in $J(\underline{2}\times\underline{k})$, and (b) as the number of elements of rank k in $J(\underline{T}_2)$. We state a third interpretation, viz., (c)

$$\frac{1}{k+1}\binom{2k}{k}$$

is the *total* number of elements in $J(S(\underline{k-1}))$, where $S(P)$ denotes the set of segments (or intervals) of P, ordered by inclusion*. Thus the Hasse diagram for $S(\underline{k-1})$ looks like the "top half" of the distributive lattice $\underline{k-1}\times\underline{k-1}$. For instance, when $k=4$ we have $S(\underline{3})$ and $J(S(\underline{3}))$ as follows:



$S(\underline{3})$



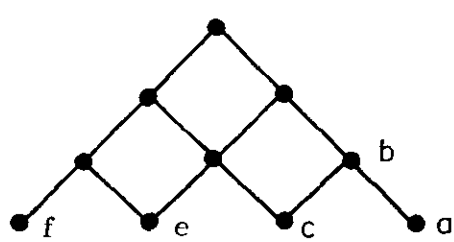
$J(S(\underline{3}))$

We leave as an exercise for the reader the result that the number of maximal chains in $J(S(\underline{k}))$ is

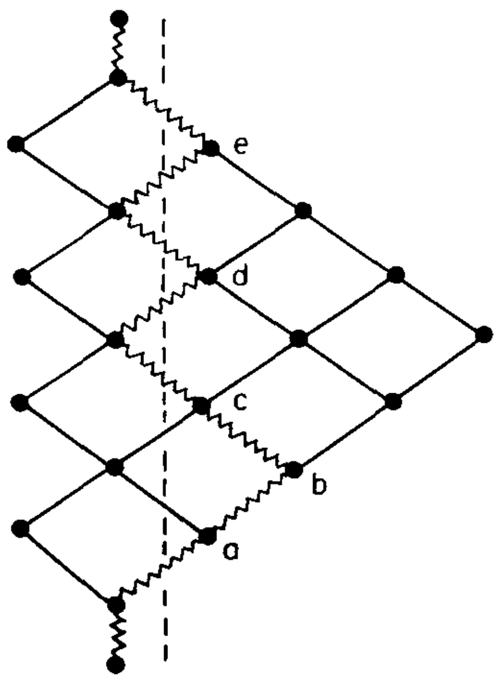
$$\frac{\binom{k+1}{2}!}{(2k-1)(2k-3)^2(2k-5)^3\cdots 3^{k-1}1^k}$$

There is an interesting way to see that the number of maximal chains in $J(\underline{2}\times\underline{k})$ is equal to the number of order ideals of $S(\underline{k-1})$. Draw the Hasse diagram of $J(\underline{2}\times\underline{k})$, pick a maximal chain C, and rotate the Hasse diagram 90° so there is one vertex on top and $k-1$ on the bottom. Remove the "bottom zigzag" of this rotated Hasse diagram. Then the resulting diagram H is the Hasse diagrams of $S(\underline{k-1})$. Let I be the smallest order ideal of H which contains all the elements in the intersection $C\cap H$. It is easily seen that this correspondence $C\to I$ between maximal chains C in $J(\underline{2}\times\underline{k})$ and order ideals I of $H\cong S(\underline{k-1})$ as a bijection. As an example, we take $k=5$ and C as shown at the top of the following page (indicated by wiggly lines).

The corresponding order ideal of $S(\underline{4})$ consists of the labeled elements on the right.



*There are two other lattices associated with the Catalan numbers, due to D. Tamari [21] (first published in [7]) and G. Kreweras [12], but since these lattices are not distributive we will not discuss them here.



The above correspondence between order ideals and maximal chains generalizes straightforwardly to show that if $L = J(P)$ is any finite *planar* distributive lattice (equivalently, P has no antichains of cardinality ≥ 3), then the number of maximal chains in L is equal to the number of order ideals in the partially ordered set obtained by rotating the Hasse diagram of L 90° and removing the "bottom zigzag." We state without proof one amusing consequence of this observation, based on a problem of Berlekamp [22, p. 341, problem 3] (see also Carlitz, Roselle, and Scoville [4]). Write down the graph of some partition λ. Let S be a square of this graph with coordinates (i,j) (i.e., S is in the i^{th} row and j^{th} column). Then the squares (i',j') satisfying $i' \geq j$ and $j' \geq j$ form the graph of a partition $\mu(S)$. In the square S write the number of elements ν of the Young lattice $\underline{Y}$ satisfying $\nu \leq \mu$. For example, if $\lambda = (3,3,2,1)$, then we get the array shown above right. The entry 9, for instance, corresponds to $\mu = (2,2,1)$ with the nine partitions $\nu \leq \mu$ given by (2,2,1), (2,1,1), (2,2), (1,1,1), (2,1), (2), (1,1), (1), ϕ. Now "border" the bottom and right of this array with a rook-wise connected line of squares containing the integer 1. Thus for the above array, we get the array shown in the lower right. For any entry in this new array, consider the largest square of which it is the upper left-hand corner. For instance, the entries 5 (either one), 9, and 28 give the square arrays

28	9	3
14	5	2
5	2	
2		

28	9	3	1
14	5	2	1
5	2	1	1
2	1	1	
1	1		

$$\begin{matrix} 5 & 2 \\ 2 & 1 \end{matrix} \qquad \begin{matrix} 9 & 3 & 1 \\ 5 & 2 & 1 \\ 2 & 1 & 1 \end{matrix} \qquad \begin{matrix} 28 & 9 & 3 \\ 14 & 5 & 2 \\ 5 & 2 & 1 \end{matrix}$$

Then we have the following result: The determinant of each of these square arrays is equal to one.

We now return to the partially ordered set $\underline{T}_2$. Here no simple expression for the generating function

$$\sum^{\infty} a(n)x^n$$

is known. On the other hand, it is easy to show (we will not do so here) that

$$\sum_{|I|=k} e(I) = k!\ .$$

The numbers $e(I)$ can be evaluated in a manner analogous to $e(\lambda)$, $\lambda \in \underline{T}$. In fact, if P is any finite rooted tree (considered as a partially ordered set) and $x \in P$, define

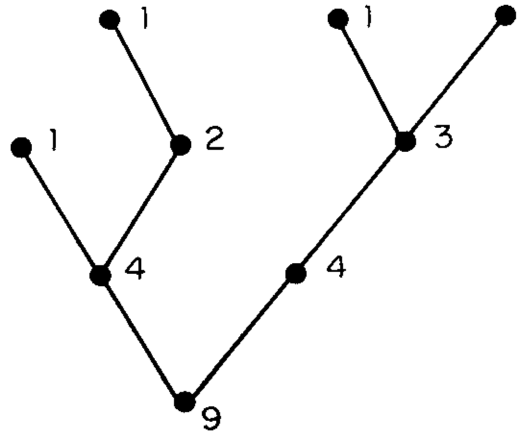
$$h(x) = \text{card}\left\{ y | y \in P,\ y \geq x \right\}.$$

Then an easy induction argument shows

$$e(P) = k!/h(x_1)h(x_2)\cdots h(x_k),$$

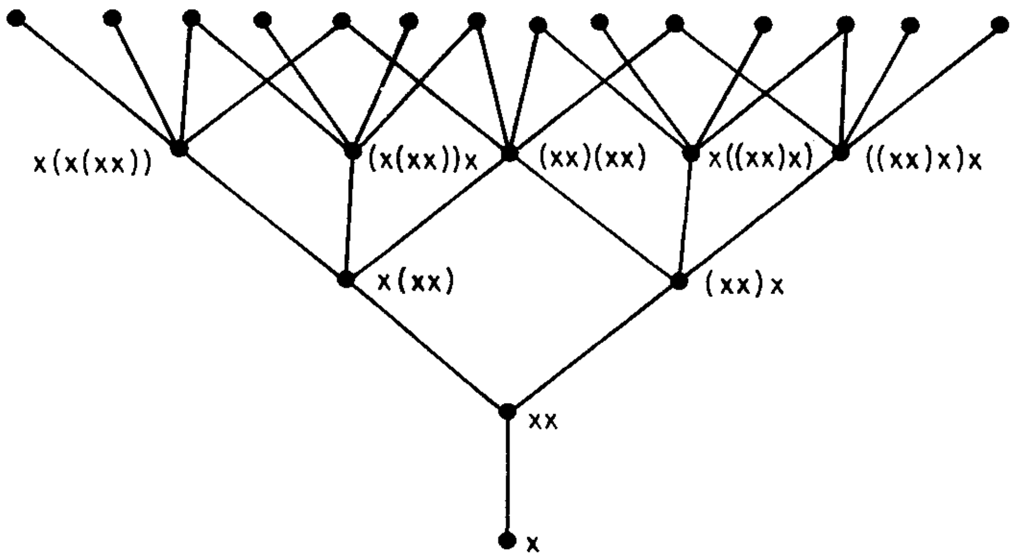
where $P = |k|$ and the x_i's are the elements of P. For example, see the array on the right. So for this partially ordered set P,

$$e(P) = 9!/9\cdot4\cdot4\cdot3\cdot2\cdot1\cdot1\cdot1\cdot1 = 420.$$



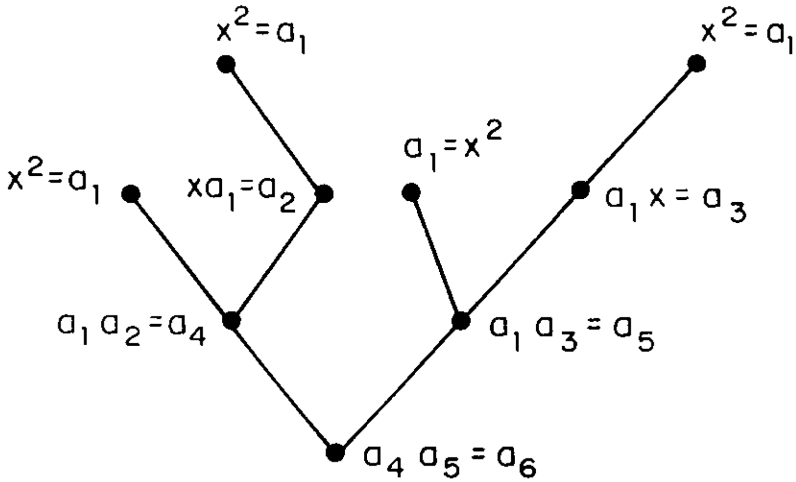
A discussion of these and related results may be found in [18, §22].

The lattice $J(\underline{T}_2)$ is closely connected with the well-known problem of parenthesizing a string of k letters (say x's). A bibliography of this problem is given by Brown [3], though the following lattice-theoretic interpretation appears to be new. We define an order relation $\underline{A}_2$ on all finite parenthesized strings of x's (excluding the void string) as follows: Given two strings S_1 and S_2, then $S_1 \leq S_2$ if and only if S_2 can be obtained from S_1 by substituting for each occurrence of x in S_1 some parenthesized string S (which depends on the particular x in S_1 being substituted for). For instance, if $S_1 = (xx)(x(xx))$ and $S_2 = (x(xx))(((xx)x)(xx))$, then $S_1 \leq S_2$ since we have substituted for the five x's in S_1 the strings x, xx, $(xx)x$, x, x. The order relation $\underline{A}_2$ looks as follows:



The basic result about $\underline{A}_2$ is that it is a distributive lattice isomorphic to $J(\underline{T}_2)$. In fact, the join-irreducible elements of $\underline{A}_2$ are elements like $x(((xx)x)x)$ which are build up from x by multiplying successively by x either on the left or on the right. Thus for instance the following order ideal of $\underline{T}_2$ corresponds to the elements

$$a_6 = a_4a_5 = (a_1a_2)(a_1a_3) = ((xx)(x(xx)))(((xx)((xx)x))) \text{ of } \underline{A}_2\ \cdot$$



In contrast to the difficulties involved in extending results about $J(\underline{N}^2)$ to $J(\underline{N}^r)$, our results on $J(\underline{T}_2)$ easily generalize to $J(\underline{T}_r)$, where $\underline{T}_r$ is the *universal r-ary tree* (whose definition is evident). For instance,

$$j_k = \frac{1}{k(r-1)+1}\binom{kr}{k}, \qquad \sum_{|I|=k} e(I) = 1\cdot r\cdot(2r-1)(3r-2)\cdots((k-1)r-(k-2)).$$

Moreover, the numbers $e(I)$ can be computed for $J(\underline{T}_r)$ in exactly the same way as for $J(\underline{T}_2)$, since I is a rooted tree. Finally if $\underline{A}_r$ denotes the set of all finite strings of x's parenthesized in accordance with an *r-ary* operation and ordered analogously to $\underline{A}_2$, then $\underline{A}_r = J(\underline{T}_r)$.

3. COVER CHARACTERIZATIONS

Most of the distributive lattices we have been considering have an interesting property which we call a "cover characterization." A *W*-distributive lattice L is said to have a *cover characterization* if there exists a function $f(k,n)$ such that if an element x of L of rank k covers n elements, then x is covered by $f(k,n)$ elements. If $f(k,n)$ is independent of k (in which case we simply write $f(n)$), then we say that L has a *strong cover characterization.* The function $f(k,n)$ (or $f(n)$) is called the *cover function of L.*

It is easy to see (by inductively building L from the bottom up) that there can be at most one distributive lattice L (up to isomorphism) with a given cover function $f(k,n)$. It is not difficult to verify that the following lattices have the indicated cover function.

L	$f(k,n)$
$\underline{N}^r = J(r\underline{N})$	r
$J(\underline{N}^2)^r = J(r\underline{N}^2)$	$n+r$
$\underline{2}^r = J(r\underline{1})$	$-n+r$
$J(\underline{T}_r)^s = J(s\underline{T}_r)$	$(r-1)k+s$

On the other hand, the lattices $J(\underline{N}^r)$, $r > 2$, do not have a cover characterization.

An interesting problem is to determine which functions $f(k,n)$ can be the cover functions of a distributive lattice. For instance, given a function $a(n)$, for what functions $b(k)$ is $f(n,k) = a(n) + b(k)$ a cover function? The following proposition is useful in ruling out various functions. The proof is left to the reader.

Proposition 1. Let L be a *W*-distributive lattice such that $u(i,j)$ elements of rank i cover exactly j elements, and $v(i,j)$ elements of rank i are covered by exactly j elements. Then for all $i \geq j \geq 0$,

$$\sum_{k=0}^{\infty} u(i,k)\binom{k}{j} = \sum_{k=0}^{\infty} v(i-j,k)\binom{k}{j}.$$

(Each sum has only finitely many non-zero terms.) □

Thus, for instance, using Proposition 1, it can be shown that if L is a W-distributive lattice with the cover function $f(n) = an + b$, then $u(5,1) = -(b/3)(a+1)(2a^3 - 2a^2 - 3)$. Hence $u(5,1) < 0$ if $|a| \geq 2$, so in this case L does not exist. We in fact conjecture that if L has a strong cover characterization with a *non-decreasing* cover function $f(n)$ (i.e., $f(i+1) \geq f(i)$), with $f(0) > 0$, then $f(n) = a$ or $f(n) = n + a$.

One positive result is the determination of all *finite* distributive lattices with a strong cover characterization.

Proposition 2. If L is a finite distributive lattice with a strong cover characterization, then L is a boolean algebra $\underline{2}^r$.

Proof. Suppose L is a finite distributive lattice with a cover function $f(n)$. Let r be the number of elements covered by the top element 1 of L. Then $f(r) = 0$. Let I be the meet of all elements covered by the top element 1 of L. Then I is covered by r elements. Suppose I covers s elements, so $f(s) = r$. Under the assumption $s > 0$, we will show that there is an element $I' > I$ such that I' covers s elements. Then I' must be covered by r elements, which is impossible since the join of these r elements would lie above 1. Hence $s = 0$, and L is a boolean algebra.

Assume $s > 0$. Let $L = J(P)$. If M is the set of maximal elements of P, then I is the order ideal $P - M$. Since $s > 0$, $I \neq \phi$. Let $x \in I$. Then there is some $x_1 \in M$ satisfying $x_1 > x$. Let $x_2, \cdots, x_r$ be the remaining elements of M (in any order). Define $I_k = M \cup \{x_1, x_2, \cdots, x_k\}$. Then each I_k is an order ideal of P, and the number of maximal elements of I_k is at most one more than the number of maximal elements of I_{k-1}. Since I_1 has $\leq s$ maximal elements and I_r has r maximal elements, some I_k has s maximal elements. This I_k is the desired I', and the proof follows. □

Using Proposition 1, one can determine the number j_k of elements of rank k of a W-distributive lattice L with a cover function $f(k,n)$, without explicitly determining L. Is there a method for computing

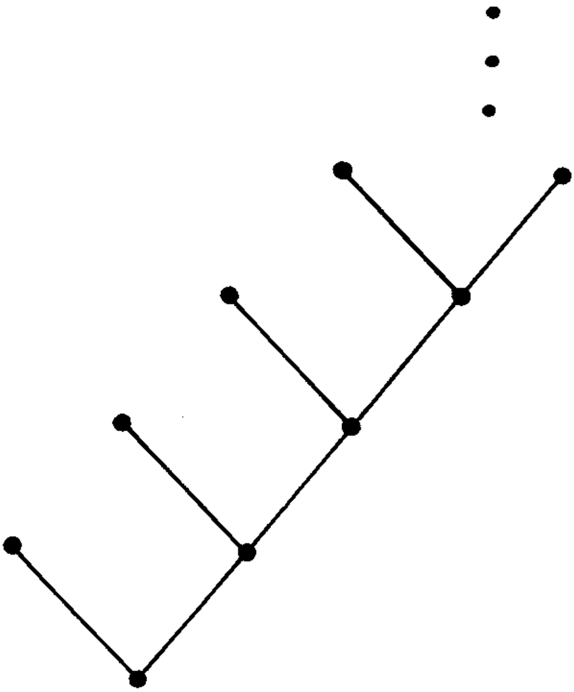
$$\sum_{|I|=k} e(I) \quad \text{and} \quad \sum_{|I|=k} e(I)^2?$$

There is some evidence for believing that these numbers will have a relatively simple form. In particular, if $f(k,n) = g(k)$ (independent of n), then it is trivial that

$$\sum_{|I|=k} e(I) = g(0)g(1)\cdots g(k-1).$$

4. THE FIBONACCI LATTICE

Let $\underline{K}_1$ denote the set of ordered pairs (m,n) of integers $1 \leq m$, $0 \leq n \leq 1$, under the order relation $(m,n) < (m',n')$ if and only if $n = 0$ and $m \leq m'$. Thus $\underline{K}_1$ looks as is shown on the right.



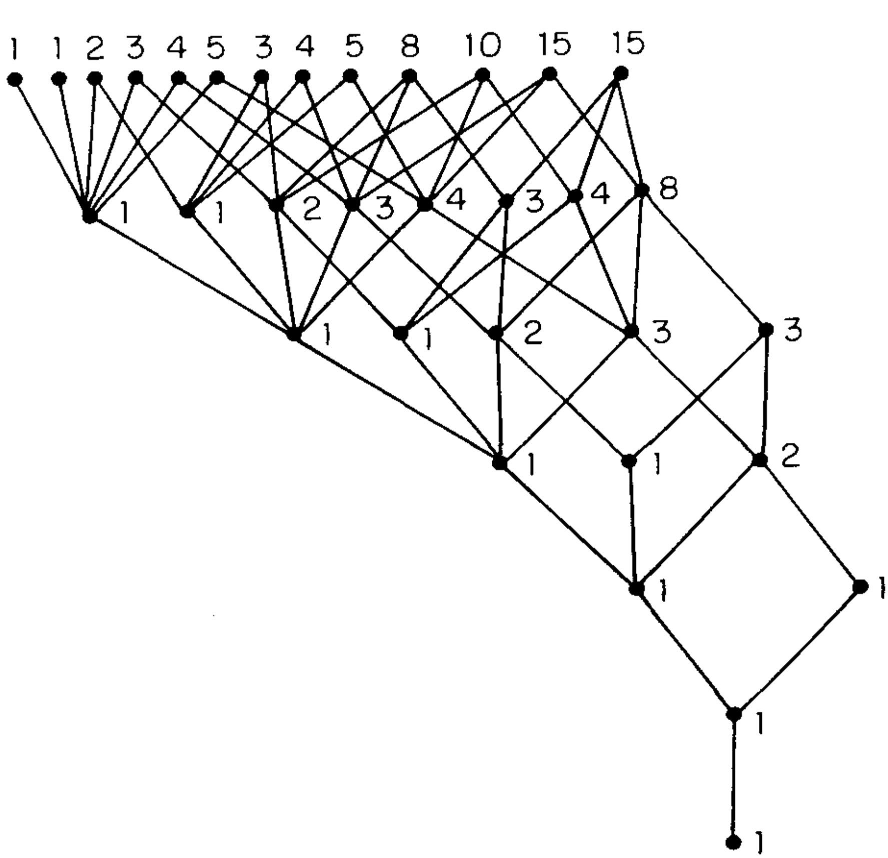
The lattice $J(\underline{K}_1)$ of finite order ideals of $\underline{K}_1$ is called the *Fibonacci lattice* and is denoted $\underline{F}_1$. Thus we have the generalized Pascal triangle at the top of the next page.

Proposition 3. The number f_k of elements of $\underline{F}_1$ of rank k is the k^{th} Fibonacci number ($f_0 = f_1 = 1$, $f_k = f_{k-1} + f_{k-2}$ if $k \geq 2$).

Proof. We will give three different proofs, reflecting three different properties of the Fibonacci numbers.

First proof. Clearly $f_0 = f_1 = 1$. Let I be an order ideal of $\underline{K}_1$ of cardinality $k > 1$. If the minimal element 0 is removed from $\underline{K}_1$, there results an isolated point x and an isomorphic copy $\underline{K}_1'$ of $\underline{K}_1$. If I contains x, then $I - \{0,x\}$ is an order ideal of $\underline{K}_1'$ of cardinality $k-2$. If I doesn't contain x, then $I - \{0\}$ is an order ideal of $\underline{K}_1'$ of cardinality $k-1$. Conversely if I' is any order ideal of $\underline{K}_1'$, then $I' \cup \{0\}$ and $I' \cup \{0,x\}$ are order ideals of $\underline{K}_1$. Hence $f_k = f_{k-1} + f_{k-2}$.

Second proof. Define $x_i = (i,0) \in \underline{K}_1$. Let I be an order ideal of $\underline{K}_1$ of cardinality k. Let i be the least integer such that $x_{k-i} \in I$. Then $x_1, x_2, \cdots, x_{k-i}$ are in I, and the remaining i elements of I are of the form $(m_j, 1)$, $j = 1, 2, \cdots, i$, where the m_j's are an arbitrary i-subset of $1, 2, \cdots, k-i$. Hence

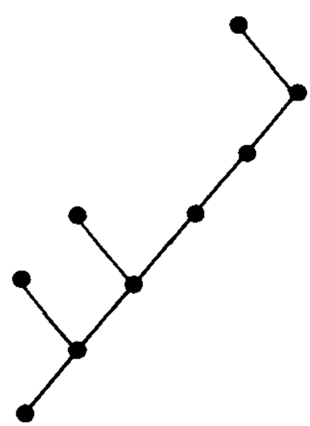


$$f_k = \sum_i \binom{k-i}{i}$$

This sum is a well-known expression for the Fibonacci numbers.

Third proof. There is a one-to-one correspondence between order ideals I of $\underline{K}_1$ of cardinality k and ordered partitions (or compositions) $k_1 + k_2 + \cdots + k_r = k$ of k into parts $k_i = 1$ or 2, as follows: $k_i = 1$ if $(i,0) \in I$ but $(i,1) \notin I$, $k_i = 2$ if $(i,1) \in I$. The number of such ordered partitions is well-known to be the k^{th} Fibonacci number f_k.□

We will denote order ideals I of $\underline{K}_1$ (or elements of $\underline{F}_1$) by the notation $k_1 k_2 \cdots k_r$, where $k_1 + \cdots + k_r$ is the ordered partition defined above. Thus for instance the order ideal $122112 \in \underline{F}_1$ is given on the right.



By modifying the second proof of Proposition 3, one can establish the following result.

Proposition 4. The number of elements of $\underline{F}_1$ of rank k which cover exactly i elements is

$$\binom{k-i-1}{i-1} + \binom{k-i}{i-1}$$

(with a binomial coefficient equaling 0 if any entry is negative). The number of elements of $\underline{F}_1$ of rank k which are covered by exactly i elements is 0 if $k-i$ is even, while if $k-1$ is odd this number is

$$\binom{(k+i-1)/2}{(k-i+1)/2}. \square$$

We now consider the problem of evaluating the sums

$$\sum_{|I|=k} e(I) \qquad \text{and} \qquad \sum_{|I|=k} e(I)^2 .$$

Surprisingly, these sums turn out to be the same as for the Young lattice $\underline{Y}$! Although coincidences in mathematics are suspect, I can offer no other explanation for this phenomenon. The evaluation of these sums for $\underline{F}_1$ is much easier than for Y.

Proposition 5. We have

$$\sum_{|I|=k} e(I) = t_k \qquad \text{and} \qquad \sum_{|I|=k} e(I)^2 = k! ,$$

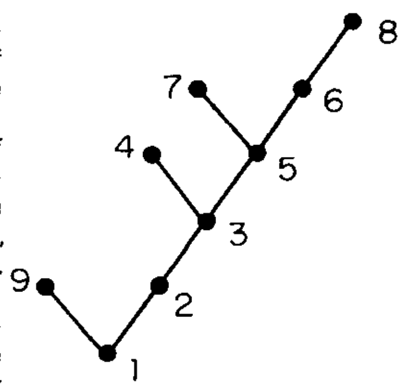
where the sums are over all order ideals I of $\underline{K}_1$ of cardinality k, and where t_k is the number of elements π in the symmetric group S_k satisfying $\pi^2 = 1$.

Proof. Let

$$h_k = \sum_{|I|=k} e(I) \qquad \text{and} \qquad g_k = \sum_{|I|=k} e(I)^2 .$$

Let x be the unique maximal element of $\underline{K}_1$ which covers 0. We divide all order-preserving bijections $\sigma : I \to \underline{k}$ (I an order ideal of $\underline{K}_1$) into two classes: (a) $x \notin I$, and (b) $x \in I$. Since $\underline{K}_1 - \{0,x\}$ is isomorphic to $\underline{K}_1$, the number of σ of type (a) is h_{k-1}. If $x \in I$, then $\sigma(x)$ can be any of $2, 3, \cdots, k$, so the number of σ of type (b) is $(k-1)h_{k-2}$. Hence $h_k = h_{k-1} + (k-1)h_{k-2}$. Moreover, by inspection $h_0 = h_1 = 1$, so $h_k = t_k$.

Similarly the number of pairs (σ,τ) of order-preserving bijections of I onto $\underline{k}$, for all I with $x \notin I$, is g_{k-1}. If $x \in I$, then there are $(k-1)^2$ ways of specifying $\sigma(x)$ and $\tau(x)$, so there are $(k-1)^2 g_{k-2}$ pairs in this case. Hence $g_k = g_{k-1} + (k-1)^2 g_{k-2}$. Since $g_0 = g_1 = 1$, we have $g_k = k!$. ☐



In analogy with the definition of a Young tableau, we define a *Fibonacci tableau* (I,σ) to be a finite order ideal I of $\underline{K}_1$, together with an order-preserving bijection $\sigma : I \to \underline{k}$, where $|I| = k$. The order ideal I is called the *shape* of the tableau, and k is called the *size* of (I,σ). Thus for example, the tableau on the right is a Fibonacci tableau of shape 212211 and size 9.

Proposition 5 can then be restated as follows: The number of Fibonacci tableaux of size k is t_k, and the number of ordered pairs of Fibonacci tableaux of size k and of the same shape is $k!$. There is a very simple alternative proof that the number of Fibonacci tableaux of size k is t_k — we construct a one-to-one correspondence $\Omega : (I,\sigma) \to \pi$ between Fibonacci tableaux (I,σ) of size k and elements $\pi \in S_k$ satisfying $\pi^2 = 1$. Namely, we define π by the condition $\pi(i) = j$ for $i > j$ if and only if some maximal element z of $\underline{K}_1$ satisfies $\sigma(z) = i$ and the unique element y covered by z satisfies $\sigma(y) = j$. Thus for the Fibonacci tableau illustrated above, $\pi = (19)(2)(34)(57)(6)(8)$. It is easily seen that this construction establishes the desired one-to-one correspondence.

Similarly one would like to prove the second formula of Proposition 5 by constructing a one-to-one correspondence $\psi : (I,\sigma,\tau) \to \pi$ between ordered pairs $((I,\sigma), (I,\tau))$ of Fibonacci tableaux of size k and of the same shape I, and elements $\pi \in S_k$. The correspondence ψ should satisfy the following two properties: (a) If $\psi(I,\sigma\,\tau) = \pi$, then $\psi(I,\tau,\sigma) = \pi^{-1}$, and (b) $\psi(I,\sigma,\sigma) = \Omega(I,\sigma)$. This correspondence would be a "Fibonacci analogue" of Schensted's correspondence for Young tableaux (see Example 3). Such a correspondence was found by E. Bender (private communication), as follows: Let $x = (m,n) \in I$, and define $x' = (m, 1-n)$. Then π is defined by the conditions

$$\pi(\sigma(x)) = \begin{cases} \tau(x), & \text{if } x' \notin I \\ \tau(x'), & \text{if } x' \in I \end{cases} .$$

We next consider the problem of evaluating the numbers $e(I)$ themselves, where I is the shape of a Fibonacci tableau. A finite order ideal I of $\underline{K}_1$ is a rooted tree, so from (5) we have

$$e(I) = k! \Big/ \prod_{x \in I} h(x),$$

where $|I| = k$, and $h(x) = \text{card}\left\{ y | y \in I,\ y \geq x \right\}$. It is easily seen that the above expression for $e(I)$ is equal to the product $n_1 \cdot n_2 \cdots n_r$ where the n_i's are those integers such that $k > n_1 > n_2 > \cdots > n_r > 0$ and $(k - n_i - i + 1, 1) \in I$. It follows that no two of the n_i's can be consecutive integers. Conversely, given a set of integers $k > n_1 > n_2 > \cdots > n_r > 0$, no two consecutive, there is a unique order ideal I of $\underline{K}_1$ of cardinality k such that $(m,1) \in I$ if and only if m has the form $k - n_i - i + 1$. We therefore obtain the following result:

Proposition 6. The set of numbers $e(I)$, including multiplicities, as I ranges over all order ideals of $\underline{K}_1$ of cardinality k is equal to the set of numbers

$$\prod_{n \in S} n,$$

where S ranges over all subsets of $\left\{1, 2, \cdots, k-1\right\}$ containing no two consecutive integers. □

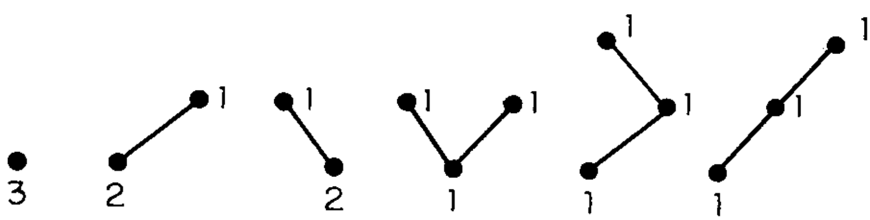
For instance, when $k=5$ we have the eight sets S given by ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,3\}$, $\{1,4\}$, $\{2,4\}$. Hence the numbers $e(I)$, $|I| = 5$, are given by 1, 1, 2, 3, 4, 3, 4, 8.

Combining Propositions 5 and 6, we obtain the formulas

$$\sum_S \prod_{n \in S} n = t_k, \qquad \sum_S \prod_{n \in S} n^2 = k!,$$

where both sums are over all subsets S of $\left\{1, 2, \cdots, k-1\right\}$ containing no two consecutive integers. Both these formulas can be easily proved directly by induction on k.

Let us now turn to the problem of counting the number $a(m,n)$ of $\underline{K}_1$-partitions of n with largest part $\leq m$. A $\underline{K}_1$-partition is called a *protruded partition* [18, §24]. For instance, there are six protruded partitions of 3, as follows:



Proposition 7. Let $a(m,n)$ be the number of protruded partitions of n with largest pair $\leq m$. Then

$$\sum_{n=0}^{\infty} a(m,n)x^n = \prod_{i=1}^{m} (1 - x^i - x^{i+1} - x^{i+2} - \cdots - x^{2i})^{-1}$$

Proof. A protruded partition of n with largest part $\leq m$ can be regarded as two sequences $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ of non-negative integers satisfying

$$\Sigma a_j + \Sigma b_j = n, \qquad m \geq a_1 \geq a_2 \geq a_3 \geq \cdots, \qquad a_j \geq b_j.$$

Let k_i be the number of a_j's which are equal to i. If some $a_j = i$, then b_j can be any of 0, 1, 2, ⋯, i, so $a_j + b_j$ is one of $i, i+1, i+2, \cdots, 2i$. Thus

$$\sum_{n=0}^{\infty} a(m,n)x^n = \prod_{i=1}^{m} \left(\sum_{k_i=0}^{\infty} (x^i + x^{i+1} + \cdots + x^{2i})^{k_i} \right) = \sum_{i=1}^{m} (1 - x^i - x^{i+1} - \cdots - x^{2i})^{-1}. \ \square$$

On the following page, we give a table of $a(m,n)$ for $m,n \leq 10$.

Many features of the theory of ordinary partitions carry over to protruded partitions. We state one such result here. For a proof, see [18, §24]. A classical identity in the theory of ordinary partitions is

n \ m	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	2	3	3	3	3	3	3	3	3	3
3	3	5	6	6	6	6	6	6	6	6
4	5	10	12	13	13	13	13	13	13	13
5	8	17	22	24	25	25	25	25	25	25
6	13	31	42	47	49	50	50	50	50	50
7	21	53	75	86	91	93	94	94	94	94
8	34	92	135	159	170	175	177	178	178	178
9	55	156	238	285	309	320	325	327	328	328
10	89	265	416	509	558	582	593	598	600	601

$$\sum_{n=0}^{\infty} \frac{q^n}{(1-x)(1-x^2)\cdots(1-x^n)} = \prod_{i=0}^{\infty} (1-qx^i)^{-1} .$$

The corresponding identity for protruded partitions is

$$\sum_{n=0}^{\infty} \frac{q^n}{(1-x-x^2)(1-x^2-x^3-x^4)\cdots(1-x^n-x^{n+1}-\cdots-x^{2n})}$$

$$= \sum_{i=0}^{\infty} (1-qx^i)^{-1} \sum_{j=0}^{\infty} \frac{x^{j(j+1)}q^j}{(1-x)(1-x^2)\cdots(1-x^j)(1-x-x^2)(1-x-x^3)\cdots(1-x-x^{j+1})} .$$

By inspection, the Fibonacci lattice $\underline{F}_1$ does not have a cover characterization. It does possess, however, a different type of property, *viz.*, it is an *extremal distributive lattice* [20]. This means that if L is any locally finite distributive lattice with 0 having the same number r_k of join-irreducibles of rank k as $\underline{F}_1$ (namely, $r_1 = 1, r_2 = r_3 = \cdots = 2$), then $j_k(L) \leq j_k(\underline{F}_1)$. In fact, $\underline{F}_1$ is precisely the distributive lattice $L(1, 2, 2, 2, \cdots)$ constructed in [20].

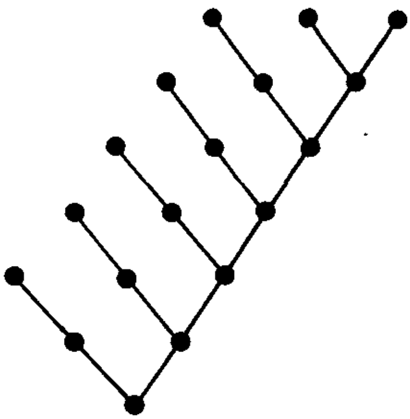
Recall the result $\underline{A}_2 \simeq J(\underline{T}_2)$ discussed in Example 4, where $\underline{A}_2$ is the lattice of parenthesized strings. Consider the related problem of parenthesizing a string of k x's subject to the commutative law (but not of course the associative law). For instance, when $k = 6$ there are 6 distinct strings, *viz.*, $x(x(x\cdot x^3))$, $x(x^2\cdot x^3)$, $x^2(x\cdot x^3)$, $x^2\cdot x^2\cdot x^2$, $x(x(x^2\cdot x^2))$, and $x^3\cdot x^3$ (an expression such as x^3 has an unambiguous meaning since $x(xx) = (xx)x$ by commutativity). The problem of counting the number N'_k of such strings was first considered by Wedderburn [23], who obtained a recursion for N'_k. It is unlikely that a simple expression for N'_k exists. For an historical survey of this problem, see Becker [1].

Let $\underline{C}_1$ be the partially ordered set of strings of x's subject to commutativity, ordered in the same way as in $\underline{A}_2$. It has been conjectured (e.g., by myself and by E. Bender) that $\underline{C}_1 \simeq J(\underline{F}_1)$. The reason for this conjecture is the following: It is not hard to see that the sub-ordered set P of $\underline{C}_1$ consisting of those elements which cover exactly one element is isomorphic to $\underline{F}_1$. Hence if $\underline{C}_1$ were a distributive lattice, we would have $\underline{C}_1 \simeq J(\underline{F}_1)$. Unfortunately, it turns out that $\underline{C}_1$ is not even a lattice. In particular, the elements $y = (x\cdot x^3)(x^3(x\cdot x^3))$ and $z = (x(x\cdot x^3))(x^3\cdot x^3)$ lie above exactly the same set of elements of P. If $\underline{C}_1$ were a lattice, the elements of P would be the join-irreducibles, so y and z would lie above the same set of join-irreducibles, which is impossible.

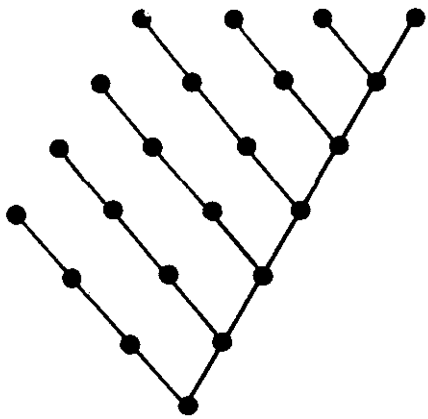
In conclusion we mention the problem of extending the lattice $\underline{F}_1 = J(\underline{K}_1)$ to a sequence of lattices $\underline{F}_r = J(\underline{K}_r)$. There are several possible definitions of $\underline{K}_r$. The one which seems to work best is the following: $\underline{K}_r$ is the unique locally finite partially ordered set with 0 such that when 0 is removed from $\underline{K}_r$, there results a partially ordered set isomorphic to a disjoint union of $\underline{r}$ and $\underline{K}_r$. For example, see the following page for what $\underline{K}_2$ and $\underline{K}_3$ look like.

Most of the results we have obtained for $\underline{F}_1$ generalize straightforwardly to $\underline{F}_r = J(\underline{K}_r)$. For instance,

(6)
$$\sum_{n=0} a(m,n)x^n = \prod_{i=1}^{m} \left(1 - x^i \binom{r+i}{r} x \right)^{-1} ,$$



$\underline{K}_2$



$\underline{K}_3$

where

$$\binom{k}{i}_x$$

denotes the *Gaussian coefficient,*

$$\binom{k}{i}_x = \frac{(1-x^k)(1-x^{k-1})\cdots(1-x^{k-i+1})}{(1-x^i)(1-x^{i-1})\cdots(1-x)} .$$

Similarly the numbers

$$\sum_{|I|=k} e(I) \qquad \text{and} \qquad \sum_{|I|=k} e(I)^2$$

satisfy simple recurrence relations, but they seem difficult to evaluate explicitly.

The limiting case $\underline{K}_\infty$ (where $\underline{K}_\infty$ with 0 removed is isomorphic to a disjoint union of $\underline{K}_\infty$ and $\underline{N}$) seems of some interest. The distributive lattice $\underline{F}_\infty = J(\underline{K}_\infty)$ is isomorphic to the set of all sequences $(n_1, n_2, \cdots)$ of non-negative integers such that all but finitely many n_i are equal to 0 and such that $n_i = 0 \Rightarrow n_{i+1} = 0$, ordered coordinatewise. The following formulas can be verified:

(7) $$j_k = 2^{k-1}, \quad k > 0, \qquad \sum_{n=0}^{\infty} a(m,n)x^n = \prod_{i=1}^{m} \left(1 - \frac{x^i}{(1-x)(1-x^2)\cdots(1-x^i)}\right)^{-1}$$

$$\sum_{|I|=k} e(I) = B_k, \qquad \sum_{|I|=k} e(I)^2 = C_k .$$

Here B_k is a *Bell number,* (also called an *exponential number*) defined by

$$B_0 = 1, \quad B_{k+1} = \sum_{0}^{k} \binom{k}{i} B_i, \qquad \text{or by} \qquad \sum_{0}^{\infty} B_k x^k/k! = e^{e^x - 1}$$

[15]. Similarly C_k is defined by

$$C_0 = 1, \quad C_{k+1} = \sum_{0}^{k} \binom{k}{i}^2 C_i, \qquad \text{or by} \qquad \sum_{0}^{\infty} C_k x^k/k!^2 = I_0(2\sqrt{I_0(2x^{1/2}) - 1}),$$

where

$$I_0(z) = \sum_{0}^{\infty} z^{2k}/2^{2k}k!^2$$

is the 0^{th}-order modified Bessel function.

Proposition 7 and Eqs. (6) and (7) are actually special cases of the following general result. Suppose P and Q are W-ordered sets such that P has a 0 which when removed results in a partially ordered set isomorphic to a disjoint union of P and Q. Let $a(m,n)$ (resp. $b(m,n)$) be the number of P-partitions (resp. Q-partitions) of n with largest part $\leqslant m$. Then

$$\sum_{n=0}^{\infty} a(m,n)x^n = \prod_{i=1}^{m} (1 - x^i U_i(x))^{-1} ,$$

where

$$U_m(x) = \sum_{n=0}^{\infty} b(m,n)x^n .$$

The proof is left to the reader.

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