Graph Colorings
And
Related
Symmetric Functions:
Ideas and
Applications

SUBTITLE

A
Description of
Results,
Interesting
Applications, &
Notable
Open problems

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1 Schur positivity.

Let G be a finite graph with no loops (edges from a vertex to itself) or multiple edges. In [36] we defined a symmetric function $X_G = X_G(x_1, x_2, \ldots)$ which generalizes the chromatic polynomial $\chi_G(n)$ of G. In this paper we will report on further work related to this symmetric function.

We first review the definition of X_G. We will denote by $V = \{v_1, \ldots, v_d\}$ the vertex set and by E the edge set of G. A *coloring* of G is any function $\kappa : V \to \mathbb{P} = \{1, 2, \ldots\}$. If κ is a coloring, then set

$$x^\kappa = \prod_{v \in V} x_{\kappa(v)}, \tag{1}$$

where $x_1, x_2, \ldots$ are commuting indeterminates. We say that the coloring κ is *proper* if there are no monochromatic edges, i.e., if $uv \in E$ then $\kappa(u) \neq \kappa(v)$. Define

$$X_G = X_G(x) = \sum_\kappa x^\kappa,$$

summed over all proper colorings κ. Thus X_G is a homogeneous symmetric function of degree d in the variables $x = (x_1, x_2, \ldots)$. Moreover, it is immediate from the definition of X_G that

$$X_G(1^n) = \chi_G(n),$$

where in general for a symmetric function f, we denote by $f(1^n)$ the substitution $x_1 = x_2 = \cdots = x_n = 1$, $x_{n+1} = x_{n+2} = \cdots = 0$.

The basic properties of the symmetric function X_G are discussed in [36]. In particular, we considered the expansion of X_G in terms of the four bases m_λ (the monomial symmetric functions), p_λ (the power sum symmetric functions), s_λ (the Schur functions), and e_λ (the elementary symmetric functions). (We are assuming a basic knowledge of symmetric functions such as may be found in Chapter I of [27].) One of the most interesting open problems concerning X_G is the following. A subposet Q of a poset (partially ordered set) P is said to be *induced* if whenever $u, v \in Q$ and $u < v$ in P, then $u < v$ in Q. A (finite) poset P is said to be $(3+1)$-*free* if it contains no induced subposet isomorphic to the disjoint union of a three-element chain and a one-element chain. We denote the incomparability graph of a poset P by $\mathrm{inc}(P)$. If b_λ is a symmetric function basis, then we say that the graph G is *b-positive* if the expansion of X_G in the basis b_λ has nonnegative coefficients.

1.1 Conjecture. [36, Conj. 5.1] *If P is a $(3+1)$-free poset, then* $\mathrm{inc}(P)$ *is e-positive.*

This conjecture is true for 3-free posets, i.e., the (edge) complement $\bar{G}$ of G is bipartite [36, Cor. 3.6].

Although the above conjecture remains open, the weaker result that incomparability graphs of $(3+1)$-free posets are s-positive was proved by V. Gasharov [15, Ch. II, Thm. 5][16], as mentioned in [36, Thm. 5.2]. In fact, Gasharov gives a combinatorial interpretation of the coefficients which we now explain (stated slightly differently from Gasharov).

1.2 Definition. Let P be a finite poset with d elements. A *P-tableau* of shape $\lambda \vdash d$ is a map $\tau : P \to \mathbb{P}$ satisfying the following three conditions:

(a) For all i we have $\lambda_i = \#\tau^{-1}(i)$.

(b) τ is a proper coloring of $\mathrm{inc}(P)$, i.e., if $\tau(u) = \tau(v)$ then $u \le v$ or $v \le u$.

(c) By (b) the elements of the set $\tau^{-1}(i)$ form a chain, say $u_1 < u_2 < \cdots < u_{\lambda_i}$. Similarly suppose that the elements of $\tau^{-1}(i+1)$ are $v_1 < v_2 < \cdots < v_{\lambda_{i+1}}$. Then for all i and all $1 \le j \le \lambda_{i+1}$ we require that $v_j \not< u_j$.

Note that if P is itself a chain $v_1 < \cdots < v_d$, then a map $\tau : P \to \mathbb{P}$ is a P-tableau of shape λ if and only if the sequence $\tau(v_1), \ldots, \tau(v_d)$ is a lattice permutation of shape λ, as defined e.g. in [27, p. 68][31, Def. 4.9.3]. Since there is a simple bijection between lattice permutations of shape λ and standard Young tableaux of shape λ [31, p. 173], we may regard a P-tableaux of shape λ (when P is a chain) as a standard Young tableau of shape λ. Hence for general P, a P-tableau of shape λ should be regarded as a generalization of a standard Young tableau of shape λ.

Let $f^\lambda(P)$ denote the number of P-tableaux of shape λ.

1.3 Theorem. (V. Gasharov) *Let P be a $(3+1)$-free poset and $G =$*

inc(P). *Then*

$$X_G = \sum_{\lambda \vdash d} f^\lambda(P) s_\lambda. \tag{2}$$

Gasharov proves (2) when P is $(3+1)$-free by an involution principle argument. Since both sides have simple combinatorial interpretations, there should be a direct bijective proof. When P is a chain the identity (2) becomes

$$(x_1 + x_2 + \cdots)^d = \sum_{\lambda \vdash d} f^\lambda s_\lambda(x),$$

where f^λ denotes the number of standard Young tableaux of shape λ. A bijective proof of this identity is provided precisely by the Robinson-Schensted correspondence, so we are seeking a generalization of Robinson-Schensted. Such a generalization can be gleaned from the work of A. Magid [28, §3], though a simpler direct bijection would be desirable.

A *claw* is a complete bipartite graph $K_{1,3}$. A graph is *clawfree* if no induced subgraph is a claw. Note that $K_{1,3}$ is the incomparability graph of the disjoint union $\mathbf{3} + \mathbf{1}$ of a three-element chain and one-element chain, and that $K_{1,3}$ is the incomparability graph of no other poset. It follows that an incomparability graph inc(P) is clawfree if and only if P is $(3+1)$-free. Thus it is natural to ask whether Conjecture 1.1 or Theorem 1.3 extends to clawfree graphs. In [36, Figure 5] we gave an example of a clawfree graph which isn't e-positive. On the other hand, the question of whether clawfree graphs might be s-positive was first raised by Gasharov (unpublished), and there now seems to be enough evidence to make it into a conjecture.

1.4 Conjecture. *If G is clawfree then G is s-positive.*

There is a nice combinatorial consequence of the s-positivity of a graph G. Recall from [36] that a *stable partition* of G of *type* $\lambda \vdash d$ is a partition of the vertex set V of G into stable (or independent) subsets of sizes $\lambda_1, \lambda_2, \ldots$. Define the graph G to be *nice* if whenever there exists a stable partition of G of type λ and whenever $\mu \leq \lambda$ (dominance or majorization order, called the "natural order" in [27, p. 6]), then there exists a stable partition of G of type μ. For instance, the claw $K_{1,3}$ is not nice, since there exists a stable partition of type $(3,1)$ but not of type $(2,2)$.

1.5 Proposition. *If G is s-positive then G is nice.*

Proof. By definition of X_G, G possesses a stable partition of type μ if and only if the coefficient of m_μ in X_G is nonzero (see [36, Prop. 2.4]). The proof now follows from the fact [24][25] that the coefficient of m_μ in the Schur function s_λ is nonzero if (and only if) $\mu \leq \lambda$. $\square$

As a small bit of evidence for Conjecture 1.4 we have the following result.

1.6 Proposition. *A graph G and all its induced subgraphs are nice if and only if G is clawfree.*

Proof. Since claws are not nice, the "only if" part follows. To prove the "if" part, we use the simple fact that if λ covers μ in dominance order, then μ is obtained from λ by subtracting 1 from some part λ_i and adding 1 to some part $\lambda_j \leq \lambda_i - 2$. (Not all such μ need be covered by λ.) Hence it suffices to prove that if a clawfree graph H has a stable partition π of type λ and if μ is as just described, then H has a stable partition of type μ. Let W be a subset of V which is the union of a block A of π of size λ_i and a block B of size λ_j. Let H_W denote the restriction of H to W. Hence H_W is bipartite. Since H is clawfree every vertex of H_W has degree one or two, so H_W is a disjoint union of paths and cycles. The vertices of each path and cycle alternate between A and B. Since $\#A > \#B$, there is a component of H_W which is a path starting and ending in A. Let P denote the vertex set of this path. Replace A and B by $(A-P)\cup(B\cap P)$ and $(A\cap P)\cup(B-P)$. This yields a stable partition of H of type μ, completing the proof. $\square$

Griggs has made a conjecture [19, Problem 3] equivalent to the statement that the incomparability graph of the boolean algebra B_n is nice. This suggests that $\text{inc}(B_n)$ might be s-positive, which is true for $n \leq 4$. Perhaps even the incomparability graph of any distributive lattice is s-positive. This seems quite unlikely, however, since in particular the distributive lattice L of Figure 1 has the property that $\text{inc}(L-\{\hat{0},\hat{1}\})$ is not s-positive (though $\text{inc}(L)$ is itself s-positive). The modular lattice of Figure 2 has an incomparability graph which isn't nice and hence isn't s-positive. (There is a partition into chains of type $(5,3,1,1,1,1)$ but not $(2,2,2,2,2,2)$.)

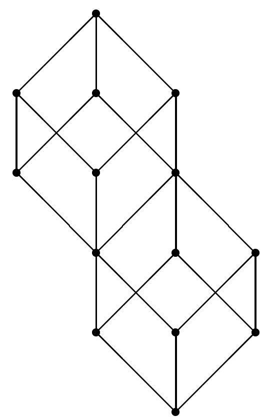

Figure 1: A distributive lattice L for which $\mathrm{inc}(L - \{\hat{0}, \hat{1}\})$ isn't s-positive

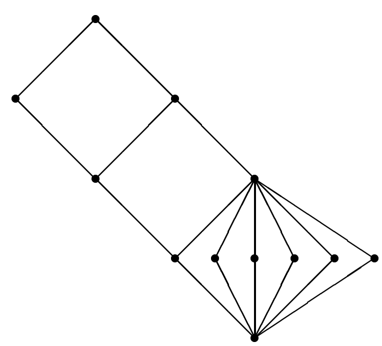

Figure 2: A modular lattice whose incomparability graph isn't nice

2 G-analogues of symmetric functions.

For each graph G we define a homomorphism φ_G from the ring of symmetric functions to the polynomial ring in the vertices of G which is closely connected with the symmetric function X_G. This homomorphism is closely related to [17], and I am grateful to Ira Gessel for calling to my attention the relevance of the paper [17]. Regard the vertices of G as commuting indeterminates, and define for each integer $i \geq 0$ a polynomial

$$e_i^G = \sum_S \left(\prod_{v \in S} v \right),$$

where S ranges over all i-element stable subsets of the vertex set V of G. In particular, $e_0^G = 1$. We regard e_i^G as a "G-analogue" of the ith elementary symmetric function e_i. Indeed, when G has no edges then $e_i^G = e_i(v_1, \ldots, v_d)$, where $V = \{v_1, \ldots, v_d\}$. Note, however, that e_i^G is not in general a symmetric function of the vertices of G.

Let Λ denote the ring of symmetric functions over $\mathbb{Z}$ in the variables $x_1, x_2, \ldots$, and let $\mathbb{Z}[V]$ denote the polynomial ring over $\mathbb{Z}$ in the vertices of G. Define a ring homomorphism $\varphi_G : \Lambda \to \mathbb{Z}[V]$ by setting $\varphi_G(e_i) = e_i^G$. (Since the e_i's for $i \geq 1$ are algebraically independent and generate Λ [27, (2.4)], φ_G is well-defined.) For $f \in \Lambda$ we write $\varphi_G(f) = f^G = f^G(v)$ and regard f^G as a "G-analogue" of f.

Closely related to G-analogues of symmetric functions are certain graphs constructed from G. If $\alpha : V \to \mathbb{N}$, then define G^α to be the graph obtained from G by replacing each vertex v of G by a clique (complete subgraph) $K_{\alpha(v)}$ of size $\alpha(v)$, and placing edges connecting every vertex of $K_{\alpha(v)}$ to every vertex of $K_{\alpha(u)}$ if uv is an edge of G. (If $\alpha(v) = 0$ then we are simply deleting the vertex v.) The graphs G^α are usually called *clan graphs*, and their chromatic polynomials have been investigated in [30].

NOTE. Given $\alpha : V \to \mathbb{N}$, a *multicoloring* of G of type α is an assignment of $\alpha(v)$ distinct colors to each vertex v. The multicoloring is *proper* if all colors assigned to adjacent vertices are different. If $\alpha(v) = 1$ for all v then a multicoloring is just an ordinary coloring. We can define a symmetric function X_G^α in exact analogy to X_G by

$$X_G^\alpha = \sum x_1^{a_1} x_2^{a_2} \cdots,$$

where the sum ranges over all multicolorings of G of type α, and where a_i is the number of vertices for which one of its colors is i. It is evident that

$$X_{G^\alpha} = X_G^\alpha \prod_{v \in V} \alpha(v)!. \tag{3}$$

Thus the theory of multicolorings of G is equivalent to the theory of ordinary colorings of the G^α's, and it is basically a matter of taste which one is preferred. Gasharov [15][16] deals with multicolorings. His result that X_G^α is s-positive for incomparability graphs of $(3+1)$-free posets actually follows from the case of ordinary colorings since if G is the incomparability graph of a $(3+1)$-free poset then so is each G^α.

The following result (pointed out to me by Ira Gessel) shows the connection between X_G and the G-analogues e_λ^G. If $\alpha : V \to \mathbb{N}$, then we write $v^\alpha = \prod_{v \in V} v^{\alpha(v)}$. Also write $[v^\alpha]f(v)$ for the coefficient of v^α in the polynomial or power series $f(v)$.

2.1 Proposition. *Let*

$$T(x, v) = \sum_\lambda m_\lambda(x) e_\lambda^G(v),$$

summed over all partitions λ. Then

$$\left(\prod_{v \in V} \alpha(v)! \right) [v^\alpha] T(x, v) = X_{G^\alpha}(x). \tag{4}$$

Proof. To obtain a monomial v^α in the expansion of $e_\lambda^G(v)$, we must choose stable sets $S_1, S_2, \ldots$ of vertices such that $\#S_i = \lambda_i$ and such that each vertex v appears in exactly $\alpha(v)$ of the S_i's. Hence

$$[v^\alpha] T(x, v) = \sum_\lambda \sum_{S_1, S_2, \ldots} m_\lambda(x),$$

where $S_1, S_2, \ldots$ have the meaning just explained. The coefficient of a monomial $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots$ in $[v^\alpha]T(x,v)$ is therefore equal to the number of sequences $S_1, S_2, \ldots$ of stable sets of vertices such that $\#S_i = \beta_i$ for all i and each vertex v appears in exactly $\alpha(v)$ of the S_i's. If we color the vertices in S_i with the color i, then we have exactly a multicoloring of G of type α. Hence $[v^\alpha]T(x,v) = X_G^\alpha(x)$. Comparing with equation (3) completes the proof. $\square$

2.2 Corollary. (a) *The following three conditions are equivalent.*

(i) *G^α is s-positive for all $\alpha : V \to \mathbb{N}$.*

(ii) *$s_\lambda^G \in \mathbb{N}[V]$ for all partitions λ.*

(iii) *Every minor of the (infinite) Toeplitz matrix $[e_{j-i}^G]_{i,j\geq 0}$ (where we set $e_k^G = 0$ if $k < 0$) has nonnegative coefficients.*

(b) *G^α is e-positive for all $\alpha : V \to \mathbb{N}$ if and only if $m_\lambda^G \in \mathbb{N}[V]$ for all partitions λ.*

Proof. (a) Consider the Cauchy product [27, (4.3′)]

$$\begin{aligned} C(x,y) &= \prod_{i,j}(1 + x_i y_j) \\ &= \sum_\lambda s_{\lambda'}(x) s_\lambda(y). \end{aligned} \tag{5}$$

When we apply the homomorphism φ_G (acting on the y variables only) we obtain

$$T(x,v) = \sum_\lambda s_{\lambda'}(x) s_\lambda^G(v).$$

By Proposition 2.1 we have

$$X_{G^\alpha}(x) = \left(\prod_{v\in V} \alpha(v)!\right) \sum_\lambda s_{\lambda'}(x)[v^\alpha] s_\lambda^G(v).$$

From this the equivalence of (i) and (ii) is immediate.

By the dual form of the Jacobi-Trudi identity [27, (5.5)], every minor of the matrix $[e_{j-i}]_{i,j\geq 0}$ is a skew Schur function $s_{\nu/\rho}$ for suitable partitions ν and ρ. Hence every minor of the matrix $[e^G_{j-i}]_{i,j\geq 0}$ is a G-analogue $s^G_{\nu/\rho}$ of a skew Schur function. Moreover, every possible $s^G_{\nu/\rho}$ occurs as a minor. Now every skew Schur function is s-positive [27, (9.1) and (9.2)], so every minor of the matrix $[e^G_{j-i}]_{i,j\geq 0}$ has nonnegative coefficients if and only if every s^G_λ has nonnegative coefficients. Hence (ii) and (iii) are equivalent.

(b) This is proved exactly as the equivalence of (i) and (ii) in (a), using the identity [27, (4.2′)]

$$C(x,y) = \sum_\lambda m_\lambda(x) e_\lambda(y). \ \square$$

We next consider the G-analogue of the power sum symmetric functions. We first note that it follows from the well known identity (equivalent to [27, (2.10′)])

$$-\log(1 - e_1 t + e_2 t^2 - e_3 t^3 + \cdots) = p_1 t + p_2 \frac{t^2}{2} + p_3 \frac{t^3}{3} + \cdots$$

that

$$-\log(1 - e^G_1 t + e^G_2 t^2 - e^G_3 t^3 + \cdots) = p^G_1 t + p^G_2 \frac{t^2}{2} + p^G_3 \frac{t^3}{3} + \cdots. \qquad (6)$$

Hence Theorem 2.3 below can be interpreted as a statement about the coefficients in the expansion of the left-hand side of (6).

2.3 Theorem. *For all graphs G and all partitions λ, we have $p^G_\lambda \in \mathbb{N}[V]$, i.e., p^G_λ is a polynomial with nonnegative (integral) coefficients.*

First proof. It suffices to prove the result for p^G_i, since $p^G_\lambda = p^G_{\lambda_1} p^G_{\lambda_2} \cdots$. A combinatorial interpretation of the coefficients of p^G_i is an immediate consequence of known results in the Cartier-Foata theory of commutation monoids, specifically the result [40, Prop. 5.10] in Viennot's development of this theory in terms of heaps of pieces. Using the terminology of [40, Def. 2.1], define P to be the set of vertices of G, and define a binary relation $\mathcal{C}$ on P by $u\mathcal{C}v$ if uv is an edge of G or $u = v$. Then the coefficient of $v^\alpha = v_1^{\alpha_1} v_2^{\alpha_2} \cdots$ in p^G_i, where $\sum \alpha_j = i$, is equal to the number of nonisomorphic pyramids (heaps with a unique maximal piece) $(E, \leq, \epsilon)$ such that $\#\epsilon^{-1}(v_i) = \alpha_i$.

Second proof. Using the notation of the proof of Corollary 2.2 and of [27], we have from [27, (4.1′)] that

$$C(x,y) = \sum_\lambda \epsilon_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y). \tag{7}$$

Hence

$$X_{G^\alpha}(x) = \left(\prod_{v\in V} \alpha(v)!\right) \sum_\lambda \epsilon_\lambda z_\lambda^{-1} p_\lambda(x) [v^\alpha] p_\lambda^G(v). \tag{8}$$

It is clear that $p_\lambda^G(v)$ has integral coefficients (since p_λ is an integral linear combination of the e_μ's). It follows from (8) that each $p_\lambda^G(v)$ has nonnegative coefficients if and only if the expansion of each X_{G^α} in terms of the basis $\epsilon_\lambda p_\lambda$ has nonnegative coefficients. But this was shown in [36, Cor. 2.7], so the proof follows. □

Examination of the proof of [36, Cor. 2.7] shows in fact that the coefficient of v^α in $p_i^G(v)$ is given by

$$[v^\alpha] p_i^G(v) = \frac{(-1)^{|\alpha|-1} \cdot |\alpha| \cdot [n]\chi_{G^\alpha}(n)}{\prod_j \alpha_j!},$$

where $|\alpha| = \sum \alpha_j$ (the number of vertices of G^α), and where $[n]\chi_{G^\alpha}(n)$ denotes the coefficient of n in the chromatic polynomial $\chi_{G^\alpha}(n)$ of the graph G^α.

There is an interesting application of Theorem 2.3 to the f-vectors of simplicial complexes. For the basic notions about simplicial complexes used here, see e.g. [34]. Let Δ be a simplicial complex on the vertex set V. Following Tits [39, p. 2], we call Δ a *flag complex* if every minimal set of vertices which is not a face of Δ has two elements. For instance, the order complex of a poset [35, p. 120] is a flag complex. If G is a graph, then the collection of stable sets of vertices (called the *stable set complex* or *independence complex* of G) is a flag complex, and every flag complex arises in this way. Equivalently (by looking at the complementary graph), flag complexes are the same as *clique complexes* of graphs, i.e., the collection of all sets of vertices which form a clique. Let $f_{i-1} = f_{i-1}(\Delta)$ denote the number of i-element faces of Δ (so $f_{-1} = 1$ unless $\Delta = \emptyset$). The vector $f(\Delta) = (f_0, f_1, \ldots)$ is called the *f-vector* of Δ. A basic problem of graph theory is to obtain information on the possible f-vectors of flag complexes. For instance, the famous theorem of Turán (e.g., [26, 10.34]) has this form. As an immediate consequence of Theorem 2.3, we have the following result, which gives some nonlinear inequalities that must be satisfied by f-vectors of flag complexes.

2.4 Corollary. *Suppose that Δ is a flag complex with f-vector $(f_0, f_1, \ldots)$. Let*

$$\sum_{n\geq 1} k_n \frac{t^n}{n} = -\log(1 - f_0 t + f_1 t^2 - f_2 t^3 + \cdots). \qquad (9)$$

Then each k_n is a nonnegative integer.

Proof. Regard Δ as the stable set complex of a graph G. Set each $v_i = 1$ in (6). Then $e_i^G(1, 1, \ldots) = f_{i-1}$, while $p_i^G(1, 1, \ldots) \in \mathbb{N}$ by Theorem 2.3. $\square$

What kind of information about the f-vector of flag complexes is implied by Corollary 2.4? We show that it is strong enough (though just barely) to establish Turán's theorem for triangles (first proved by Mantel [29]), stated below as Corollary 2.6. Similar reasoning may be found in [14], where Cartier-Foata theory (mentioned in our first proof of Theorem 2.3) is used to prove some strengthenings of Corollary 2.6. The results in [14] only use the fact that the exponential of the right-hand side of (9) has nonnegative coefficients, so it would be interesting to see whether Corollary 2.4 itself (or the stronger Theorem 2.3) can lead to even more general results.

2.5 Lemma. *Let a and b be positive real numbers, and set*

$$\sum_{n\geq 1} k_n \frac{t^n}{n} = -\log(1 - at + bt^2).$$

If each $k_n \geq 0$, then $b \leq a^2/4$.

Proof. Suppose that the polynomial $1 - at + bt^2$ has real zeros. Then the discriminant $a^2 - 4b$ is nonnegative, as desired. So assume that $1 - at + bt^2 = (1 - \theta t)(1 - \bar{\theta} t)$, where $\theta \in \mathbb{C}$, $\theta \notin \mathbb{R}$, and $\bar{}$ denotes complex conjugation.

Then $k_n = \theta^n + \bar{\theta}^n = 2\Re(\theta^n)$, where $\Re$ denotes the real part of a complex number. Since $\theta \notin \mathbb{R}$, it is easy to see that some power θ^n has negative real part, contradicting the hypothesis that $k_n \geq 0$. $\square$

2.6 Corollary. *Let G be a triangle-free (i.e., no induced K_3) graph on d vertices, without loops or multiple edges. Then G has at most $d^2/4$ edges.*

Proof. Let Δ be the clique complex of G, with f-vector $(f_0, f_1, \ldots)$. By hypothesis $f_2 = f_3 = \cdots = 0$, so the proof follows from Corollary 2.4 and Lemma 2.5. $\square$

Note that Lemma 2.5 fails if we only assume that some finite number $k_1, k_2, \ldots, k_N$ of the k_i's are nonnegative, no matter how large N is. For we can choose θ to have a large real part and an imaginary part very close to zero, in which case $\Re(\theta^n)$ will be positive unless n is large. Thus Corollary 2.4 is "just sufficient" to imply Turán's theorem for triangles. It is therefore no surprise that Corollary 2.4 fails to imply Turán's theorem for K_4-free graphs. For instance, a graph with 6 vertices and no K_4 can contain at most 12 edges, yet all coefficients of $-\log(1 - 6t + 13t^2 - 11t^3)$ are positive.

As a final application of G-analogues of symmetric functions, we give a connection with the theory of total positivity. We will use the following fundamental result of Aissen, Schoenberg, and Whitney [1] characterizing when a polynomial has negative real zeros.

2.7 Lemma. (Aissen–Schoenberg–Whitney) *Let $a_0, a_1, \ldots, a_d \in \mathbb{R}$, with some $a_i > 0$. The following two conditions are equivalent.*

(i) *Every zero of the polynomial $a_0 + a_1 t + \cdots + a_d t^d$ is a nonpositive real number.*

(ii) *Every minor of the (infinite) Toeplitz matrix $[a_{j-i}]_{i,j\geq 0}$ (where we set $a_k = 0$ if $k < 0$ or $k > d$) is nonnegative.*

2.8 Theorem. *Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$ such that for every $\alpha : V \to \mathbb{N}$, the graph G^α is s-positive. Equivalently (by Corollary 2.2(a)), $s_\lambda^G \in \mathbb{N}[V]$ for all partitions λ. Let c_i be the number of*

i-element stable sets of vertices of G. Then all the zeros of the polynomial $C_G(t) = \sum_i c_i t^i$ (called the stable set polynomial *of G) are real.*

Proof. By Corollary 2.2(a), every minor of the Toeplitz matrix $\mathbf{A}(v) = [e^G_{j-i}]_{i,j\geq 0}$ has nonnegative coefficients. If we set each $v_i = 1$ in $\mathbf{A}$ then we obtain the matrix $\mathbf{A}(1,1,\ldots) = [c_{j-i}]_{i,j\geq 0}$. Hence every minor of $\mathbf{A}(1,1,\ldots)$ is nonnegative, so by Lemma 2.7 every zero of the polynomial $\sum_i c_i t^i$ is real (and nonpositive). $\square$

Combining Theorems 1.3 and 2.8 yields the following result.

2.9 Corollary. *Let P be a $(3+1)$-free poset. Let c_i be the number of i-element chains of P. Then every zero of the polynomial $\sum c_i t^i$ is real.* $\square$

For a general discussion of the use of Lemma 2.7 to show that combinatorially defined polynomials have real zeros, see [9]. For additional information on stable set polynomials, see [14][22] and the references given there.

A special case of Corollary 2.9 are the stable set polynomials $\sum c_i t^i$ of indifference graphs (also called unit interval graphs), which are the incomparability graphs of posets that are both $(3+1)$-and $(2+2)$-free (see e.g. [13, p. 51]). These graphs have such a simple structure that there might be a proof of Corollary 2.9 for them that avoids Lemma 2.7, perhaps similar to [32, Thm. 1].

If G is a clawfree graph then every G^α is also clawfree. Hence an immediate consequence of Theorem 2.8 is the following.

2.10 Corollary. *If Conjecture 1.4 is true, then the stable set polynomial of a clawfree graph has only real zeros.*

The conclusion to the above corollary was first suggested by Hamidoune [20, p. 242]. It is true for line graphs (a special class of clawfree graphs) by a result of Heilmann and Lieb [21] (see also [18, Cor. 6.1.2]), as mentioned by Hamidoune.

A more precise connection than Theorem 2.8 between Schur positivity and the reality of the zeros of the stable set polynomial is given as follows.

2.11 Theorem. *Let $P(t)$ be a polynomial with real coefficients satisfying $P(0) = 1$. Define*

$$F_P(x) = \prod_i P(x_i),$$

an inhomogeneous symmetric formal power series. The following three conditions are equivalent.

- (i) *$F_P(x)$ is s-positive. (Equivalently, every homogeneous component of $F_P(x)$ is s-positive.)*
- (ii) *$F_P(x)$ is e-positive.*
- (iii) *All the zeros of $P(t)$ are negative real numbers.*

Proof. If $P(t) = \prod_{j=1}^{d}(1 + \theta_j t)$ with $\theta_j \neq 0$, then by (5) we have

$$F_P(x) = \sum_\lambda m_\lambda(\theta) e_\lambda(x),$$

where in general $f(\theta) = f(\theta_1, \ldots, \theta_d)$. From this it is clear that (iii)⇒(ii), while (ii)⇒(i) is obvious since each e_λ is s-positive. Now also from (5) we have

$$F_P(x) = \sum_\lambda s_{\lambda'}(\theta) s_\lambda(x).$$

Arguing as in the proof of Corollary 2.2, it follows from Lemma 2.7 that $F_P(x)$ is s-positive (if and) only if each θ_i is a positive real number. Hence (i)⇒(iii) and the proof follows. □

2.12 Corollary. *The following three conditions on a graph G with vertex set V are equivalent.*

- (i) *The symmetric function*

$$Y_G = \sum_{\alpha : V \to \mathbb{N}} X_G^\alpha$$

is s-positive.

(ii) *Y_G is e-positive.*

(iii) *All the zeros of the stable set polynomial $C_G(t)$ of G are real.*

Proof. A simple combinatorial argument shows that

$$Y_G(x) = \prod_i C_G(x_i).$$

The proof follows from Theorem 2.11 (and the fact that $C_G(t)$ has positive coefficients, so every real zero is negative). $\square$

3 Generalizations.

There are a number of possible generalizations of the symmetric function X_G. These generalizations are largely unexplored territory. We will sketch what is known about three such generalizations in this section.

3.1 The Tutte polynomial.

The Tutte polynomial $T_G(x, y)$ is a polynomial in two variables associated with a graph G (or more generally any matroid). It specializes to the chromatic polynomial *via* the identity (11). Unlike the chromatic polynomial, the Tutte polynomial does not vanish when the graph has loops, and is not unaffected by replacing a multiple edge by a single edge. Hence we will allow G to have loops and multiple edges. For a good survey of Tutte polynomials, see [10]. One of the formulas [10, Prop. 6.3.26] for the Tutte polynomial of a graph (though not the original definition) is given by

$$t^{\rho(G)} T_G\left(\frac{t+n}{t}, t+1\right) = \frac{1}{n^{c(G)}} \sum_{\kappa : V \to [n]} (t+1)^{m(\kappa)}, \tag{10}$$

where (a) $c(G)$ denotes the number of connected components of G, (b) $\rho(G)$ denotes the rank of the bond lattice L_G, i.e., $\rho(G) = \#V - c(G)$, (c) κ ranges over *all* colorings of G with the n colors $[n] = \{1, 2, \dots, n\}$, and (d) $m(\kappa)$ denotes the number of monochromatic edges of κ (number of edges of G whose vertices are colored the same). Note that if we set $t = -1$ in (10) the right-hand side becomes $n^{-c(G)}\chi_G(n)$, so

$$\chi_G(n) = (-1)^{\rho(G)} n^{c(G)} T_G(-n+1, 0). \tag{11}$$

Equation (10) suggests the following symmetric function generalization of the Tutte polynomial.

3.1 Definition. Let G be a graph on the vertex set V (allowing loops and multiple edges). Let $x = (x_1, x_2, \dots)$ and t be indeterminates, and define

$$X_G(x;t) = \sum_{\kappa : V \to \mathbb{P}} (1+t)^{m(\kappa)} x^\kappa,$$

where the sum is over *all* colorings $\kappa : V \to \mathbb{P}$ of G with positive integers, and where x^κ is given by (1) and $m(\kappa)$ is as in (10).

Note that $X_G(x; -1) = X_G(x)$. Moreover, it follows from (10) that

$$X_G(1^n; t) = n^{c(G)} t^{\rho(G)} T_G\left(\frac{t+n}{t}, t+1\right). \tag{12}$$

The only interesting results we know about $X_G(x;t)$ concern its expansion in terms of power sum symmetric functions. We will just state the main result here, first observed by Timothy Chow. The proof is a straightforward generalization of [36, Thm. 2.5] (the case $t = -1$).

3.2 Theorem. *We have*

$$X_G(x;t) = \sum_{S \subseteq E} t^{\#S} p_{\lambda(S)}(x), \tag{13}$$

where the sum ranges over all subsets of the edges of G, and where $\lambda(S)$ is the partition whose parts are the number of vertices of the connected components of the spanning subgraph of G with edge set S. In particular, the coefficients of $X_G(x;t)$ when expanded in terms of power sum symmetric functions are polynomials in t with nonnegative integer coefficients.

It is not difficult to compute $X_G(x;t)$ when G is the complete graph K_d. To obtain a coloring κ satisfying $\#\kappa^{-1}(i)=\alpha_i$, choose the sets $B_i=\kappa^{-1}(i)$ in $\binom{d}{\alpha_1,\alpha_2,\ldots}$ ways. Each such coloring κ satisfies $m(\kappa)=\sum\binom{\alpha_i}{2}$. Hence

$$X_{K_d}(x;t)=\sum_{\lambda\vdash d}\binom{d}{\lambda_1,\lambda_2,\ldots}(1+t)^{\sum\binom{\lambda_i}{2}}m_\lambda.$$

Equivalently,

$$\sum_{d\geq 0}\frac{X_{K_d}(x;t)}{d!}=\prod_{i\geq 1}\left(\sum_{m\geq 0}\frac{x_i^m(1+t)^{\binom{m}{2}}}{m!}\right). \tag{14}$$

Now consider equation (13). We can choose a subset $S\subseteq E$ by choosing a partition $\pi=\{B_1,B_2,\ldots,B_k\}$ of V and placing a connected graph on each block B_i. The contribution of a fixed partition π of type λ (i.e., with block sizes $\lambda_1,\lambda_2,\ldots$) to the right-hand side of (13) is $C_{\lambda_1}(t)C_{\lambda_2}(t)\cdots$, where

$$C_m(t)=\sum_{i=m-1}^{\binom{m}{2}}c_{mi}t^i,$$

and where c_{mi} is the number of connected (simple) graphs with i edges on an m-element vertex set. Hence

$$X_{K_d}(x;t)=\sum_{\lambda\vdash d}b_\lambda C_{\lambda_1}(t)C_{\lambda_2}(t)\cdots p_\lambda,$$

where b_λ is the number of partitions of type λ of a d-element set. The numbers b_λ are given explicitly by

$$b_\lambda=\frac{d!}{(1!)^{m_1}m_1!(2!)^{m_2}m_2!\cdots},$$

where λ has m_i parts equal to i. A simple application of the exponential formula (e.g., [33, §VI]) yields

$$\sum_{d\geq 0}X_{K_d}(x;t)\frac{u^d}{d!}=\exp\sum_{m\geq 1}C_m(t)p_m(x)\frac{u^m}{m!}. \tag{15}$$

One can also easily derive (15) directly from (14).

3.2 Directed graphs.

Let D be a directed graph, allowing loops (edges (u,u)) and bidirected edges (edges (u,v) and (v,u), $u \neq v$), but not multiple edges. Recently Chung and Graham [12] defined a polynomial $C_D(m,n)$ associated with the directed graph D. $C_D(m,n)$ has many properties comparable with the Tutte polynomial, though it is not a true analogue of the Tutte polynomial. One of the formulas for $C_D(m,n)$ (though not the original definition) is given as follows. Define a *path-cycle cover* of D to be a subset S of the edges such that every component of the spanning subgraph D_S of D with edge set S is a directed path (possibly of length zero, i.e., a single vertex) or directed cycle (possibly of length one, i.e., a loop from a vertex to itself). Let $c_D(i,j)$ denote the number of path-cycle covers with i paths and j cycles. Then

$$C_D(m,n) = \sum_{i,j} c_D(i,j)(m)_i n^j,$$

where $(m)_i = m(m-1)\cdots(m-i+1)$. This formula suggests defining a function $\Xi_D(x,y)$ which is symmetric separately in the two sets of variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ as follows. If S is a path-cycle cover, then define $\lambda(S)$ (respectively, $\mu(S)$) to be the partition whose parts are the number of vertices in the components of D_S that are directed paths (respectively, directed cycles). Hence $|\lambda| + |\mu| = d$, the number of vertices of D. We now define

$$\Xi_D(x,y) = \sum_S \tilde{m}_{\lambda(S)}(x) p_{\mu(S)}(y),$$

where the sum is over all path-cycle covers of D, and where $\tilde{m}$ denotes the augmented monomial symmetric function (as defined in [36, §2]). It follows immediately that

$$\Xi_D(1^m, 1^n) = C_D(m,n).$$

The path-cycle symmetric function $\Xi_D(x,y)$ was investigated by Chow [11]. We will state one of his more interesting results here, which when specialized to $x = 1^m$ and $y = 1^n$ answers a question raised by Chung and Graham [12, §8(c)], and which has no counterpart for the symmetric function $X_G(x)$.

3.3 Theorem. *Let D be a digraph with vertex set V and edge set $E \subseteq V \times V$. Let $\bar{D}$ denote the complement of D, i.e., the digraph with vertex set V and edge set $V \times V - E$. Then*

$$\Xi_D(x,y) = [\omega_x \Xi_{\bar{D}}(x,-y)]_{x \to (x,y)}, \tag{16}$$

where (a) ω_x denotes the involution ω acting on the x variables only, (b) $-y = (-y_1, -y_2, \ldots)$, and (c) $x \to (x,y)$ means that we replace the x variables with the union of the x and y variables.

3.3 Hypergraphs.

A (simple) graph may be regarded as a set of vertices and two-element subsets of vertices. What happens if we can take arbitrary subsets of vertices? A collection $\mathcal{H}$ of subsets of a vertex set V is called a *hypergraph*. The elements of $\mathcal{H}$ are still called *edges*. From now on we will assume that every edge has at least two elements. (We do not require, as is sometimes done, that the union of the edges is V.) A *proper coloring* of $\mathcal{H}$ with positive integers is a map $\kappa : V \to \mathbb{P}$ such that no edge is monochromatic[2]. This is equivalent to assuming that no minimal edge is monochromatic, so we might as well assume $\mathcal{H}$ is an *antichain*, i.e., no two elements of $\mathcal{H}$ are comparable (with respect to inclusion).

There is an extensive theory of hypergraph coloring (e.g., [4, Ch. 19]), but little of this theory is enumerative. Given an antichain $\mathcal{H}$ of subsets of V, we can define a symmetric function $X_{\mathcal{H}}$ exactly in analogy with graphs, i.e.,

$$X_{\mathcal{H}}(x) = \sum_{\kappa} x^{\kappa}, \tag{17}$$

where the sum ranges over all proper colorings $\kappa : V \to \mathbb{P}$ of $\mathcal{H}$. The only results of any significance we know at present about $X_{\mathcal{H}}$ concern its expansion into power sum symmetric functions. Let Π_V be the lattice of partitions of

[2]It may seem more natural to define a coloring to be proper if every edge has all its vertices colored differently. However, a proper coloring of $\mathcal{H}$ would then just be a proper coloring of the ordinary graph whose edges are the two-element subsets of edges of $\mathcal{H}$, so nothing new would be obtained.

V, and define $L_{\mathcal{H}}$ to be the join-sublattice of Π_V generated by all partitions with a unique nonsingleton block $B \in \mathcal{H}$ (including the empty join $\hat{0}$, the partition of V all of whose blocks are singletons). Thus if $\mathcal{H}$ is a graph, then $L_{\mathcal{H}}$ is just the lattice of contractions (or bond lattice) of $\mathcal{H}$. There is a further interpretation of the poset (actually a lattice, since it is a finite join-semilattice with $\hat{0}$) $L_{\mathcal{H}}$. Let $V = \{v_1, \dots, v_d\}$. If $S = \{v_{i_1}, \dots, v_{i_j}\} \in \mathcal{H}$, then let H_S denote the subspace of K^d (where K is a field, usually taken to be $\mathbb{R}$ or $\mathbb{C}$) given by

$$H_S = \{(z_1, \dots, z_d) \in K^d : z_{i_1} = \cdots = z_{i_j}\}.$$

Then $L_{\mathcal{H}}$ is just the *intersection lattice*, as defined in [5], of the subspace arrangement $\mathcal{A}_{\mathcal{H}} = \{H_S : S \in \mathcal{H}\}$.

3.4 Theorem. *With $\mathcal{H}$ as above, we have*

$$X_{\mathcal{H}} = \sum_{\pi \in L_{\mathcal{H}}} \mu(\hat{0}, \pi) p_{\mathrm{type}(\pi)}, \tag{18}$$

where type(π) *is the partition of d whose parts are the block sizes of π.*

The proof is exactly analogous to that of [36, Thm. 2.6]. Unlike the case of graphs, the sign of the integer $\mu(\hat{0}, \pi)$ does not depend only on type(π), so we cannot conclude that $\omega X_{\mathcal{H}}$ is p-positive as was the case for graphs [36, Cor. 2.7]. If we set $x = 1^n$ in (18) (i.e., $x_1 = x_2 = \cdots = x_n = 1, x_{n+1} = x_{n+2} = \cdots = 0$), then $X_{\mathcal{H}}(1^n)$ is just the chromatic polynomial $\chi_{\mathcal{H}}(n)$ of $\mathcal{H}$, i.e., the number of proper colorings of $\mathcal{H}$ with n colors. The polynomial $\chi_{\mathcal{H}}(n)$ is also known as the *characteristic polynomial* of the subspace arrangement $L_{\mathcal{H}}$.

Our second result concerning the expansion of $X_{\mathcal{H}}$ in terms of power sums is a generalization of [36, Thm. 2.5]. In fact, it applies to an even more general situation which generalizes $X_{\mathcal{H}}(x)$ in exactly the same way that $X_G(x;t)$ (defined in Definition 3.1) generalizes $X_G(x)$. Namely, for any hypergraph $\mathcal{H}$ with vertex set V define

$$X_{\mathcal{H}}(x;t) = \sum_{\kappa : V \to \mathbb{P}} (1+t)^{m(\kappa)} x^{\kappa},$$

where the sum ranges over all colorings $\kappa : V \to \mathbb{P}$ of $\mathcal{H}$, and where $m(\kappa)$ is the number of monochromatic edges of $\mathcal{H}$. Thus $X_{\mathcal{H}}(x;-1) = X_{\mathcal{H}}(x)$. Unlike the situation for $X_{\mathcal{H}}(x)$, the symmetric function $X_{\mathcal{H}}(x;t)$ is not determined by the minimal elements of $\mathcal{H}$, so we should no longer assume that $\mathcal{H}$ is an antichain. Comparing with (12) suggests that we *define* the Tutte polynomial $T_{\mathcal{H}}(x,y)$ of $\mathcal{H}$ by

$$X_{\mathcal{H}}(1^n;t) = n^{c(\mathcal{H})}t^{\rho(\mathcal{H})}T_{\mathcal{H}}\left(\frac{t+n}{t}, t+1\right), \tag{19}$$

where $c(\mathcal{H})$ is the number of connected components of $\mathcal{H}$ and $\rho(\mathcal{H})$ is the rank of the intersection lattice of the arrangement $\mathcal{A}_{\mathcal{H}}$. It might be interesting to investigate this "hypergraph Tutte polynomial" further. (It is actually not always a polynomial, so perhaps the factor $n^{c(\mathcal{H})}t^{\rho(\mathcal{H})}$ in (19) needs to be modified. An alternative definition of the Tutte polynomial of a hypergraph has been offered by Athanasiades [3, §3].)

Theorem 3.2 extends to $X_{\mathcal{H}}(x;t)$ in an obvious way. We omit the proof, which is completely analogous to that of [36, Thm. 2.5].

3.5 Theorem. *Let $\mathcal{H}$ be a hypergraph with edge set E. Then*

$$X_{\mathcal{H}}(x;t) = \sum_{S\subseteq E} t^{\#S} p_{\lambda(S)}(x), \tag{20}$$

where the sum ranges over all subsets of the edges of $\mathcal{H}$, and where $\lambda(S)$ is the partition whose parts are the number of vertices of the connected components of the spanning subhypergraph of $\mathcal{H}$ with edge set S.

As an explicit example, if $\mathcal{H}$ has vertices a,b,c,d and edges ac, cd, abc, then

$$\begin{aligned} X_{\mathcal{H}}(x;t) &= \tilde{m}_{1111} + (2t+6)\tilde{m}_{211} + (2t^2+5t+4)\tilde{m}_{31} + (2t+3)\tilde{m}_{22} + (t+1)^3\tilde{m}_4 \\ &= p_{1111} + 2tp_{211} + (2t^2+t)p_{31} + (t^2+t^3)p_4. \end{aligned}$$

Note that if we set $t=-1$ in (20) then we obtain a second expansion (the first being Theorem 3.4) of $X_{\mathcal{H}}(x)$ in terms of power sums.

As an interesting example of a hypergraph, fix $k \geq 1$ and let $\mathcal{H} = \mathcal{H}_{d,k}$ consist of *all* k-element subsets of the d-element set V. The arrangement $\mathcal{A}_{\mathcal{H}_{d,k}}$ is called a *k-equal arrangement* and has been extensively studied

[6][7][8][38]. By definition we have

$$X_{\mathcal{H}_{d,k}}(x) = \sum_{\kappa} x^{\kappa},$$

summed over all colorings $\kappa : V \to \mathbb{P}$ such that $\#\kappa^{-1}(i) < k$ for all i. Standard properties of exponential generating functions (e.g., [33, Cor. 6.2]) yield

$$\sum_{d\geq 0} X_{\mathcal{H}_{d,k}}(x)\frac{u^d}{d!} = \prod_{i\geq 1}\left(1 + x_i u + x_i^2\frac{u^2}{2!} + \cdots + x_i^{k-1}\frac{u^{k-1}}{(k-1)!}\right) \tag{21}$$

Setting $x_1 = \cdots = x_n = 1$, $x_{n+1} = x_{n+2} = \cdots = 0$ gives

$$\sum_{d\geq 0} \chi_{\mathcal{H}_{d,k}}(n)\frac{u^d}{d!} = \left(1 + u + \frac{u^2}{2!} + \cdots + \frac{u^{k-1}}{(k-1)!}\right)^n,$$

a result first obtained in [6, Cor. 4.5 and second equation on p. 693] (see also [5, Thm. 4.4.1(iii)]) using less combinatorial reasoning. If we define complex numbers $\theta_1, \ldots, \theta_{k-1}$ by

$$1 + u + \frac{u^2}{2!} + \cdots + \frac{u^{k-1}}{(k-1)!} = \prod_{j=1}^{k-1}(1 + \theta_j u),$$

then it follows from (21) and the Cauchy formula (7) that

$$\sum_{d\geq 0} X_{\mathcal{H}_{d,k}}(x)\frac{u^d}{d!} = \sum_{\lambda} \epsilon_\lambda z_\lambda^{-1} p_\lambda(\theta) p_\lambda(x) u^{|\lambda|}.$$

Hence by (18), for fixed $\lambda \vdash d$ we have

$$\sum_{\substack{\pi\in L_{\mathcal{H}_{d,k}}\\ \text{type } \pi=\lambda}} \mu(\hat{0}, \pi) = d!\epsilon_\lambda z_\lambda^{-1} p_\lambda(\theta).$$

Similarly, from (5) we obtain

$$X_{\mathcal{H}_{d,k}}(x) = d! \sum_{\lambda\vdash d} s_{\lambda'}(\theta) s_\lambda(x), \tag{22}$$

the expansion of $X_{\mathcal{H}_{d,k}}(x)$ in terms of Schur functions. Alternatively, since

$$e_i(\theta) = [0 \le i \le k-1]/i!,$$

where $[P] = 1$ if P is true and 0 if P is false (see [23] for a discussion of this notation), it follows from the dual Jacobi-Trudi identity [27, p. 25, (3.5)] that the coefficient $s_{\lambda'}(\theta)$ in (22) is given by

$$s_{\lambda'}(\theta) = d! \cdot \det \left[\frac{[0 \le \lambda_i - i + j \le k-1]}{(\lambda_i - i + j)!} \right]_{i,j=1}^{\lambda'_1}.$$

If $\lambda_1 + \lambda'_1 < k$ then no index (i,j) of an entry of the above determinant satisfies $\lambda_i - i + j \ge k$. It is not difficult to deduce that in this case we have $s_{\lambda'}(\theta) = f^\lambda$, the number of standard Young tableaux of shape λ. This fact is also easy to obtain from (18) and the Murnagham-Nakayama rule.

It is also not difficult to compute the "Tutte symmetric function" $X_{\mathcal{H}_{d,k}}(x;t)$ of the hypergraph $\mathcal{H}_{d,k}$. It is an immediate consequence of the definition of $X_{\mathcal{H}_{d,k}}(x;t)$ that

$$X_{\mathcal{H}_{d,k}}(x;t) = \sum_{\kappa: V \to \mathbb{P}} (1+t)^{\sum_i \binom{\#\kappa^{-1}(i)}{k}} x^\kappa.$$

Equivalently,

$$\sum_{d \ge 0} X_{\mathcal{H}_{d,k}}(x;t) \frac{u^d}{d!} = \prod_{i \ge 1} \left(\sum_{m \ge 0} \frac{(ux_i)^m (1+t)^{\binom{m}{k}}}{m!} \right),$$

an immediate generalization of both (14) and (21).

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