Flag-symmetry of the poset of shuffles and a local action of the symmetric group

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Abstract

We show that the posets of shuffles introduced by Greene in 1988 are flag-symmetric, and we describe a permutation action of the symmetric group on the maximal chains which is local and yields a representation of the symmetric group whose character has Frobenius characteristic closely related to the flag symmetric function. A key tool is provided by a new labeling of the maximal chains of a poset of shuffles. This labeling and the structure of the orbits of maximal chains under the local action lead to combinatorial derivations of enumerative properties obtained originally by Greene. As a further consequence, a natural notion of type of shuffles emerges and the monoid of multiplicative functions on the poset of shuffles is described in terms of operations on power series. The main results concerning the flag symmetric function and the local action on the maximal chains of a poset of shuffles are obtained from new general results regarding chain labelings of posets.

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0. Introduction

In [St2], Stanley initiated an investigation of posets which involve two algebraic objects related to the order structure of the poset — a certain symmetric function (flag symmetric function) and a certain associated representation of the symmetric group. In Section 1 we give precise definitions and summarize the results of [St2] that will be used later in this paper. Briefly, [St2] is concerned with classes of posets whose order structure leads to a symmetric function derived from the enumeration of rank-selected chains, and which turns out to be the Frobenius characteristic of a representation of the symmetric group, of degree equal to the number of maximal chains of the poset; moreover, this representation can be realized via an action of the symmetric group on the maximal chains of the poset, under which each adjacent transposition $\sigma_i = (i, i+1)$ acts on chains locally, that is, modifying at most the chain element of rank i.

The goal of this paper is to add a new infinite family of posets to the examples appearing in [St2] and [St3], namely, the posets of shuffles introduced and investigated by Greene [Gre1]. In the process, several general results emerge. In Section 1 we give the necessary background on locally rank-symmetric posets affording a local action of the symmetric group (based on [St2]). Section 2 contains the necessary preliminaries concerning the posets of shuffles (i.e., shuffles of subwords of two given words). In Section 3 we give a new labeling of the posets of shuffles and establish its properties which are instrumental in the remainder of the paper. In Section 4 we describe a local action of the symmetric group on the maximal chains of a poset of shuffles, such that the Frobenius characteristic for the corresponding representation character is (essentially) the flag symmetric function. The desired results regarding the posets of shuffles follow from more general results motivated by the properties of the new labeling of these posets. Section 5 is devoted to the enumeration of shuffles according to a natural notion of type. As a consequence we describe the monoid of multiplicative functions on the poset of shuffles in terms of operations on power series.

As a by-product of the present investigation of the posets of shuffles, we obtain alternative, purely combinatorial, derivations of enumerative results obtained in [Gre1]. The present work parallels that of [St3] regarding the lattice of noncrossing partitions, thus adding to previously known structural analogies between the posets of non-crossing partitions and those of shuffles. It is hoped that this work will facilitate the development of a systematic general theory of the posets with a local group action concordant with the flag symmetric function.

1. Preliminaries

Let P be a finite poset with a minimum element $\hat{0}$, and a maximum element $\hat{1}$. Throughout this paper, we will consider only such posets that are *ranked*, that is, there exists a function $\rho: P \to \mathbf{Z}$ such that $\rho(\hat{0}) = 0$ and $\rho(t') = \rho(t) + 1$ whenever $t \lessdot t'$ (the notation $t \lessdot t'$ means that t is *covered* by t', i.e., $t < t'$ and there is no element $u \in P$ such that $t < u < t'$).

Let $\rho(P) := \rho(\hat{1}) = n$. For $S \subseteq [n-1]$, where $[n-1] := \{1, 2, \ldots, n-1\}$, let $\alpha_P(S)$ denote the number of rank-selected chains in P whose elements (other than $\hat{0}$ and $\hat{1}$) have rank set equal to S. Thus,

$$\alpha_P(S) := \ \#\{\hat{0} < t_1 < t_2 < \cdots < t_{|S|} < \hat{1} \ : \ \{\rho(t_1), \rho(t_2), \ldots, \rho(t_{|S|})\} = S\}.$$

The function $\alpha_P: 2^{[n-1]} \to \mathbf{Z}$ is the *flag f-vector* of P. It contains information equivalent to that of the *flag h-vector* β_P whose values are given by

$$\beta_P(S) := \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T), \qquad \text{for all } S \subseteq [n-1]. \tag{1}$$

For example, writing $\alpha_P(t_1, \ldots, t_k)$ for $\alpha_P(\{t_1, \ldots, t_k\})$ and similarly for β_P, the poset of Figure 1 has $\alpha(\emptyset) = 1, \alpha(1) = \alpha(2) = 5, \alpha(1,2) = 12$, and $\beta(\emptyset) = 1, \beta(1) = \beta(2) = 4, \beta(1,2) = 3$. The poset of Figure 2 has $\alpha(\emptyset) = 1, \alpha(1) = \alpha(3) = 2, \alpha(2) = 3, \alpha(1,2) = \alpha(1,3) = \alpha(2,3) = 4, \alpha(1,2,3) = 6$, and $\beta(\emptyset) = 1, \beta(1) = \beta(3) = 1, \beta(2) = 2, \beta(1,2) = \beta(2,3) = 0, \beta(1,3) = 1, \beta(1,2,3) = 0$.

The flag f- and h-vectors appear in numerous contexts in algebraic and geometric combinatorics; for instance, the values $\beta_P(S)$ have topological significance related to the order complex of the rank-selected subposet $P_S := \{\hat{0}, \hat{1}\} \cup \{t \in P : \ \rho(t) \in S\}$ (see, e.g., [St1, section 3.12] for additional information and references).

Consider now the formal power series

$$F_P(x) := F_P(x_1, x_2, \ldots) = \sum_{\hat{0} \le t_1 \le t_2 \le \cdots \le t_k < \hat{1}} x_1^{\rho(t_1)} x_2^{\rho(t_2) - \rho(t_1)} \cdots x_{k+1}^{n - \rho(t_k)}.$$

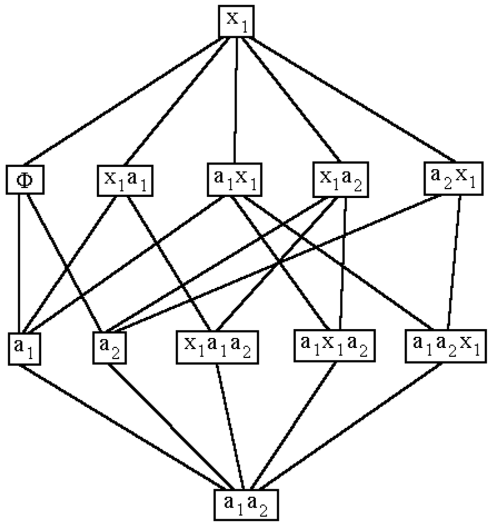


Figure 1. The poset of shuffles W_{21}.

This definition was suggested for investigation by Richard Ehrenborg [E] and is one of the central objects in [St2] and in this paper. Alternatively,

$$F_P(x) = \sum_{\substack{S\subseteq[n-1] \\ S=\{s_1<s_2<\cdots<s_k\}}} \alpha_P(S) \cdot \sum_{1\le i_1<i_2<\cdots<i_{k+1}} x_{i_1}^{s_1} x_{i_2}^{s_2-s_1} \cdots x_{i_{k+1}}^{n-s_k}. \tag{2}$$

It is easy to see that the series $F_P(x)$ is homogeneous of degree n and that it is a *quasisymmetric function*, that is, for every sequence $n_1, n_2, \ldots, n_m$ of exponents, the monomials $x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_m}^{n_m}$ and $x_{j_1}^{n_1} x_{j_2}^{n_2} \cdots x_{j_m}^{n_m}$ appear with equal coefficients whenever $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$. Through a simple counting argument and using the relation (1), the series $F_P(x)$ can also be rewritten as

$$F_P(x) = \sum_{S\subseteq[n-1]} \beta_P(S)\ L_{S,n}(x), \tag{3}$$

where the $L_{S,n}(x)$ are Gessel's quasisymmetric functions

$$L_{S,n}(x) := \sum_{\substack{1\le i_1\le i_2\le\cdots\le i_n \\ i_j<i_{j+1} \text{ if } j\in S}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

which constitute a basis for the (2^{n-1}-dimensional) space of quasisymmetric functions of degree n (for more on quasisymmetric functions and symmetric functions we refer the interested reader to [Re] and [M]).

A first question discussed in [St2] is that of conditions under which $F_P(x)$ is actually a symmetric function, in which case we refer to $F_P(x)$ as the *flag symmetric function of* P and to P as a *flag-symmetric poset.* An immediate necessary condition is that P be rank-symmetric (i.e., $\#\{t \in P : \rho(t) = r\} = \#\{t \in P : \rho(t) = n - r\}$ for every $0 \leq r \leq n$). A necessary and sufficient condition can be deduced readily from (2) [St2, Corollary 1.2]. Namely, for every $S \subseteq [n-1]$ the value of $\alpha_P(S)$ depends only on the (multi)set of differences $s_1 - 0, s_2 - s_1, \ldots, s_k - s_{k-1}, n - s_k$ and not on their ordering. If this is the case, then the symmetric function $F_P(x)$ can be expressed in terms of the basis of monomial symmetric functions $\{m_\lambda(x)\}_{\lambda \vdash n}$ as

$$F_P(x) = \sum_{\lambda \vdash n} \alpha_P(\lambda) m_\lambda(x), \tag{4}$$

where $\lambda \vdash n$ denotes a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ of n, and $\alpha_P(\lambda) := \alpha_P(\{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \lambda_2 + \ldots + \lambda_{l-1}\})$.

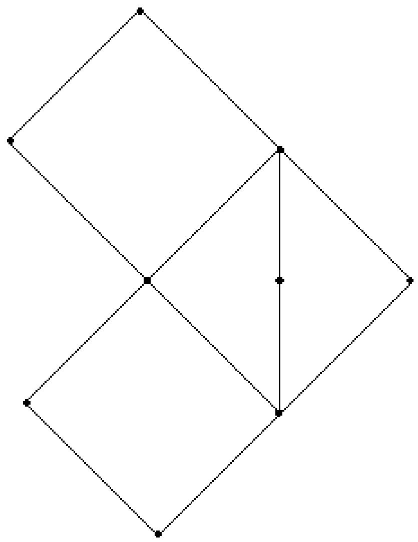


Figure 2. A flag-symmetric poset which is not locally rank-symmetric.

For example, the earlier calculation of the flag f-vector of the poset of Figure 2 shows that it is flag-symmetric, with flag symmetric function $F_P(x) = m_4(x) + 2m_{31}(x) + 3m_{22}(x) + 4m_{211}(x) + 6m_{1111}(x)$.

The following sufficient condition for $F_P(x)$ to be a symmetric function introduces the class of *locally rank-symmetric posets.* This condition is not necessary for flag-symmetry, as shown by the poset of Figure 2, but it is necessary and sufficient for every interval of P to be flag-symmetric.

Proposition 1.1 [St2, Theorem 1.4] *Let P be a ranked poset with $\hat{0}$ and $\hat{1}$. If P is locally rank-symmetric, i.e., if every interval in P is a rank-symmetric (sub)poset, then $F_P(x)$ is a symmetric function.*

Locally rank-symmetric posets turn out to be a rich source of examples yielding flag symmetric functions. The examples of flag-symmetric posets provided in [St2] include products of chains (shown to be the only flag-symmetric distributive lattices, and identical to the class of locally rank-symmetric distributive lattices), and Hall lattices (a "q-analogue" of a product of chains), as well as a discussion of some other classes of posets. If $F_P(x)$ is a symmetric function, homogeneous of degree n, and if it turns out to be Schur positive (i.e., its expression in terms of the Schur functions basis has nonnegative coefficients only), then it follows from the general theory of representations and symmetric functions that it is the Frobenius characteristic

$$\mathrm{ch}(\psi) := \sum_{\lambda \vdash n} \frac{\psi(\lambda)}{z_\lambda} p_\lambda(x) \tag{5}$$

of a character ψ of the symmetric group S_n. In the preceding display line, λ runs over all partitions of n, $\psi(\lambda)$ is the value of ψ on the conjugacy class of type λ, $z_\lambda = 1/(\lambda_1\lambda_2 \cdots m_1! m_2! \cdots)$ with m_i being the multiplicity of i as a part of λ, and $p_\lambda(x)$ is the power symmetric function indexed by λ (that is, $p_\lambda(x) = p_{\lambda_1}(x)p_{\lambda_2}(x) \cdots$ with $p_j(x) := x_1^j + x_2^j + \cdots$). It is known that when the Frobenius characteristic of a character ψ of S_n is expanded in terms of Schur functions $\{s_\lambda(x)\}_{\lambda \vdash n}$, then the coefficient of $s_\lambda(x)$ is the multiplicity with which the irreducible character χ^λ of S_n occurs in ψ. Thus, $F_P(x)$ describes a representation of S_n, whose degree $\psi(1^n)$ can be recovered as the coefficient of m_{1^n} in $F_P(x)$. In view of (4), the degree of ψ is $\alpha_P(1^n)$, the number of maximal chains in P.

The preceding discussion suggests seeking a natural action of S_n on the complex vector space $\mathbf{C}\mathcal{M}(P)$ with the set $\mathcal{M}(P)$ of maximal chains in P as a basis, giving rise to a representation of S_n with character ψ as in (5). Of particular interest would be a *local action* with this property (defined in [St2] and motivated by the notion of local stationary algebra appearing in [V]); that is, an action such that for every adjacent transposition $\sigma_i = (i, i+1)$ and every maximal chain m of P we have

$$\sigma_i(m) = \sum_{m' \in \mathbf{C}\mathcal{M}(P)} c_{mm'} m',$$

with nonzero coefficient $c_{mm'}$ only if m' differs from m at most in the element of rank i. Following [St2], we call such an action *good.* Good actions of the symmetric group are discussed in [St2] in the case of posets whose rank-two intervals are isomorphic to C_3 or $C_2 \times C_2$ (where C_i denotes an i-element chain), and for posets whose rank-three intervals are isomorphic to C_4 or $C_3 \times C_2$ or $C_2 \times C_2 \times C_2$. These results are based on work of David Grabiner [Gra]. Another illustration in [St2] gives a local action of the Hecke algebra of S_n on $\mathbf{C}\mathcal{M}(B_n(q))$, where $B_n(q)$ denotes the lattice of subspaces of an n-dimensional vector space over GF_q. In [St3] a good action is exhibited for the lattice of noncrossing partitions. To these classes of examples this paper adds the posets of shuffles.

We note that related results were recently obtained by Patricia Hersh [He] (generalizing the local S_n-action on noncrossing partitions), and Jonathan Farley and Stefan Schmidt [FaSc] (generalizing the work of Grabiner [Gra]).

2. Flag-symmetry of the posets of shuffles

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_M\}$ and $\mathcal{X} = \{x_1, x_2, \ldots, x_N\}$ be two (finite) disjoint sets which we will call the lower and the upper alphabets, respectively. Consider the collection of *shuffles* over $\mathcal{A}$ and $\mathcal{X}$, that is, words $w = w_1 w_2 \cdots w_k$ with distinct letters from $\mathcal{A} \cup \mathcal{X}$ satisfying the *shuffle property:* the subset of letters belonging to each alphabet appears in increasing order of the letter-subscripts in the appropriate alphabet. For instance, if $M = 4$ and $N = 3$, then $w = x_2 a_1 a_3 x_3$ is a shuffle word, but $w = a_1 x_2 a_2 a_3 x_1$ is not a shuffle word. Note that the empty word $\emptyset$ is a shuffle word.

The *poset of shuffles* W_{MN} consists of the shuffle words over alphabets $\mathcal{A}$ and $\mathcal{X}$ with $\#\mathcal{A} = M$ and $\#\mathcal{X} = N$ with the order relation given by $w \lessdot w'$ iff w' is obtained from w either by deleting a letter belonging to $\mathcal{A}$ or by inserting (in an allowable position) a letter belonging to $\mathcal{X}$. In particular, $\hat{0} = a_1 a_2 \cdots a_M$ and $\hat{1} = x_1 x_2 \cdots x_N$. Figure 1 shows the Hasse diagram of W_{21}. Clearly, W_{0N} and W_{M0} are isomorphic to the boolean lattices of rank N and M, respectively. We will write $\{w\}$ for the set of letters of a shuffle word w.

Greene [Gre1] investigated the posets of shuffles, whose definition was motivated by an idealized model considered in mathematical biology. The results established in [Gre1] include structural properties of W_{MN} (e.g., W_{MN} is a ranked poset; it admits a decomposition into symmetrically embedded boolean lattices and, hence, a symmetric chain decomposition; W_{MN} is an EL-shellable poset), as well as expressions for key invariants of W_{MN} (the zeta polynomial, the number of maximal chains, the Möbius function, the rank generating function, the characteristic polynomial). Two of the formulas in [Gre1] will arise later in this paper.

Proposition 2.1. [Gre1, Theorem 3.4] *The number of maximal chains in W_{MN} is given by*

$$C_{MN} = (M+N)! \sum_{k\geq 0} \binom{M}{k}\binom{N}{k}\frac{1}{2^k}. \tag{6}$$

The Möbius function of W_{MN} is

$$\mu_{MN} = \mu_{W_{MN}}(\hat{0},\hat{1}) = (-1)^{M+N}\binom{M+N}{M}. \tag{7}$$

We now turn to the interval structure of the posets of shuffles.

Lemma 2.2. *Every interval in a poset of shuffles is isomorphic to a product of posets of shuffles.*

Proof. Let $[u,w]$ be an arbitrary interval in W_{MN}, and write $u = u_1u_2\cdots u_r$, $w = w_1w_2\cdots w_s$. Let $u_{i_1}u_{i_2}\cdots u_{i_t}$ and $w_{j_1}w_{j_2}\cdots w_{j_t}$ be the subwords of u and w, respectively, formed by the letters common to the two words. Because $u < w$, the shuffle property implies $u_{i_p} = w_{j_p}$ for each $p = 1,2,\ldots,t$. Moreover, the remaining letters of u belong to the alphabet $\mathcal{A}$ and the remaining letters of w belong to the alphabet $\mathcal{X}$. Therefore the interval $[u,w]$ is isomorphic to the product of the posets of shuffles $W_{i_p-i_{p-1}-1,j_p-j_{p-1}-1}$ for $p = 1,2,\ldots,t+1$, where we set $i_0 = j_0 = 0$, $i_{t+1} = r+1$ and $j_{t+1} = s+1$. $\square$

For example, if $u = a_2x_3a_4a_5a_{10}x_6x_8$ and $v = x_1x_2x_3x_5a_{10}x_6x_8x_{10}x_{11}$ in $W_{10,15}$, then $r = 7, s = 9$ and there are $t = 4$ letters common to the two words. These form the word $x_3a_{10}x_6x_8 = u_2u_5u_6u_7 = v_3v_5v_6v_7$ so we have $[u,v] \simeq W_{12}\times W_{21}\times W_{00}\times W_{00}\times W_{02} \simeq W_{12}\times W_{21}\times W_{02}$.

Remark 2.3. Of course, factors of the form W_{00} are singleton posets and can be discarded from the product, and $W_{i0} \simeq W_{0i} \simeq B_i$, the boolean lattice with i atoms. Using the notation from the proof of Lemma 2.2, we will write $[u,w] \simeq_{\mathbf{c}} \prod_p W_{i_p - i_{p-1}-1, j_p - j_{p-1}-1}$, the *canonical isomorphism type* of the interval $[u,w]$. The notion of canonical isomorphism type of an interval will be used in Section 5.

Proposition 2.4. *For every $M, N \geq 0$, the poset of shuffles W_{MN} is locally rank-symmetric.*

Proof. Since the posets of shuffles are rank-symmetric [Gre1, Corollary 4.9] and since the product of posets preserves rank-symmetry, Lemma 2.2 implies that that every interval in W_{MN} is rank-symmetric. □

It therefore follows from Proposition 1.1 [St2, Theorem 1.4] that each poset of shuffles W_{MN} has a flag symmetric function $F_{MN} = F_{MN}(x) := F_{W_{MN}}(x)$. An explicit expression for F_{MN} can be obtained by extending Greene's notion of $\mathcal{A}$- and $\mathcal{X}$-maximal chains in W_{MN} [Gre1] to $\mathcal{A}$- and $\mathcal{X}$-maximal chains in rank-selected subposets of W_{MN}. An argument similar to Greene's yields a recurrence relation for the numbers $\alpha_{W_{MN}}(\lambda)$, which in turn implies that

$$F_{MN} = F_{M-1,N}\, e_1 + F_{M,N-1}\, e_1 - F_{M-1,N-1}\, e_2 - F_{M-1,N-1}\, p_2, \tag{8}$$

leading to

$$\sum_{M,N \geq 0} F_{MN} u^M v^N = \frac{1}{(1-ue_1)(1-ve_1) - uve_2}, \tag{9}$$

where $e_j = \sum_{1 \leq i_1 < i_2 < \cdots < i_j} x_{i_1} x_{i_2} \cdots x_{i_j}$, the jth elementary symmetric function in variables $x_1, x_2, \ldots$, and $p_2 = \sum x_i^2$. Consequently,

$$F_{MN} = \sum_{k \geq 0} \binom{M}{k} \binom{N}{k} e_2^k\, e_1^{M+N-2k}. \tag{10}$$

For example, the calculation of the flag f-vector of W_{21} done in Section 1 gives $F_{21}(x) = m_3(x) + 5m_{21}(x) + 12m_{111}(x)$. Since $e_1^3(x) = m_3(x) + 3m_{21}(x) + 6m_{111}(x)$ and $e_2(x)e_1(x) = m_{21}(x) + 3m_{111}(x)$, we have $F_{21}(x) = e_1^3(x) + 2e_2(x)e_1(x)$.

We omit the details of this argument. Instead, we will obtain the expression (10) for the flag-symmetric function of a poset of shuffles as a consequence (Corollary 4.5) of a general result (Theorem 4.4) concerning chain labelings of flag-symmetric posets.

3. A labeling for posets of shuffles

To describe an action of S_{M+N} on the maximal chains of W_{MN}, it would be natural to resort to a labeling of the chains and have the symmetric group act on the chains by acting on their label sequences simply by permuting coordinates. The poset W_{MN} is already known to be EL-shellable [Gre1], through the labeling of each covering relation $u \lessdot w$ by the unique letter in the symmetric difference of the sets of letters $\{u\}$ and $\{w\}$, and with the ordering $a_1 < a_2 < \cdots < a_M < x_1 < x_2 \cdots < x_N$ for the labels. Under this labeling each maximal chain is labeled by a permutation in S_{M+N}. However, this does not serve well the goal of describing an S_{M+N} action on the maximal chains. A similar situation occurred in [St3], where the standard EL-labeling of the noncrossing partition lattice was not suitable for describing a local action of the symmetric group on the maximal chains, and a new EL-labeling was produced for this purpose. Here too, we will define a new labeling for a poset of shuffles which lends itself naturally to the description of the desired local action of S_{M+N}.

By a *labeling* we mean a map $\Lambda : \mathcal{M}(P) \to L^n$, written

$$\Lambda(c) = (\Lambda_1(c), \Lambda_2(c), \ldots, \Lambda_n(c)),$$

where n is the length of the maximal chains of P, and L is a totally ordered set. The labeling of interest in the present paper is a C*-labeling*, that is, for every maximal chain $c = (\hat{0} = w^0 \lessdot w^1 \lessdot \cdots \lessdot w^n = \hat{1})$ and every $r \in [n]$, the label $\Lambda_r(c)$ depends only on the initial subchain $(\hat{0} = w^0 \lessdot w^1 \lessdot \cdots \lessdot w^r)$. If the label $\Lambda_r(c)$ depends only on the covering $w^{r-1} \lessdot w^r$, and not on the maximal chain c itself, then Λ is an E*-labeling*.

Three properties of labelings will play a role in this paper: the R^*-, R-, and S-labeling properties. A C-labeling Λ is an R^**-labeling* if every chain of the form

$$(\hat{0} = w^0 \lessdot w^1 \lessdot \cdots \lessdot w^r < u)$$

has a unique completion by covering relations

$$(\hat{0} = w^0 \lessdot w^1 \lessdot \cdots \lessdot w^r \lessdot w^{r+1} \lessdot \cdots \lessdot w^s = u)$$

such that

$$\Lambda_{r+1}(c) < \Lambda_{r+2}(c) < \cdots < \Lambda_s(c),$$

where c is any maximal chain beginning $\hat{0} = w^0 \lessdot w^1 \lessdot \cdots \lessdot w^s$. (By the definition of C-labeling, the remaining elements of c do not affect the labels $\Lambda_i(c)$ for $1 \le i \le s$.) In the same setting as for an R^*-labeling, the requirement for an *R-labeling* is the existence of a unique *weakly* increasing completion of the chain:

$$\Lambda_{r+1}(c) \le \Lambda_{r+2}(c) \le \cdots \le \Lambda_s(c).$$

A labeling Λ of the maximal chains of a poset is an *S-labeling* if it is one-to-one and if for every maximal chain $c = (\hat{0} = w^0 \lessdot w^1 \lessdot \cdots \lessdot w^n = \hat{1})$ and for every rank $i \in [n-1]$ such that $\Lambda_i(c) \ne \Lambda_{i+1}(c)$, there is a unique chain $c' = (\hat{0} = w^0 \lessdot w^1 \lessdot \cdots \lessdot w^{i-1} \lessdot t^i \lessdot w^{i+1} \lessdot \cdots \lessdot w^n = \hat{1})$ differing from c only at rank i, with the following property: the label sequence $\Lambda(c')$ differs from $\Lambda(c)$ only in that $\Lambda_i(c') = \Lambda_{i+1}(c)$ and $\Lambda_{i+1}(c') = \Lambda_i(c)$.

We now turn to the definition of a labeling Λ for the poset of shuffles, and then show that it has the properties R^* and S. In the next section we will see the implications of an RS- or R^*S-labeling with regard to a local action on the poset.

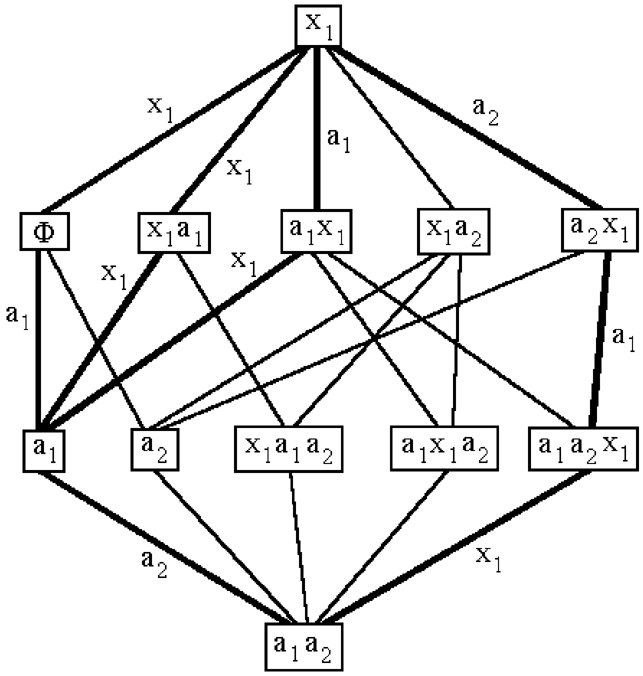


Figure 3. The labeling Λ
on four of the maximal chains of W_{21}.

To each maximal chain $c = (\hat{0} = w^0 \lessdot w^1 \lessdot \cdots \lessdot w^{M+N} = \hat{1})$ in W_{MN} we give a label sequence

$$\Lambda(c) = (\Lambda_1(c), \Lambda_2(c), \ldots, \Lambda_{M+N}(c)),$$

by assigning a label from $\mathcal{A} \cup \mathcal{X}$ to each covering relation on c. In defining Λ we distinguish three types of covering relations, (x), (xa), and (a), as follows:

(x) $w^i \lessdot w^{i+1}$ with w^{i+1} obtained from w^i by inserting a letter $x_k \in \mathcal{X}$ in a position consistent with the shuffle property; then we set $\Lambda_{i+1}(c) = x_k$.

(xa) $w^i \lessdot w^{i+1}$ with w^i of the form $w^i = ux_k a_m v$ and $w^{i+1} = ux_k v$, where this is the first deletion along c, starting from $\hat{0}$, of a letter (necessarily belonging to $\mathcal{A}$) located immediately after x_k; then we set $\Lambda_{i+1}(c) = x_k$.

(a) $w^i \lessdot w^{i+1}$ with w^{i+1} obtained from w^i by deleting a letter $a_j \in \mathcal{A}$, and this deletion is not of type (xa); then we set $\Lambda_{i+1}(c) = a_j$.

Figure 3 shows the labeling Λ of four of the maximal chains in W_{21}.

Lemma 3.1. *The labeling Λ is injective on the maximal chains of W_{MN}, for all M, N. Its range consists of the (multi)permutations of all multisets of the form $A \cup 2X \cup (\mathcal{X} - X)$, where $A \subseteq \mathcal{A}$, $X \subseteq \mathcal{X}$, $|A| + |X| = M$, and $2X$ denotes the multiset consisting of two copies of each element of X.*

Proof. From the definition of Λ it is clear that all letters in $\mathcal{X}$ appear in the label sequence of any maximal chain c and that for every $a_j \in \mathcal{A}$ which does not appear in the label sequence, there is an $x_k \in \mathcal{X}$ which appears twice. Thus, the label sequence of every maximal chain c is of the claimed form.

Conversely, we claim that given a multipermutation σ of $A \cup 2X \cup (\mathcal{X} - X)$ for some A and X as in the statement of the Lemma, there is a unique maximal chain in W_{MN} having label sequence $\Lambda(c) = \sigma$. Indeed, first note that if $A = \mathcal{A}$ and $X = \emptyset$ (that is, σ is a permutation of $\mathcal{A} \cup \mathcal{X}$), then only coverings of type (a) and (x) are possible. Thus, starting from $\hat{0} = w^0$, σ dictates a sequence of deletions of letters from $\mathcal{A}$ and insertions of letters from $\mathcal{X}$, each insertion being made in the rightmost possible position. This determines a unique maximal chain c as desired. For example, for W_{23}, the permutation $\sigma = a_2x_3x_1a_1x_2$ determines the chain $c = (\hat{0} = a_1a_2 \lessdot a_1 \lessdot a_1x_3 \lessdot a_1x_1x_3 \lessdot x_1x_3 \lessdot x_1x_2x_3 = \hat{1})$.

Next, suppose that $\mathcal{A} - A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ and $X = \{x_{j_1}, x_{j_2}, \ldots, x_{j_k}\}$, for some $1 \le k \le \min\{M, N\}$, where $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. Observe that the shuffle condition implies that if the pairs x_m, a_p and x_n, a_q are involved in coverings of type (xa), then $m \neq n$ and $p \neq q$, and $m < n$ if and only if $p < q$. Therefore, in the multipermutation σ, the second occurrence of x_{j_r} must correspond to a covering of type (xa) involving the pair of letters x_{j_r}, a_{i_r}, for each $r = 1, \ldots, k$. The first occurrence of x_{j_r} in σ is forced to correspond to the insertion of x_{j_r} immediately in front of a_{i_r}, and for each $x_t \notin X$, its unique occurrence in σ forces the insertion of x_t in the rightmost position possible to the left of a_{i_r} and/or x_{j_r}, if $j_r = \min\{s > t : x_s \in X\}$ (if this set is empty, then x_t is inserted in the rightmost position possible). As in the preceding case, a unique maximal chain c is determined by σ. For example, for W_{45}, let $\sigma = x_3x_5a_2x_4x_2x_1x_2x_4a_4$. The coverings of type (xa) must involve the pairs x_2, a_1 and x_4, a_3. From σ we reconstruct the chain

$\hat{0} = a_1a_2a_3a_4 \lessdot a_1a_2x_3a_3a_4 \lessdot a_1a_2x_3a_3a_4x_5 \lessdot a_1x_3a_3a_4x_5 \lessdot a_1x_3x_4a_3a_4x_5 \lessdot$
$x_2a_1x_3x_4a_3a_4x_5 \lessdot x_1x_2a_1x_3x_4a_3a_4x_5 \lessdot x_1x_2x_3x_4a_3a_4x_5 \lessdot x_1x_2x_3x_4a_4x_5 \lessdot x_1x_2x_3x_4x_5 = \hat{1}$. $\square$

The behavior of Λ on intervals of rank two can be easily described.

Lemma 3.2. *For every rank-two interval in a poset of shuffles W_{MN}, the labeling Λ conforms to one of the following cases:*

3.2.1) If a rank-two interval is isomorphic to $C_2 \times C_2$, then its two chains c_1 and c_2 have label sequences of the form $\Lambda(c_1) = (l_1, l_2)$ and $\Lambda(c_2) = (l_2, l_1)$, where l_1 and l_2 are distinct letters from $\mathcal{A} \cup \mathcal{X}$.

3.2.2) If a rank-two interval is isomorphic to Π_3, then its three chains γ_1, γ_2 and γ_3 have label sequences of the form $\Lambda(\gamma_1) = (x_j, l)$, $\Lambda(\gamma_2) = (l, x_j)$, and $\Lambda(\gamma_3) = (x_j, x_j)$, for suitable letters $x_j \in \mathcal{X}$ and $l \in \mathcal{A} \cup (\mathcal{X} - \{x_j\})$.

Proof. Each rank-two interval of a poset of shuffles has either 4 or 5 elements. That is, each rank-two interval is isomorphic either to $C_2 \times C_2$ or to the lattice Π_3 of partitions of a 3-element set. Specifically, an interval of rank 2 is of one of the following forms:

i) $[ua_mva_nw, uvw]$ or $[uvw, ux_pvx_qw]$, for some shuffle words u, v, w;
or $[ua_mvw, uvx_pw]$ or $[uva_mw, ux_pvw]$, for some words u, v, w with $v \neq \emptyset$. By Lemma 2.2, these

intervals are isomorphic to $W_{10} \times W_{10}$, $W_{01} \times W_{01}$, or $W_{10} \times W_{01}$, all of which are isomorphic to $C_2 \times C_2$.

ii) $[ua_pv, ux_mv]$ for some words u, v. Such an interval is isomorphic to $W_{11} \simeq \Pi_3$.

The definition of the chain labeling Λ and the two possible structures of the intervals of rank 2 yield the two cases in the desired conclusion. $\square$

Proposition 3.3. *The labeling Λ is an S-labeling on the poset of shuffles.*

Proof. This follows immediately from Lemma 3.1 and Lemma 3.2. $\square$

Proposition 3.4. *Consider the ordering $a_1 < a_2 < \cdots < a_M < x_1 < x_2 < \cdots < x_N$ on the union of the two alphabets. Then the labeling Λ is an R^*-labeling of the poset of shuffles.*

Proof. Let $\hat{0} = w^0 \lessdot w^1 \lessdot \cdots \lessdot w^r < v$ be a chain in W_{MN}. Let B be the set of pairs of letters x_{j_s}, a_{i_s} which occur as consecutive letters in w^r and such that $a_{i_s} \notin \{v\}$. By the definition of Λ, every covering along any w^r-v-chain where such an a_{i_s} is removed will be a covering of type (xa). Thus, consider the w^r-v-chain obtained by first deleting, in increasing order of their indices, the letters in $(\mathcal{A} - B) \cap (\{w^r\} - \{v\})$; then inserting, also in increasing order of the indices, every $x_t \in \{v\} - \{w^r\}$, and deleting each letter $a_{i_s} \in B$ after the insertion of $x_t \in \{v\} - \{w^r\}$ if an only if $t < j_s$. The label sequence of this chain is clearly strictly increasing. Since any other label sequence with distinct entries is a permutation of the same set of labels $(\mathcal{A} - B) \cup \mathcal{X}$, this is the only strictly increasingly labeled w^r-v-chain. $\square$

Remark 3.5. The reader familiar with the theory of shellable posets may be interested in the observation that *the labeling Λ readily gives rise to a CL-labeling of W_{MN}* (in the sense of [BjWa], [BjGaSt]). Indeed, the unique strictly increasing $\hat{0}$-u-chain guaranteed by the preceding result can be taken as the "root" of each interval $[u, v]$, and the labeling Λ^* defined by $\Lambda^*(u \lessdot v) = (\Lambda(u \lessdot v), M + N - \rho(u))$ is a CL-labeling valued in $((\mathcal{A} \cup \mathcal{X}) \times [M+N])^{M+N}$, under lexicographic order on $(\mathcal{A} \cup \mathcal{X}) \times [M+N]$.

Remark 3.6. *The proof of Proposition 3.4 shows that the R^*-labeling Λ has a stronger property: the unique increasingly labeled extension of a chain $\hat{0} = w^0 \lessdot w^1 \lessdot \cdots \lessdot w^r < v$ depends only on w^r and v. We will write $\gamma(w^r, v)$ to denote this chain.*

The remainder of this section is devoted to enumerative consequences of the labeling Λ, yielding combinatorial proofs of results from [Gre1]. We begin with a bijective proof of the local rank-symmetry of the posets of shuffles (an inductive proof was given in Proposition 2.4). In particular, this is a bijective proof of the rank-symmetry of a poset of shuffles. An alternative bijective proof of the rank-symmetry of W_{MN} is implicit in the symmetric chain decomposition which appears in [Gre1].

Corollary 3.7. *For every two elements $u < w$ in a poset of shuffles W_{MN}, there is a bijection between the elements of rank $\rho(u) + i$ and the elements of rank $\rho(w) - i$ in the interval $[u, w]$.*

Proof. Let $v \in [u, w]$ be an element of rank $\rho(u)+i$. Consider the maximal chain $c(u, v, w)$ formed by concatenating $\gamma(\hat{0}, u)$, $\gamma(u, v)$, $\gamma(v, w)$, and $\gamma(w, \hat{1})$. Let $c'(u, v, w)$ be the unique maximal chain whose label sequence is the concatenation of $\Lambda(\gamma(\hat{0}, u))$, $\Lambda(\gamma(v, w))$, $\Lambda(\gamma(u, v))$, and $\Lambda(\gamma(w, \hat{1}))$. Define $\varphi(v)$ to be the element of rank $\rho(w) - i$ on the chain $c'(u, v, w)$. It is easy to see (from the definition and injectivity of Λ) that $c'(u, v, w)$ contains the elements u and w and that φ establishes a bijection between the rank-$(\rho(u) + i)$ and the rank-$(\rho(w) - i)$ elements in the interval $[u, w]$. $\square$

Corollary 3.8. *The number of elements of the poset W_{MN} is equal to $\sum_{k\geq 0} \binom{M}{k}\binom{N}{k} 2^{M+N-2k}$.*

Proof. We may count the increasingly labeled chains $\gamma(\hat{0}, w)$ since they are in bijection with the elements $w \in W_{MN}$. For a prescribed number $k \geq 0$ of coverings of type (xa), the set of labels along such a chain is determined by the choice of k pairs from $\mathcal{A} \times \mathcal{X}$ for the coverings of type (xa), and an arbitrary subset of the complement in $\mathcal{A} \cup \mathcal{X}$ of the letters chosen for the k pairs. $\square$

Lemma 3.1 yields readily the number of maximal chains in a poset of shuffles, giving a more direct derivation of the formula (6) due to C. Greene.

Corollary 3.9. *The number of maximal chains in the poset of shuffles W_{MN} is*

$$C_{MN} = \sum_{k\geq 0} \binom{M}{k}\binom{N}{k} \frac{(M+N)!}{2^k}.$$

Proof. By Lemma 3.1, we can count the maximal chains in W_{MN} by counting the possible label sequences $\Lambda(c)$. For each value $k \geq 0$, the kth term in the sum gives the number of multipermutations $\Lambda(c)$ in which k letters of $\mathcal{X}$ appear with multiplicity 2, while k of the letters of $\mathcal{A}$ do not occur in σ. $\square$

From the R^*-labeling Λ we can recover the formula (7) for the Möbius function of a poset of shuffles, which was obtained in [Gre1] using an EL-labeling as well as through an alternative computation.

Corollary 3.10. *The Möbius function of the poset of shuffles W_{MN} is given by*

$$\mu_{W_{MN}}(\hat{0}, \hat{1}) = (-1)^{M+N}\binom{M+N}{M}.$$

Proof. By the general theory of [BjW], the Möbius function is $(-1)^{M+N}$ times the number of maximal chains to which the R^*-labeling Λ assigns weakly decreasing label sequences. From Lemma 3.1 it follows that such chains have label sequences of the form

$$\Lambda(c) = (x_{j_{N+k}}, x_{j_{N+k-1}}, \ldots, x_{j_1}, a_{i_{M-k}}, a_{i_{M-k-1}}, \ldots, a_{i_1})$$

for some $0 \leq k \leq \min\{M, N\}$, where $i_{M-k} > i_{M-k-1} > \cdots > i_1$ and $j_{N+k} \geq j_{N+k-1} \geq \cdots \geq j_1$ with k non-consecutive equalities. Therefore, the decreasingly labeled maximal chains correspond bijectively to the selections of $M-k$ letters from $\mathcal{A}$ and k letters from $\mathcal{X}$ for some k. It is an easy exercise to show that the number of such selections is $\binom{M+N}{M}$ yielding the desired formula for the Möbius function. $\square$

4. A local action of the symmetric group

We begin with two general results which imply that the posets of shuffles have a local action of the symmetric group and establish the relation between the Frobenius characteristic of the corresponding character and the flag symmetric function of the poset.

Theorem 4.1. *Suppose P is a finite ranked poset of rank n, with $\hat{0}$ and $\hat{1}$. If P has an S-labeling Λ, then the action of S_n on labels by permuting coordinates induces a local (permutation) action on the maximal chains of P.*

Proof. Let $\Lambda(c) = (\Lambda_1, \ldots, \Lambda_n)$ be the label of a maximal chain c, and let $1 \le i \le n-1$. The adjacent transposition $\sigma_i = (i, i+1)$ acts on $\Lambda(c)$ by interchanging Λ_i and Λ_{i+1}. By definition of S-labeling, there is a unique maximal chain c' such that $\Lambda(c') = \sigma_i \cdot \Lambda(c)$. Since the σ_i's generate S_n, we get an action of S_n on the set of labels of maximal chains, and hence on $\mathcal{M}(P)$. Moreover, this action is local by the definition of an S-labeling. $\square$

Observation 4.2.1. Suppose S_n acts on the maximal chains of a labeled poset P of rank n by permuting the coordinates of the labels. Then each orbit of maximal chains consists of the chains labeled by the permutations of a multiset, and the Frobenius characteristic of the S_n-action is $\sum_{\nu \vdash n} N(\nu) h_\nu$, where $N(\nu)$ denotes the number of orbits of maximal chains which are labeled by the permutations of a multiset of type ν (i.e., $\nu_1, \nu_2, \ldots$ are the multiplicities of the distinct elements of the multiset).

Observation 4.2.2. A special case is when the maximal chains in each orbit form a subposet isomorphic to a product of chains, $C_{\nu_1+1} \times C_{\nu_2+1} \times \cdots$. It is not hard to show that this is the case for the posets of shuffles and the action discussed here, as well as for the lattice of noncrossing partitions discussed in [St3]. Thus, in addition to admitting a partition of the elements into boolean lattices (as shown in [Gre1] for poset of shuffles and in [SiU] for the noncrossing partition lattice), these posets also admit a partition of their maximal chains into products of chains. Figure 4 shows the orbits of maximal chains in W_{21}. In general, in W_{MN}, each orbit of maximal chains is isomorphic to a product of chains of the form $C_3^k \times C_2^{M+N-2k}$.

Observations 4.2.1 and 4.2.2 are generalized by the following result.

Theorem 4.3. *Suppose that P is a ranked poset (with $\hat{0}$ and $\hat{1}$) of rank n with a local S_n-action.*

(a) *Let $c \in \mathcal{M}(P)$. Then the stabilizer stab(c) of c is a Young subgroup $S_{B_1} \times S_{B_2} \times \cdots \times S_{B_k}$ of S_n, where $\pi = \{B_1, B_2, \ldots, B_k\}$ is a partition of n.*

(b) *If ψ is the character of the S_n-action, then* $\mathrm{ch}(\psi)$ *is h-*positive, *i.e.,* $\mathrm{ch}(\psi) = \sum_{\nu \vdash n} a_\nu h_\nu$, *where $a_\nu \ge 0$.*

(c) *If* $\mathrm{ch}(\psi) = h_\nu$ *for some $\nu \vdash n$, then $P \simeq C_{\nu_1+1} \times C_{\nu_2+1} \times \cdots$.*

Proof. (a) Let $\theta \in \mathrm{stab}(c)$, and let i be the least element of $[n]$ for which $\theta^{-1}(i) = j > i$. We claim that $(i, j) \in \mathrm{stab}(c)$. Let $(i_1, i_2, \ldots, i_r)$ be the cycle of θ containing i, where $i_1 = i$

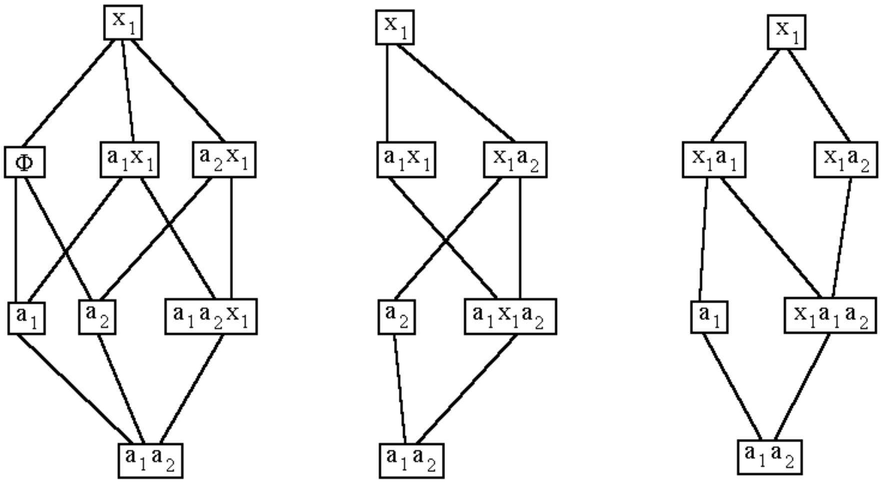


Figure 4. The orbits of maximal chains in W_{21}.

and $i_r = j$. Let $\tau_1, \tau_2, \ldots, \tau_s$ be the remaining cycles of θ. Then (multiplying right-to-left)

$$\begin{aligned}\theta &= \tau_s \cdots \tau_2 \tau_1 (i_3, i_2)(i_4, i_3) \cdots (i_r, i_{r-1})(i_1, i_r) \\ &= \tau_s \cdots \tau_2 \tau_1 (i_3, i_2)(i_4, i_3) \cdots (i_r, i_{r-1}) \sigma_{i_r -1} \sigma_{i_r -2} \cdots \sigma_{i_1+1} \sigma_{i_1} \sigma_{i_1+1} \cdots \sigma_{i_r-2} \sigma_{i_r-1}. \end{aligned} \tag{11}$$

Note that only one factor of the last product above moves i_1, namely, σ_{i_1}. Let t be the element of c of rank i. It follows from the definition of local action that t is also an element of the chain $c' = \sigma_{i_2} \cdots \sigma_{i_r-2} \sigma_{i_r-1} \cdot c$. Let s be the element of the chain $c'' = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_r-1} \cdot c$ of rank i. Then again by definition of local action, s is an element of the chain $\theta \cdot c$ (since the factors to the left of σ_{i_1} in (11) can be written as products of σ_p's with $p > i$). Since $\theta \cdot c = c$, we have $s = t$. Thus $\sigma_{i_1} \cdot c' = c'$, so we can remove the factor σ_{i_1} from the product (11) and still get a permutation $\theta' \in \mathrm{stab}(c)$. But

$$\begin{aligned}\theta' &= \tau_s \cdots \tau_2 \tau_1 (i_3, i_2) \cdots (i_{r-1}, i_{r-2})(i_r, i_{r-1}) \\ &= \theta(i_1, i_r).\end{aligned}$$

Hence $(i_1, i_r) \in \mathrm{stab}(c)$, as claimed.

It follows by induction on i (as defined above) that if for any $a, b \in [n]$ we have $\theta(a) = b$, then $(a, b) \in \mathrm{stab}(c)$. From this it is clear that $\mathrm{stab}(c)$ is a Young subgroup of S_n.

(b) By (a), the S_n-action on $\mathcal{M}(P)$, when restricted to an orbit $\mathcal{O} \in \mathcal{M}(P)/S_n$, is equivalent to the action of S_n on the set S_n/S_ν of cosets of some Young subgroup $S_\nu = S_{\nu_1} \times S_{\nu_2} \times \cdots$, where $\nu = \nu_{\mathcal{O}} \vdash n$. If ψ^ν is the character of this action of S_n on S_n/S_ν, then $\mathrm{ch}(\psi^\nu) = h_\nu$. Hence

$$\mathrm{ch}(\psi) = \sum_{\mathcal{O} \in \mathcal{M}(P)/S_n} h_{\nu_{\mathcal{O}}}.$$

(c) Let M be the multiset $\{1^{\nu_1}, 2^{\nu_2}, \ldots\}$. The action of S_n on the set S_M of permutations of M obtained by permuting coordinates is equivalent to the natural action of S_n on S_n/S_ν. Hence by (a) there is an S_n-equivariant bijection $\rho : \mathcal{M}(P) \to S_M$. Let $t \in P$, say $\mathrm{rank}(t) = k$, and let c be a maximal chain of P containing t. Let $\rho(c) = a = a_1 a_2 \cdots a_n \in S_M$. Let $b = b_1 b_2 \cdots b_n \in S_M$ have the property that $\{a_1, a_2, \ldots, a_k\} = \{b_1, b_2, \ldots, b_k\}$ (as multisets), so also $\{a_{k+1}, \ldots, a_n\} = \{b_{k+1}, \ldots, b_n\}$. Since a can be transformed to b by adjacent transpositions all different from σ_k, it follows from the definition of local action that the chain $c' \in \mathcal{M}(P)$ satisfying $\rho(c') = b$ contains t. Hence for any submultiset $N \subseteq M$, we can define t_N to be the *unique* element of P for which there exists $c \in \mathcal{M}(P)$ containing t_N and such that N is equal to the first $k = \mathrm{rank}(t_N)$ elements of $\rho(c)$. We thus have a well-defined surjection $\tau : B_M \to P$, $\tau(N) = t_N$, where B_M is the lattice of submultisets of M ordered by inclusion. Since $B_M \simeq C_{\nu_1+1} \times C_{\nu_2+1} \times \cdots$, it suffices to show that τ is a poset isomorphism.

By construction, τ is order-preserving (i.e., $N \subseteq N' \Rightarrow \tau(N) \leq \tau(N')$), and the induced map $\tau : \mathcal{M}(B_M) \to \mathcal{M}(P)$ is injective. Since $\#\mathcal{M}(B_M) = \#\mathcal{M}(P) = n!/\nu_1!\,\nu_2! \cdots$, it follows that $\tau : \mathcal{M}(B_M) \to \mathcal{M}(P)$ is a bijection. Suppose that $N, N' \in B_M$ with $N \neq N'$ and $\tau(N) = \tau(N')$. Let c_1 be a maximal chain of the interval $[\emptyset, N]$ of B_M, and c_2 a maximal chain of $[N', M]$. Then $\tau(c_1 \cup c_2)$ is a maximal chain of P not belonging to $\tau(\mathcal{M}(B_M))$, contradicting the surjectivity of $\tau : \mathcal{M}(B_M) \to \mathcal{M}(P)$. Thus τ is injective on B_M. Since B_M and P have the same number of maximal chains and τ is injective, it is easy to see that τ must be an isomorphism. □

Theorem 4.4. *Suppose P is a flag-symmetric poset of rank n with flag symmetric function F_P and having an S-labeling Λ. Let ψ be the character of the action of S_n on $\mathbf{C}\mathcal{M}(P)$ induced from the S_n-action on labels.*

(a) If Λ is an RS-labeling then $F_P = \mathrm{ch}(\psi) = h_\nu$ for some partition ν of n, and $P \simeq C_{\nu_1+1} \times C_{\nu_2+1} \times \cdots$.

(b) If Λ is an R^*S-labeling, then $\mathrm{ch}(\psi) = \omega F_P$*, where ω is the standard involution on symmetric functions [M, p.21]. Hence F_P is an e-positive symmetric function.*

Proof. (a) Let $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in \mathbf{P}^\ell$, with $\gamma_1 + \cdots + \gamma_\ell = n$. For a finite multiset M, let $\mathcal{U}_\gamma(M)$ denote the collection of all sequences $\pi = (M_1, \ldots, M_\ell)$ of (nonempty) multisets M_i such that $\#M_i = \gamma_i$ and $\bigcup M_i = M$. Let

$$\mathcal{U}_\gamma(\Lambda) = \bigcup_M \mathcal{U}_\gamma(M),$$

where M ranges over all distinct multisets $M = \{\Lambda_1, \ldots, \Lambda_n\}$ of entries of labels $\Lambda(c) = (\Lambda_1, \ldots, \Lambda_n)$ of maximal chains of P. For each $\pi = (M_1, \ldots, M_\ell) \in \mathcal{U}_\gamma(\Lambda)$, it follows from the definition of RS-labeling that there is a unique chain $t(\pi) = (\hat{0} = t_0 < t_1 < \ldots < t_\ell = \hat{1})$ of P with the following properties:

(i) $\rho(t_i) - \rho(t_{i-1}) = \gamma_i$ for $1 \le i \le \ell$, and

(ii) if c is the unique completion of t to a maximal chain of P whose label $\Lambda(c) = (\Lambda_1, \ldots, \Lambda_\ell)$ satisfies

$$\Lambda_1 \le \cdots \le \Lambda_{\gamma_1},\ \Lambda_{\gamma_1+1} \le \cdots \le \Lambda_{\gamma_1+\gamma_2}, \ldots,\ \Lambda_{\gamma_1+\cdots+\gamma_{\ell-1}+1} \le \cdots \le \Lambda_n, \tag{12}$$

then $M_i = \{\Lambda_{\gamma_1+\cdots+\gamma_{i-1}+1}, \ldots, \Lambda_{\gamma_1+\cdots\gamma_i}\}$.

The map $\pi \mapsto t(\pi)$ is a bijection from $\mathcal{U}_\gamma(\Lambda)$ to the set of all chains of P whose elements have ranks $0,\ \gamma_1,\ \gamma_1+\gamma_2, \ldots,\ \gamma_1+\cdots+\gamma_{\ell-1},\ n$. Hence

$$F_P = \sum_{\lambda \vdash n} \alpha_P(\lambda) m_\lambda = \sum_{\lambda \vdash n} \#\mathcal{U}_\lambda(\Lambda) \cdot m_\lambda. \tag{13}$$

Let $\nu = \mathrm{type}(M)$, i.e., $\nu \vdash n$ and the part multiplicities of M (in weakly decreasing order) are $\nu_1,\ \nu_2, \ldots$. It is well-known (equivalent to [M,(6.7)(ii)]) that

$$\sum_{\lambda \vdash n} \#\mathcal{U}_\lambda(M) \cdot m_\lambda = h_\nu. \tag{14}$$

Let $\mathcal{L}(\Lambda)$ be the collection of all multisets $M = \{\Lambda_1, \ldots, \Lambda_n\}$ of entries of a maximal chain label of P. Summing (14) over all $M \in \mathcal{L}_\lambda$ and comparing with (13) gives

$$F_P = \sum_{M \in \mathcal{L}(\Lambda)} h_{\mathrm{type}(M)}.$$

Since the action of S_n by permuting coordinates of permutations of multiset of type ν has Frobenius characteristic h_ν, we get $F_P = \mathrm{ch}(\psi)$.

Since there is a *unique* weakly increasing maximal chain of P from $\hat{0}$ to $\hat{1}$ (equivalently, since $\alpha_P(\emptyset) = 1$), we get $F_P = h_\nu$ for some $\nu \vdash n$.

It now follows from Theorem 4.3(c) that $P \simeq C_{\nu_1+1} \times C_{\nu_2+1} \times \cdots$.

(b) The argument is parallel to (a), except that the inequalities $\leq$ of (12) become strict inequalities $<$. Hence $\mathcal{U}_\gamma(M)$ is replaced by the collection $\mathcal{V}_\gamma(M)$ of sequences $\pi = (M_1, \ldots, M_\ell)$ of *sets*, rather than multisets, so (14) becomes

$$\sum_{\lambda \vdash n} \#\mathcal{V}_\lambda(M) \cdot m_\lambda = e_\nu.$$

Since $\omega e_\nu = h_\nu$, we get $F_P = \omega(\mathrm{ch}(\psi))$. $\square$

The expression (10) for the flag symmetric function $F_{W_{MN}}$, follows now immediately from the general Theorem 4.4(b).

Corollary 4.5. *The action of S_{M+N} on the label sequences of the maximal chains in the poset of shuffles W_{MN} induces a local action on the poset. The Frobenius characteristic for the character ψ of the corresponding representation of S_{M+N} and the flag symmetric function of W_{MN} are related by*

$$\mathrm{ch}(\psi) = \sum_{k \geq 0} \binom{M}{k} \binom{N}{k} h_2^k(x) h_1^{M+N-2k}(x) = \omega F_{W_{MN}}(x).$$

In the remainder of this section we make comments regarding the preceding results and discuss other possible directions for generalizations.

Remark 4.6. Theorems 4.1 and 4.4 apply to posets which are products of chains and also to the lattice of noncrossing partitions. The corresponding conclusions are established directly in [St2] and [St3].

Remark 4.7. The power series $F_P(x)$ may be viewed in a broader context. For a function φ in the incidence algebra (see, e.g., [St1, section 3.6]) of a ranked poset P, define

$$\alpha_P(\varphi, S) = \sum_{\hat{0}=t_0<t_1<\cdots<t_k<\hat{1}} \varphi(\hat{0}, t_1)\varphi(t_1, t_2)\cdots\varphi(t_k, \hat{1}),$$

where the sum ranges over the chains in P whose rank-support is the set $S \subseteq [n-1]$, and n is the rank of P. Now define

$$F_P(\varphi, x) := \sum_{\substack{S\subseteq[n-1] \\ S=\{s_1<s_2<\cdots<s_k\}}} \alpha_P(\varphi, S) \cdot \sum_{1\leq i_1<i_2<\cdots<i_{k+1}} x_{i_1}^{s_1} x_{i_2}^{s_2-s_1}\cdots x_{i_{k+1}}^{n-s_k}.$$

Note that $F_P(\varphi, x)$ is a quasisymmetric function, homogeneous of degree n. In particular, if $\varphi = \zeta$, the zeta function of P (i.e., $\zeta(u, v) = 1$ if $u \leq v$, and $\zeta(u, v) = 0$ otherwise), we recover $\alpha_P(\zeta, S) = \alpha_P(S)$ and $F_P(\zeta, x)$ is the function $F_P(x)$ of (2). We intend to pursue this generalization elsewhere, mentioning here only one result – the next proposition. We note that the same result holds for an arbitrary invertible element φ from the incidence algebra of P and its inverse. Here we present a direct proof for the special case $\varphi = \zeta$ and $\varphi^{-1} = \mu$ which is the instance occurring in the context of this paper.

Proposition 4.7.1. *Let P be a ranked poset of rank n, having elements $\hat{0}$ and $\hat{1}$. If ζ and μ are, as usual, the zeta function and the Möbius function of P, then*

$$F_P(\mu, x) = (-1)^n \omega F_P(\zeta, x),$$

where ω is the involution on quasisymmetric functions defined by $\omega L_{S,n}(x) = L_{\overline{S},n}(x)$, with $\overline{S}$ denoting the complement of S in $[n-1]$.

Proof. Using (3) and then (1), we have

$$\omega F_P(\zeta, x) = \omega \sum_{S\subseteq[n-1]} \beta_P(S) L_{S,n}(x) = \sum_{S\subseteq[n-1]} \sum_{T\subseteq S} (-1)^{|S-T|} \alpha_P(T) L_{\overline{S},n}(x).$$

Using Hall's theorem (e.g., [St1, Prop. 3.8.5]), the sum over T evaluates to $(-1)^{|S|-1}\mu_{P_S}(\hat{0}, \hat{1})$. Next, using Baclawski's theorem for the Möbius function of a subposet (see [Gre2, formula (7.2)]), the sum over T can be expressed as

$$(-1)^{|S|-1} \sum_{k\geq 0} \sum_{\substack{\hat{0}<t_1<t_2<\cdots<t_k<\hat{1} \\ \text{in } P_{\overline{S}}}} (-1)^k \mu(\hat{0}, t_1)\mu(t_1, t_2)\cdots\mu(t_k, \hat{1}).$$

(When $k = 0$, the chain $\hat{0} < \hat{1}$ gives the term $\mu(\hat{0}, \hat{1})$.) By grouping the chains in $P_{\overline{S}}$ according to their rank-support, $U \subseteq \overline{S}$, we obtain

$$\omega F_P(\zeta, x) = (-1)^n \sum_{U \subseteq [n-1]} \alpha_P(\mu, U) \cdot \sum_{\overline{S} \supseteq U} (-1)^{|\overline{S} - U|} L_{\overline{S}, n}(x).$$

Finally, by the definition of $L_{V,n}(x)$ and an inclusion-exclusion argument, the inner sum over $\overline{S}$ is equal to $\sum_{1 \leq i_1 < i_2 < \cdots < i_j} x_{i_1}^{u_1} x_{i_2}^{u_2 - u_1} \cdots x_{i_{j+1}}^{n - u_j}$, where $U = \{u_1 < u_2 < \cdots < u_j\}$. Thus, $\omega F_P(\zeta, x) = (-1)^n F_P(\mu, x)$ as claimed. $\square$

Since ω restricted to symmetric functions agrees with the standard involution ω, the expressions for the characteristic $\mathrm{ch}(\psi)$ from Theorem 4.4 can be restated as (a) $\mathrm{ch}(\psi) = F_P(\zeta, x) = h_\nu$ when P is RS-labeled, and (b) $\mathrm{ch}(\psi) = (-1)^n F_P(\mu, x)$ when P is R^*S-labeled.

Remark 4.8. *The S_{M+N}-action of Corollary 4.5 is a permutation action on the maximal chains of W_{MN}, for which each orbit consists of those maximal chains whose label sequence $\Lambda(c)$ is a permutation of the same multiset of letters. Thus, the explanation of formula (6) for the maximal chains of W_{MN} provided in the proof of Corollary 3.9 amounts to counting the maximal chains according to their orbit, and grouping the orbits according to the type $2^k 1^{M+N-2k}$ of the multiset of the chain labels.*

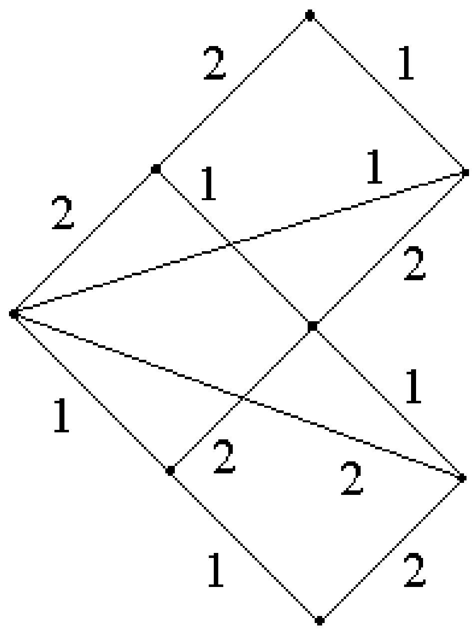
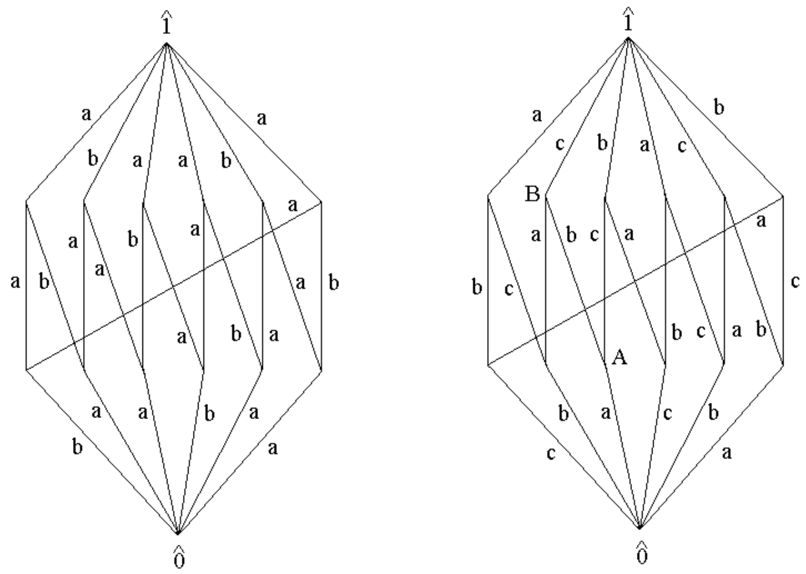


Figure 5. A poset with an S-labeling
and an orbit of maximal chains which is not a product of chains.

Remark 4.9. *An S-labeling does not ensure that each orbit of maximal chains is isomorphic to a product of chains.* The poset shown in Figure 5, suggested to us by H. Barcelo, has a local action induced from the action of S_4 on the labels, but the orbit of chains labeled by the multiset 1122 is not a product of chains. In fact, no product of chains other than the trivial one, C_5, occurs as a subposet of rank four in this poset.

Remark 4.10. *The converse of Theorem 4.1 does not hold. That is, a chain labeling such that the action of S_n on labels induces a local action is not necessarily an S-labeling.* The poset P shown in Figure 6(a) has a labeling of its maximal chains which, although not injective, gives rise to an S_3 local action on the maximal chains of P. The orbits are four copies of $C_2 \times C_3$, each labeled in the standard way with the multiset a, b, b. On the other hand, the noninjective labeling of Figure 6(b) does not produce an S_3 action on the maximal chains (e.g., $\sigma_1\sigma_2\sigma_1(\hat{0} \lessdot A \lessdot B \lessdot \hat{1}) \neq \sigma_2\sigma_1\sigma_2(\hat{0} \lessdot A \lessdot B \lessdot \hat{1})$).



(a) (b)

Figure 6. Non-injective labelings.
a) The action on labels induces a local action.
b) The action on labels does not induce a local action.

Finally, we give a local condition which characterizes labeled posets with a local action induced from the action on labels. This, of course, can be seen to apply to the earlier examples.

Theorem 4.11. *Let P be a finite ranked poset of rank n, having a $\hat{0}$ and $\hat{1}$, and with a labeling Λ of its maximal chains. For a maximal chain $c = (\hat{0} = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_n = \hat{1})$ and a rank $i \in \{1, 2, \ldots, n-2\}$, let $\tau := (\hat{0} = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_{i-1})$ and let $\theta := (t_{i+2} \lessdot t_{i+3} \lessdot \cdots \lessdot t_n = \hat{1})$. A local action is induced from the S_n-action on labels if and only if Λ satisfies the following condition for every maximal chain c and every value of $i \in \{1, 2, \ldots, n-2\}$:*

The length-three chains δ from t_{i-1} to t_{i+2} with labels induced by restricting $\Lambda(\tau\delta\theta)$ can be partitioned so that

(a) each class is isomorphic to $C_2 \times C_2 \times C_2$ or $C_3 \times C_2$ or C_4, and

(b) the labeling in each class coincides with the standard labeling of a product of chains by join-irreducibles.

Proof. The conditions (a) and (b) on Λ imply readily the Coxeter relations for S_n, showing that the local action is well-defined. Conversely, within each orbit of chains δ, the local action fixes the chains labeled aaa, so these form classes isomorphic to C_4; a chain δ labeled with aba is mapped under the local action to chains with the same τ and θ and label sequences aab and baa, structured as a copy of $C_2 \times C_3$ and forming a class as claimed; similarly, a chain δ labeled as abc is mapped by the subgroup generated by σ_i and σ_{i+1} to six chains forming a copy of $C_2 \times C_2 \times C_2$, with labels as claimed. □

5. Multiplicative functions on the poset of shuffles

Consider now the poset $W_{\infty\infty}$ whose elements are the shuffles of finite words using the lower alphabet $\mathcal{A}_\infty = \{a_1, a_2, \ldots\}$ and the upper alphabet $\mathcal{X}_\infty = \{x_1, x_2, \ldots\}$. The comparability relation is as in the case of finite alphabets. A *multiplicative function* on $W_{\infty\infty}$ is a function f defined on the intervals in $W_{\infty\infty}$ for which $f_{00} = 1$ and which has the property that if $[u, v] \simeq_{\mathbf{c}} \prod_{i,j} W_{ij}^{c_{ij}}$ then $f(u, v) = \prod_{ij} f_{ij}^{c_{ij}}$, where we write f_{ij} for the value of f on an interval canonically isomorphic to W_{ij} (see Remark 2.3).

Let f and g be two multiplicative functions on $W_{\infty\infty}$, and let

$$F \;=\; F(x, y) \;=\; \sum_{i,j \geq 0} f_{ij}\, x^i y^j$$

$$G \;=\; G(x, y) \;=\; \sum_{i,j \geq 0} g_{ij}\, x^i y^j$$

$$F * G \;=\; (F * G)(x, y) \;=\; \textstyle\sum_{i,j\geq 0} (f * g)_{ij}\, x^i y^j,$$

where $f * g$ denotes convolution in the incidence algebra $I(W_{\infty\infty})$. The main result of this section, Theorem 5.2, shows how to express $F * G$ in terms of F and G, and hence "determines" the monoid of multiplicative functions on $W_{\infty\infty}$.

We begin by establishing an expression for the number of elements $w \in W_{MN}$ of a given *type* $((a_{ij}), (b_{ij}))$, that is, such that $[\hat{0}, w] \simeq_{\mathbf{c}} \prod_{i,j} W_{ij}^{a_{ij}}$ and $[w, \hat{1}] \simeq_{\mathbf{c}} \prod_{i,j} W_{ij}^{b_{ij}}$. Note that the canonical isomorphism (Remark 2.3) implies that $1 + \sum i b_{ij} = \sum a_{ij}$ and $1 + \sum j a_{ij} = \sum b_{ij}$, and we can recover from the type of a word w the values $m{:}= \#(\{w\} \cap \mathcal{A}) = \sum_{i,j} i\,(a_{ij} + b_{ij})$ and $n{:}= \#(\{w\} \cap \mathcal{X}) = \sum_{i,j} j\,(a_{ij} + b_{ij})$. Also, the type of $w \in W_{MN}$ determines M and N, so the enumeration of shuffle words by type can be done in $W_{\infty\infty}$.

Proposition 5.1. *Let (a_{ij}) and (b_{ij}) be nonnegative integers such that*

$$\epsilon := \sum_{\substack{i,j\\ j\neq 0}} a_{ij} - \sum_{\substack{i,j\\ i\neq 0}} b_{ij} \;\in \{1, 0, -1\}.$$

Set

$$m = \sum_{i,j} i\,(a_{ij} + b_{ij}), \quad n = \sum_{i,j} j\,(a_{ij} + b_{ij}),$$

and $r = \sum_{\substack{i,j\\ j\neq 0}} a_{ij}$, *and* $s = \sum_{\substack{i,j\\ i\neq 0}} b_{ij}$.

Then the number of elements $w \in W_{\infty\infty}$ whose type is $((a_{ij}), (b_{ij}))$ is given by

$$\begin{cases} (2 - \epsilon^2)\dfrac{\binom{m+1}{\ldots, a_{ij}, \ldots}\binom{n+1}{\ldots, b_{ij}, \ldots}}{\binom{m+1}{r}\binom{n+1}{s}} & \text{if } w \neq \emptyset \text{ (i.e., } r + s > 0) \\ 1 & \text{if } w = \emptyset \text{ (i.e., } r = s = 0). \end{cases}$$

Proof. Each $w \in W_{\infty\infty} - \{\emptyset\}$ is of the form $ULUL\cdots UL$, or $LULU\cdots LU$, or $LU\cdots LUL$, or $UL\cdots ULU$, where each U is a nonempty factor whose letters are from the upper alphabet, and each L is a nonempty factor whose letters are from the lower alphabet. If the type of w is $((a_{ij}), (b_{ij}))$ then, for each $j \geq 1$, the number of U-factors of length j is $\sum_i a_{ij}$ and, for each $i \geq 1$, the number of L-factors of length i in w is equal to $\sum_j b_{ij}$. The alternation of nonempty L- and U-factors imposes the condition $\epsilon \in \{1, 0, -1\}$ appearing in the hypothesis. To construct a word w of the prescribed type, we begin by deciding the length of each U- and L-factor. The number of possibilities is the number of (multi)permutations of the nonzero lengths, so that U- and L-factors alternate:

$$\binom{r}{a_{01}, a_{02}, \ldots, a_{11}, a_{12}, \ldots}\binom{s}{b_{10}, b_{20}, \ldots, b_{11}, b_{21}, \ldots}(1 + \chi(\epsilon = 0)), \tag{15}$$

where, following A. Garsia [Ga] (see also [Kn]), if p is a proposition then we write $\chi(p) = 1$ if p is true, and $\chi(p) = 0$ if p is false. To complete the construction of w, we need to choose the location of the W_{i0}'s and W_{0j}'s required by entries a_{i0} and b_{0j} in the type of w. A factor W_{i0} in the canonical product for $[\hat{0}, w]$ must arise between two successive lower alphabet letters of w, or in front of the first L-factor if w begins with an L-factor, or after the last L-factor if w ends with an L-factor. Therefore such a factor W_{i0} occurs in one of $m - s + 1 - \epsilon$ positions. Similarly, a factor W_{0j} in the canonical product for $[w, \hat{1}]$ can arise from any of $n - r + 1 + \epsilon$ positions (between two successive letters from the upper alphabet, in front of the first U-factor if w begins with a U-factor, or after the last U-factor if w ends with a U-factor). In conclusion, the word w can be completed in

$$\binom{m - s + 1 - \epsilon}{a_{10}, a_{20}, \ldots, m - s + 1 - \epsilon - \sum_i a_{i0}}\binom{n - r + 1 + \epsilon}{b_{01}, b_{02}, \ldots, n - r + 1 + \epsilon - \sum_j b_{0j}}$$

ways.

The remainder of the proof is a calculation. After multiplying (15) by the preceding expression, the relations $r - s = \epsilon$, $1 + \sum ib_{ij} = \sum a_{ij}$, and $1 + \sum ja_{ij} = \sum b_{ij}$ allow some simplifications. For example, $m - s + 1 - \epsilon - \sum_i a_{i0} = m + 1 - r - \sum_i a_{i0} = \sum_i ib_{ij} + 1 - (r + \sum_i a_{i0}) = 0$. Similarly, $n - r + 1 + \epsilon - \sum_j b_{0j} = 0$.

The first case in the conclusion of the Proposition now follows from a simple manipulation with binomial coefficients. The case $w = \emptyset$ is trivial, so the proof is complete. $\square$

Theorem 5.2. *Let $F_0 = F(x, 0)$, $G_0 = G(0, y)$, and*

$$\begin{aligned} \tilde{F}(x, y) &= F(x, G_0 y) \\ \tilde{G}(x, y) &= G(F_0 x, y). \end{aligned}$$

Then

$$\frac{1}{F*G} = \frac{1}{\tilde{F}G_0} + \frac{1}{F_0\tilde{G}} - \frac{1}{F_0G_0}. \tag{16}$$

Proof. For fixed $r, s, m, n \geq 0$ write

$$Q_{r,s,m,n} = \sum \frac{\prod f_{ij}^{a_{ij}} g_{ij}^{b_{ij}} x^{\sum i(a_{ij}+b_{ij})} y^{\sum j(a_{ij}+b_{ij})}}{(\prod a_{ij}!)(\prod b_{ij}!)},$$

where the sum ranges over all a_{ij} and b_{ij} satisfying

$$\sum_{j\neq 0} a_{ij} = r, \quad \sum_{i\neq 0} b_{ij} = s$$

$$\sum i b_{ij} = m, \quad \sum j a_{ij} = n$$

$$m+1 = \sum a_{ij}, \quad n+1 = \sum b_{ij}.$$

By Proposition 5.1, the convolution $F*G$ is given by

$$F*G = (F*G)_{-1} + (2(F*G)_0 - F_0G_0) + (F*G)_1,$$

where

$$\begin{aligned}
(F*G)_{-1} &= \sum_{k,m,n} k!\,(k+1)!\,(m-k)!\,(n+1-k)!\,Q_{k+1,k,m,n} \\
(F*G)_0 &= \sum_{k,m,n} k!^2\,(m+1-k)!\,(n+1-k)!\,Q_{k,k,m,n} \\
(F*G)_1 &= \sum_{k,m,n} (k+1)!\,k!\,(m+1-k)!\,(n-k)!\,Q_{k,k+1,m,n}.
\end{aligned}$$

If M is a monomial then write $[M]Q$ for the coefficient of M in the power series Q. We first consider $(F*G)_0$. We have

$$\begin{aligned}
(F*G)_0 &= \sum_{k,m,n} k!^2\,(m+1-k)!\,(n+1-k)!\,[q^{m+1}r^{n+1}s^m t^n u^k v^k] \\
&\quad \cdot \sum_{a_{ij},b_{ij}} \frac{\prod_{j\neq 0}\left(q\,t^j u f_{ij} x^i y^j\right)^{a_{ij}} \prod_i \left(q\,f_{i0} x^i\right)^{a_{i0}} \prod_{i\neq 0}\left(r s^i v g_{ij} x^i y^j\right)^{b_{ij}} \prod_j \left(r g_{0j} y^j\right)^{b_{0j}}}{(\prod a_{ij}!)\,(\prod b_{ij}!)}
\end{aligned}$$

$$= \sum_{k,m,n} k!^2\,(m+1-k)!\,(n+1-k)![q^{m+1}r^{n+1}s^m t^n u^k v^k]$$

$$\cdot \prod_{i,j}\left(\sum_{a_{ij}\geq 0} \frac{\left(q\,t^j\,u^{\chi(j\neq 0)} f_{ij}\,x^i y^j\right)^{a_{ij}}}{a_{ij}!}\right)\cdot\left(\sum_{a_{ij}\geq 0}\frac{\left(rs^i v^{\chi(i\neq 0)} g_{ij} x^i y^j\right)^{b_{ij}}}{b_{ij}!}\right)$$

$$= \sum_{k,m,n} k!^2\,(m+1-k)!\,(n+1-k)!$$

$$\cdot[q^{m+1}r^{n+1}s^m t^n u^k v^k]\prod_{i,j}\exp\left(q\,t^j u^{\chi(j\neq 0)} f_{ij}\,x^i y^j + rs^i v^{\chi(i\neq 0)} g_{ij} x^i y^j\right)$$

$$= \sum_{k,m,n} k!^2\,(m+1-k)!\,(n+1-k)!$$

$$\cdot[q^{m+1}r^{n+1}s^m t^n u^k v^k]\exp\sum_{i,j}\left(q\,t^j u^{\chi(j\neq 0)} f_{ij}\,x^i y^j + r\,s^i v^{\chi(i\neq 0)} g_{ij}\,x^i y^j\right)$$

$$= \sum_{k,m,n} k!^2\,(m+1-k)!\,(n+1-k)!$$

$$\cdot[s^m t^n u^k v^k]\frac{\left(\displaystyle\sum_{i,j} t^j u^{\chi(j\neq 0)} f_{ij}\,x^i y^j\right)^{m+1}}{(m+1)!}\cdot\frac{\left(\displaystyle\sum_{i,j} s^i v^{\chi(i\neq 0)} g_{ij}\,x^i y^j\right)^{n+1}}{(n+1)!}$$

$$= \sum_{k,m,n}\frac{k!^2\,(m+1-k)!\,(n+1-k)!}{(m+1)!\,(n+1)!}[s^m t^n]\binom{m+1}{k}\left(\sum_i f_{i0}\,x^i\right)^{m+1-k}$$

$$\cdot\left(\sum_{\substack{i,j\\ j\neq 0}} f_{ij}\,t^j x^i y^j\right)^k\binom{n+1}{k}\left(\sum_i g_{0j}\,y^j\right)^{n+1-k}\left(\sum_{\substack{i,j\\ i\neq 0}} g_{ij}\,s^i x^i y^j\right)^k$$

$$= \sum_{k,m,n}[s^m t^n]\left(\sum_i f_{i0}\,x^i\right)^{m+1-k}\left(\sum_{\substack{i,j\\ j\neq 0}} f_{ij}\,t^j x^i y^j\right)^k\left(\sum_i g_{0j}\,y^j\right)^{n+1-k}\left(\sum_{\substack{i,j\\ i\neq 0}} g_{ij}\,s^i x^i y^j\right)^k$$

$$= \sum_k \left(\sum_i f_{i0}\, x^i\right)^{1-k} \left(\sum_j g_{0j}\, y^j\right)^{1-k}$$

$$\cdot \sum_{m,n} [s^m t^n] \left(\sum_{\substack{i,j\\ j\neq 0}} f_{ij}\, t^j x^i y^j\right)^k \left(\sum_{\substack{i,j\\ i\neq 0}} g_{ij}\, s^i x^i y^j\right)^k \cdot \left(\sum_i f_{i0}\, x^i\right)^m \left(\sum_j g_{0j}\, y^j\right)^n$$

$$= \sum_k \left(\sum_i f_{i0}\, x^i\right)^{1-k} \left(\sum_j g_{0j}\, y^j\right)^{1-k}$$

$$\cdot \left(\sum_{\substack{i,j\\ j\neq 0}} \left(\sum_r g_{0r}\, y^r\right)^j f_{ij}, x^i y^j\right)^k \left(\sum_{\substack{i,j\\ i\neq 0}} \left(\sum_s f_{s0}\, x^s\right)^i g_{ij}\, x^i y^j\right)^k .$$

Note that

$$\sum_r g_{0r}\, y^r = G_0$$

and

$$\sum_{\substack{i,j\\ j\neq 0}} z^j f_{ij}\, x^i y^j = F(x, yz) - F_0,$$

and similarly for F_0 and $G(xz, y) - G_0$. Hence

$$\begin{aligned}(F * G)_0 &= F_0 G_0 \cdot \sum_k F_0^{-k} G_0^{-k} \left(\sum_{\substack{i,j\\ j\neq 0}} G_0^j\, f_{ij}\, x^i y^k\right)^k \left(\sum_{\substack{i,j\\ i\neq 0}} F_0^i\, g_{ij}\, x^i y^k\right)^k \\ &= \frac{F_0 G_0}{1 - \frac{(\tilde{F} - F_0)(\tilde{G} - G_0)}{F_0 G_0}}.\end{aligned}$$

Exactly analogous reasoning applies to $(F * G)_{-1}$ and $(F * G)_1$. For instance, $(F * G)_{-1}$ can be written

$$\begin{aligned}(F * G)_{-1} &= \sum_{k,m,n} k!\,(k+1)!\,(m-k),(n+1-k)!\,[q^{m+1} r^{n+1} s^m t^n u^{k+1} v^k] \\ &\cdot \sum_{a_{ij}, b_{ij}} \frac{\prod_{j\neq 0} \left(q\, t^j u\, f_{ij}\, x^i y^j\right)^{a_{ij}} \prod_i \left(q\, f_{i0}\, x^i\right)^{a_{i0}} \prod_{i\neq 0} \left(r\, s^i v\, g_{ij}\, x^i y^j\right)^{b_{ij}} \prod_j \left(r\, g_{0j}\, y^j\right)^{b_{0j}}}{\left(\prod a_{ij}!\right)\left(\prod b_{ij}!\right)}.\end{aligned}$$

Reasoning as above yields

$$(F * G)_{-1} = \frac{(\tilde{F} - F_0)G_0}{1 - \frac{(\tilde{F}-F_0)(\tilde{G}-G_0)}{F_0 G_0}}.$$

Similarly (or by symmetry),

$$(F * G)_1 = \frac{F_0(\tilde{G} - G_0)}{1 - \frac{(\tilde{F}-F_0)(\tilde{G}-G_0)}{F_0 G_0}}.$$

From this we easily obtain

$$\frac{1}{F * G} = \frac{1}{(F * G)_{-1} + (2(F * G)_0 - F_0 G_0) + (F * G)_1} = \frac{1}{\tilde{F} G_0} + \frac{1}{F_0 \tilde{G}} - \frac{1}{F_0 G_0},$$

as desired. □

Example 5.3. In general it seems difficult to understand the operation $F * G$. It is not even obvious from (16) that $*$ is associative! A special case for which $F * G$ can be explicitly evaluated is the following. Let $a_1, b_1, \ldots, a_k, b_k$ be real numbers (or indeterminates). Then it is straightforward to prove from Theorem 5.2 by induction on k that

$$\frac{1}{(1 - a_1 x)(1 - b_1 y)} * \cdots * \frac{1}{(1 - a_k x)(1 - b_k y)} =$$

$$\frac{1}{1 - (\sum a_i)x - (\sum b_i)y + (\sum (a_1 + a_2 + \cdots + a_i)b_i)xy}. \tag{17}$$

For instance, if ζ denotes the zeta function of $W_{\infty\infty}$ (whose value is 1 on every interval of $W_{\infty\infty}$), then

$$\sum_{i,j} \zeta_{ij} x^i y^j = \frac{1}{(1 - x)(1 - y)}.$$

Hence the left-hand side of (17) becomes the generating function for ζ^k, whose value at W_{ij} is the number $Z_{ij}(k)$ of k-element multichains $\hat{0} = t_0 \leq t_1 \leq \cdots \leq t_k = \hat{1}$ in W_{ij}. Equivalently, $Z_{ij}(k)$ is the value of the zeta polynomial of W_{ij} at k [St1]. We obtain from (17) that

$$\sum_{i,j} Z_{ij}(k) x^i y^j = \frac{1}{1 - kx - ky - \binom{k+1}{2}xy},$$

a result of Greene [Gre1].

Remark 5.4. *The identity (17) can be deduced purely combinatorially.*

The left-hand side provides a refinement of the flag f-vector in the sense that the coefficient of $a_1^{i_1} b_1^{j_1} a_2^{i_2} b_2^{j_2} \cdots x^m y^n$, where $m = i_1 + i_2 + \cdots$ and $n = j_1 + j_2 + \cdots$, is the number of multichains $\hat{0} = t_0 \leq t_1 \leq \cdots \leq t_{m+n-1} < t_{m+n} = \hat{1}$ in W_{mn} such that t_{r-1} and t_r differ by i_r letters from the alphabet $\mathcal{A}$ and by j_r letters from the alphabet $\mathcal{X}$. The right-hand side of (17) can be interpreted as the generating function for the language $\mathcal{L}$ over the alphabet $\{a_1, a_2, \ldots, b_1, b_2, \ldots\}$ consisting of the words in which there is no occurrence of a letter a_k immediately followed by a letter b_l with $k \leq l$. (This follows directly through a simple sign-reversing-involution argument or from the general theory of Cartier and Foata [CaFo].) The number of such words formed with the multiset of letters $a_1^{i_1} b_1^{j_1} a_2^{i_2} b_2^{j_2} \cdots$, where $m = i_1 + i_2 + \cdots$ and $n = j_1 + j_2 + \cdots$, is the coefficient of $a_1^{i_1} b_1^{j_1} a_2^{i_2} b_2^{j_2} \cdots x^m y^n$ on the right-hand side of (17). For example, the coefficient of $a_1^2 b_1 b_2 x^2 y^2$ is 2, accounting for the words $b_1 b_2 a_1 a_1$ and $b_2 b_1 a_1 a_1$ in $\mathcal{L}$.

Now each word $w \in \mathcal{L}$ consisting of m a_i's and n b_i's determines a shuffle-word $s(w)$ in W_{mn} by placing the alphabet $\mathcal{A}$ in the positions of the a_i's and the alphabet $\mathcal{X}$ in the positions of the b_i's. For example, $w = a_2 b_3 b_3 a_1 a_3 b_5 b_1 b_2 a_3$ gives rise to the shuffle word $s(w) = a_1 x_1 x_2 a_2 a_3 x_3 x_4 x_5 a_4$ in W_{45}. Of course, $w \rightarrow s$ is not a one-to-one map.

Given a word $w \in \mathcal{L}$, we construct a unique multichain $t(w)$ in W_{mn} by starting with $t_0 = \hat{0}$ and obtaining t_{r+1} from t_r by removing the letters appearing in $s(w)$ in those positions where a_r occurs in w, and inserting the letters appearing in $s(w)$ in those positions where b_r occurs in $s(w)$. For instance, the word w from the preceding example yields $t(w) = (\hat{0} = t_0 < t_1 < t_2 < t_3 = t_4 < t_5 = \hat{1})$, where $t_0 = a_1 a_2 a_3 a_4$, $t_1 = a_1 a_3 x_4 a_4$, $t_2 = a_3 x_4 x_5 a_4$, $t_3 = t_4 = x_1 x_2 x_4 x_5$, $t_5 = x_1 x_2 x_3 x_4 x_5$.

Recall from Section 3 that there is a unique order in which the i_r deletions and j_r insertions can be performed such that t_{r+1} is reached from t_r via the Λ-increasing chain $\gamma(t_r, t_{r+1})$. The condition defining the words $w \in \mathcal{L}$ ensures that each insertion of a letter from $\mathcal{X}$ is made in the leftmost position permitted by the shuffle word $s(w)$, thus not multiply-counting chains which involve covering relations of the type (xa). From these observations it follows that $w \rightarrow t(w)$ is a bijection, completing a combinatorial proof of (17).

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