SOME ASPECTS OF (r, k)-PARKING FUNCTIONS

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ABSTRACT. An (r,k)-*parking function* of length n may be defined as a sequence $(a_1,\dots,a_n)$ of positive integers whose increasing rearrangement $b_1 \le \cdots \le b_n$ satisfies $b_i \le k+(i-1)r$. The case $r=k=1$ corresponds to ordinary parking functions. We develop numerous properties of (r,k)-parking functions. In particular, if $F_n^{(r,k)}$ denotes the Frobenius characteristic of the action of the symmetric group $\mathfrak{S}_n$ on the set of all (r,k)-parking functions of length n, then we find a combinatorial interpretation of the coefficients of the power series $\left(\sum_{n\ge 0} F_n^{(r,1)} t^n\right)^k$ for any $k \in \mathbb{Z}$. For instance, when $k>0$ this power series is just $\sum_{n\ge 0} F_n^{(r,k)} t^n$. We also give a q-analogue of this result. For fixed r, we can use the symmetric functions $F_n^{(r,1)}$ to define a multiplicative basis for the ring Λ of symmetric functions. We investigate some of the properties of this basis.

1. Introduction

Parking functions were first defined by Konheim and Weiss as follows. We have n cars $C_1,\dots,C_n$ and n parking spaces $1,2,\dots,n$. Each car C_i has a preferred space a_i. The cars go one at a time in order to their preferred space. If it is empty they park there; otherwise they park at the next available space (in increasing order). If all the cars are able to park, then the sequence $\alpha=(a_1,\dots,a_n)$ is called a *parking function* of length $\ell(\alpha)=n$. For instance, $(3,1,4,3)$ is not a parking function since the last car will go to space 3, but spaces 3 and 4 are already occupied. It is easy to see that $(a_1,\dots,a_n)\in[n]^n$ (where $[n]=\{1,2,\dots,n\}$) is a parking function if and only if its increasing rearrangement $b_1\le b_2\le\cdots\le b_n$ satisfies $b_i\le i$.

Let PF_n denote the set of all parking functions of length n. A fundamental result of Konheim and Weiss [2] (earlier proved in an equivalent form by Steck [7]—see Yan [8, §1.4] for a discussion) states that $\#\mathrm{PF}_n=(n+1)^{n-1}$. An elegant proof of this result was given by Pollak (reported in [3]), which we now sketch since it will be generalized later. Suppose that we have the same n cars, but now there are $n+1$ spaces $1,2,\dots,n+1$. The spaces are arranged on a circle. The cars follow the same algorithm as before, but once a car reaches space $n+1$ and is unable to park, it can continue around the circle to spaces $1,2,\dots$ until it can finally park. Of course all the cars can park this way, so at the end there will be one empty space. Note that their preferences $(a_1,\dots,a_n)\in[n+1]^n$ will be a parking function if and only if the empty space is $n+1$. If the empty space is e and the preferences are changed to $(a_1+i,\dots,a_n+i)$ for some i, where a_j+i is taken modulo $n+1$ so that $a_j+i\in[n+1]$, then the empty space becomes $e+i$. Hence given $(a_1,\dots,a_n)\in[n+1]^n$, exactly one of the vectors $(a_1+i,\dots,a_n+i)$ will be a parking function. It follows that $\#\mathrm{PF}_n=\frac{1}{n+1}(n+1)^n=(n+1)^{n-1}$.

Date: March 14, 2023.

The first author was partially supported by NSF grant DMS-1068625.

We will use notation and terminology on symmetric functions from [6, Chap. 7]. The symmetric group $\mathfrak{S}_n$ acts on PF_n by permuting coordinates. Let $F_n := \mathrm{ch}\,\mathrm{PF}_n$ denote the Frobenius characteristic of this action of $\mathfrak{S}_n$, as defined in [6, §7.18]. Hence F_n is a homogeneous symmetric function of degree n, called the *parking function symmetric function.* If $\alpha = (a_1, \dots, a_n)$ is a sequence of positive integers with m_i i's (so $\sum m_i = n$), then the Frobenius characteristic of the action of $\mathfrak{S}_n$ on the set of permutations of the terms of α is the complete symmetric function $h_{m_1}h_{m_2}\cdots$ (with $h_0 = 1$). Hence to compute F_n, take all vectors $(b_1, \dots, b_n) \in \mathrm{PF}_n$ with $b_1 \le b_2 \le \cdots \le b_n$ (the number of such vectors is the Catalan number C_n) and add the corresponding h_λ for each. For instance, when $n = 3$ the weakly increasing parking functions are 111, 112, 113, 122, 123, so $F_3 = h_3 + 3h_2h_1 + h_1^3$.

The symmetric function F_n has many remarkable properties, summarized (in a dual form, and with equation (1.2) below not included) in [6, Exer. 7.48(f)].

Proposition 1.1. *We have*

$$\begin{aligned}
F_n &= \sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)-1} z_\lambda^{-1} p_\lambda \\
&= \frac{1}{n+1} \sum_{\lambda \vdash n} s_\lambda(1^{n+1}) s_\lambda \\
&= \frac{1}{n+1} \sum_{\lambda \vdash n} \left[\prod_i \binom{\lambda_i + n}{\lambda_i} \right] m_\lambda \\
&= \sum_{\lambda \vdash n} \frac{n(n-1)\cdots(n-\ell(\lambda)+2)}{d_1(\lambda)!\cdots d_n(\lambda)!} h_\lambda \qquad (1.1)\\
&= \sum_{\lambda \vdash n} \varepsilon_\lambda \frac{(n+2)(n+3)\cdots(n+\ell(\lambda))}{d_1(\lambda)!\cdots d_n(\lambda)!} e_\lambda \qquad (1.2)\\
\omega F_n &= \frac{1}{n+1} \left[\prod_i \binom{n+1}{\lambda_i} \right] m_\lambda,
\end{aligned}$$

where $d_i(\lambda)$ denotes the number of parts of λ equal to i and $\varepsilon_\lambda = (-1)^{n-\ell(\lambda)}$. Moreover,

$$F_n = \frac{1}{n+1}[t^n] H(t)^{n+1}, \qquad (1.3)$$

where $[t^n]f(t)$ denotes the coefficient of t^n in the power series $f(t)$, and

$$H(t) = \sum_{n \ge 0} h_n t^n = \frac{1}{(1-x_1t)(1-x_2t)\cdots}.$$

Note in particular that the coefficient of h_λ in equation (1.3) is the number of weakly increasing parking functions of length n whose entries occur with multiplicities $\lambda_1, \lambda_2, \dots$.

A further important property of F_n, an immediate consequence of equation (1.3) and the Lagrange inversion formula, is the following. Let

$$E(t) = \sum_{n \ge 0} e_n t^n = \prod_i (1 + x_i t), \qquad (1.4)$$

and let $G(t)^{\langle -1\rangle}$ denote the compositional inverse of the power series $G(t)$ (which will exist as a formal power series if $G(t) = a_1 t + a_2 t^2 + \cdots$, where $a_1 \neq 0$). Then

$$\sum_{n\geq 1} F_n t^n = (tE(-t))^{\langle -1\rangle}. \tag{1.5}$$

There are several known generalizations of parking functions. In particular, if $\boldsymbol{u} = (u_1, \ldots, u_n)$ is a weakly increasing sequence of positive integers, then a ***u**-parking function* is a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ (where $\mathbb{P} = \{1, 2, \ldots\}$) such that its increasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies $b_i \leq u_i$. Thus an ordinary parking function corresponds to $\boldsymbol{u} = (1, 2, \ldots, n)$. For the general theory of $\boldsymbol{u}$-parking functions, see the survey [8, §13.4]. We will be interested here in the special case $\boldsymbol{u} = (k, r+k, 2r+k, \ldots, (n-1)r+k)$, where $r, k \geq 1$. We call such a $\boldsymbol{u}$-parking function an *(r,k)-parking function.* With this terminology, an ordinary parking function is a $(1,1)$-parking function. We call an $(r,1)$-parking function simply an *r-parking function.*

NOTE. Our terminology is not universally used. For instance, if $(a_1, \ldots, a_n)$ is what we call an (r,r)-parking function, then Bergeron [1] would call $(a_1 - 1, \ldots, a_n - 1)$ an r-parking function.

Pollak's proof that $\#\mathrm{PF}_n = (n+1)^{n-1}$ extends easily to (r,k)-parking functions. Namely, we now have rn cars $C_1, \ldots, C_{rn}$ and $rn+k-1$ spaces $1, 2, \ldots, rn+k-1$. We consider preferences $\alpha = (a_1, \ldots, a_n)$, $1 \leq a_i \leq rn+k-1$, where cars $C_{r(i-1)+1}, \ldots, C_{ri}$ all prefer a_i. The cars use the same parking algorithm as before. It is not hard to check that all the cars can park if and only if α is an (r,k)-parking function. Now arrange $rn+k$ spaces on a circle, allow the preferences $1 \leq a_i \leq rn+k$, and park as in Pollak's proof. Then α is an (r,k)-parking function if and only if the space $rn+k$ is empty. Reasoning as in Pollak's proof gives the following result, which in an equivalent form is due to Steck [7].

Theorem 1.2. *Let $\mathrm{PF}_n^{(r,k)}$ denote the set of (r,k)-parking functions of length n. Then*

$$\#\mathrm{PF}_n^{(r,k)} = k(rn+k)^{n-1}.$$

The results in Proposition 1.1 can be extended to (r,k)-parking functions (Theorem 2.1). Most of them appear in Bergeron [1, Prop. 1] for the case $k = r$. (Bergeron and his collaborators have gone on to generalize their results in a series of papers on rectangular parking functions.) One of our key results (Theorem 3.1) connects r-parking functions to (r,k)-parking functions as follows.

Let $\mathrm{PF}_n^{(r,k)}$ denote the set of all (r,k)-parking functions of length n, and let $F_n^{(r,k)}$ denote the Frobenius characteristic $\mathrm{ch}\,\mathrm{PF}_n^{(r)}$ of the action of $\mathfrak{S}_n$ on $\mathrm{PF}_n^{(r,k)}$ by permuting coordinates. Define

$$\begin{aligned} \mathcal{P}^{(r,k)}(t) &= \sum_{n\geq 0} F_n^{(r,k)} t^n \\ \mathcal{P}^{(r)}(t) &= \mathcal{P}^{(r,1)}(t), \end{aligned}$$

Then (Theorem 3.1)

$$\mathcal{P}^{(r)}(t)^k = \mathcal{P}^{(r,k)}(t). \tag{1.6}$$

Equation (1.6) suggests looking at $\mathcal{P}^{(r)}(t)^k$ for negative integers k. We obtain parking function interpretations of the coefficients of such power series in Section 4. As some motivation for what to expect, consider two power series $A(t), B(t)$, with $B(0) = 0$, that are related by

$$A(t) = \frac{1}{1 - B(t)} = 1 + B(t) + B(t)^2 + \cdots .$$

Thus

$$B(t) = 1 - \frac{1}{A(t)}, \tag{1.7}$$

and often $B(t)$ will be a generating function for certain "prime" objects, while $A(t)$ will be a generating function for all objects, i.e., products of primes. See for instance [5, Prop. 4.7.11]. We will see examples of this relationship with our generating functions for parking functions.

For instance, if we set

$$\mathcal{P}^{(r,k)}(t)^{-1} = 1 - \sum_{n \geq 1} G_n^{(r,k)} t^n, \tag{1.8}$$

then $G_n^{(1,1)}$ is the Frobenius characteristic of the action of $\mathfrak{S}_n$ on *prime* parking functions of length n, i.e., parking functions that remain parking functions when some term equal to 1 is deleted (a concept due to Gessel [6, Exer. 5.49(f)]). An increasing parking function $b_1 b_2 \cdots b_n$ can be uniquely factored $\beta_1 \cdots \beta_k$, such that (1) if b_j is the first term of β_i then $b_j = j$, and (2) if we subtract from each term of β_i one less than its first element (so it now begins with a 1), then we obtain a prime parking function.

As a direct generalization of the previous example, $G_n^{(r,1)}$ is the Frobenius characteristic of the action of $\mathfrak{S}_n$ on sequences $a_1 a_2 \cdots a_n$ such that some $a_i = 1$, and if remove this term then we obtain an (r, r)-parking function. More generally, if $1 \leq k \leq r$ then $G_n^{(r,k)}$ is the Frobenius characteristic of the action of $\mathfrak{S}_n$ on sequences $a_1 a_2 \cdots a_n$ such that we can remove some term less than $k+1$ and obtain an (r, r) parking function (Theorem 4.3). For instance, when $r = 2$ and $n = 3$ the increasing sequences with this property are 111, 112, 113, 114, 122, 123, 124, 222, 223, 224. Hence $G_3^{(2,2)} = 2h_1^3 + 6h_2h_1 + 2h_3$. The situation for $\mathcal{P}^{(r,k)}(t)^{-j}$ when $j > r$ is more complicated (Theorem 4.1).

2. Expansions of $F_n^{(r,k)}$

In this section we consider the expansion of $F_n^{(r,k)}$ into the six classical bases for symmetric functions. These expresssions are defined even when k is an indeterminate, so we can use any of them to define $F_n^{(r,k)}$ in this situation. For later combinatorial applications we will only consider the case when k is an integer. We use notation from [6, Ch. 7] regarding symmetric functions. We also use multinomial coefficient notation such as

$$\binom{k}{d_1, \ldots, d_n, k - \sum d_i} = \frac{k(k-1)\cdots(k - \sum d_i + 1)}{d_1! \cdots d_n!},$$

where $d_1, \ldots, d_n$ are nonnegative integers and k may be an indeterminate. As usual we abbreviate $\binom{k}{d, k-d}$ as $\binom{k}{d}$.

Theorem 2.1. *Recall that $d_i(\lambda)$ denotes the number of parts of λ equal to i. Then $F_0^{(r,k)} = 1$, and for $n \geq 1$ we have*

$$F_n^{(r,k)} = \frac{k}{rn+k} \sum_{\lambda \vdash n} \binom{rn+k}{d_1(\lambda), \dots, d_n(\lambda), rn+k-\ell(\lambda)} h_\lambda \tag{2.1}$$

$$= \frac{k}{rn+k} \sum_{\lambda \vdash n} \varepsilon_\lambda \binom{rn+k+\ell(\lambda)-1}{d_1(\lambda), \dots, d_n(\lambda), rn+k-1} e_\lambda \tag{2.2}$$

$$= \frac{k}{rn+k} \sum_{\lambda \vdash n} \left[\prod_i \binom{\lambda_i + rn + k - 1}{\lambda_i} \right] m_\lambda$$

$$= \frac{k}{rn+k} \sum_{\lambda \vdash n} s_\lambda(1^{rn+k}) s_\lambda$$

$$= k \sum_{\lambda \vdash n} z_\lambda^{-1} (rn+k)^{\ell(\lambda)-1} p_\lambda \tag{2.3}$$

$$\omega F_n^{(r,k)} = \frac{k}{rn+k} \sum_{\lambda \vdash n} \left[\prod_i \binom{rn+k}{\lambda_i} \right] m_\lambda,$$

Moreover,

$$F_n^{(r,k)} = \frac{k}{rn+k} [t^n] H(t)^{rn+k}. \tag{2.4}$$

Proof. Define two elements α and β of $[rn+k]^n$ to be *equivalent* if their difference is a multiple of $(1, 1, \dots, 1)$ mod $rn+k$. This defines an equivalence relation on $[rn+k]^n$, and each equivalence class contains $rn+k$ elements. It follows from the proof of Theorem 1.2 that each equivalence class contains exactly k (r,k)-parking functions. Moreover, all the elements α in each equivalence class have the same multiset of part multiplicities, i.e., the multiset $\{d_1, \dots, d_{rn+k}\}$, where d_i is the number of i's in α.

For $n \geq 1$ let $D_n^{(r,k)}$ denote the Frobenius characteristic of the action of $\mathfrak{S}_n$ on $[rn+k]^n$ by permuting coordinates. It follows that

$$F_n^{(r,k)} = \frac{k}{rn+k} D_n^{(r,k)}.$$

Hence if we set $q = 1, k = n$, and $n = rn + k$ in Exercise 7.75(a) of [6] then we get

$$D_n^{(r,k)} = \sum_{\lambda \vdash n} s_\lambda(1^{rn+k}) s_\lambda.$$

(Exercise 7.75 deals with $\mathfrak{S}_k$ acting on submultisets M of $\{1^n, \dots, k^n\}$. Replace M with the vector $(d_1, \dots, d_k\}$, where d_i is the multiplicity of i in M, to get our formulation.) Therefore

$$F_n^{(r,k)} = \frac{k}{rn+k} \sum_{\lambda \vdash n} s_\lambda(1^{rn+k}) s_\lambda.$$

The remainder of the proof is routine symmetric function manipulation. □

A further important property of $F_n^{(r,k)}$ in the case $k = r$, an immediate consequence of equation (2.4) and the Lagrange inversion formula [6, Thm. 5.4.2], is the following.

Let $E(t)$ be given by equation (1.4). Then

$$\sum_{n\geq 0} F_n^{(r,r)} t^{n+1} = (tE(-t)^r)^{\langle -1\rangle} \tag{2.5}$$

3. A relation between r-parking functions and (r,k)-parking functions

In this section we give a combinatorial proof of the following result.

Theorem 3.1. *Let $k, r \in \mathbb{P}$. Then $\mathcal{P}^{(r)}(t)^k = \mathcal{P}^{(r,k)}(t)$.*

Proof. We need to give a bijection $\psi\colon (\mathrm{PF}_n^{(r,1)})^k \to \mathrm{PF}_n^{(r,k)}$ such that if $\psi(\alpha_1,\dots,\alpha_k) = \beta$, then $\ell(\alpha_1)+\cdots+\ell(\alpha_k) = \ell(\beta)$. Note that we consider the empty sequence $\emptyset$ to be an (r,j)-parking function for any r and j.

Given $(\alpha_1,\dots,\alpha_k) \in (\mathrm{PF}_n^{(r,1)})^k$, define α_i' to be the sequence obtained by adding $r(\ell(\alpha_1)+\cdots+\ell(\alpha_{i-1}))+i-1$ to every term of α_i. For instance, if $r=2$ and

$$(\alpha_1,\dots,\alpha_5) = ((1,2),\emptyset,\emptyset,(1),(1,3,4)),$$

then $\alpha_1' = (1,2)$, $\alpha_2' = \alpha_3' = \emptyset$, $\alpha_4' = (8)$, and $\alpha_5' = (11,13,14)$.

It is easily seen that ψ is the desired bijection. In particular, the inverse ψ^{-1} has the following description. Given $\beta = (b_1,\dots,b_n) \in \mathrm{PF}_n^{(r,k)}$, let $c_i = b_i - ri + r - 1$. (The term $r-1$ could be replaced by any constant independent from i; we made the choice so $c_1 = 0$.) Let $c_{j_1} < \cdots < c_{j_r}$ be the left-to-right maxima of the sequence $c_1,\dots,c_n$, so $j_1 = 1$. Factor β (regarded as a word $b_1\cdots b_n$) as $\beta_1\cdots\beta_r$, where β_i begins with b_{j_i}. Subtract a constant t_i from each term of β_i so that we obtain a sequence (or word) β_i' beginning with a 1. Insert $c_{j_{i+1}} - c_{j_i} - 1$ empty words $\emptyset$ between β_i' and β_{i+1}', and place empty words at the end so that there are k words in all. These words $\alpha_1,\dots,\alpha_k$ then satisfy $\psi^{-1}(\beta) = (\alpha_1,\dots,\alpha_k)$. □

Example 3.2. Suppose that $r=2, k=7$, and

$$\beta = (1,2,2,10,12,14,15,19,22).$$

Then $(c_1,\dots,c_9) = (0,-1,-3,3,3,3,3,4,5)$. The left-to-right maxima are $c_1 = 0$, $c_4 = 3$, $c_8 = 4$, $c_9 = 5$. Thus $\beta_1 = (1,2,2)$, $\beta_2 = (10,12,14,15)$, $\beta_3 = (19)$, and $\beta_4 = (22)$. Hence $\beta_1' = (1,2,2)$, $\beta_2' = (1,3,5,6)$, $\beta_3' = \beta_4' = (1)$. Between β_1' and β_2' insert $c_4 - c_1 - 1 = 2$ copies of $\emptyset$. Similarly since $c_8 - c_4 - 1 = c_9 - c_8 - 1 = 0$ we insert no further copies of $\emptyset$ between remaining β_i''s. We now have the six words $\beta_1',\emptyset,\emptyset,\beta_2',\beta_3',\beta_4'$, Since $k=7$ we insert one $\emptyset$ at the end, finally obtaining

$$\psi^{-1}(\beta) = ((1,2,2),\emptyset,\emptyset,(1,3,5,6),(1),(1),\emptyset).$$

Theorem 3.1 has a natural q-analogue. We simply state the relevant result since the bijection in the proof of Theorem 3.1 is compatible with our q-analogue, so the proof carries over. More specifically, using the notation of equation (3.1) below it is easy to check that if $\beta \in \mathrm{PF}_n^{(r,k)}$ and $\psi^{-1}(\beta) = (\alpha_1,\dots,\alpha_k)$, then

$$s^{(r,k)}(\beta) = \sum_{j=1}^{k} (s^{(r,1)}(\alpha_j) + (k-j)\ell(\alpha_j)).$$

Given an (r,k)-parking function $\alpha = (a_1, \ldots, a_n)$ of length n, note that the largest possible value of $\sum a_i$ is $k + (k+r) + \cdots + (k+(n-1)r) = kn + \binom{n}{2}r$. Define

$$s^{(r,k)}(\alpha) = kn + \binom{n}{2}r - \sum_{i=1}^{n} a_i. \tag{3.1}$$

When $k = r$ this is a well-known statistic on parking functions, sometimes used in the variant form $\sum a_i$. See for instance [4][8, §§1.2.2,1.3.3]. Note that the action of $\mathfrak{S}_n$ on (r,k)-parking functions α of length n is compatible with this statistic, i.e., if $w \in \mathfrak{S}_n$ then $s^{(r,k)}(w \cdot \alpha) = w \cdot s^{(r,k)}(\alpha)$.

Given a sequence $\beta = (b_1, \ldots, b_n) \in \mathbb{P}^n$, let U_β denote the Frobenius characteristic of the action by permuting coordinates of $\mathfrak{S}_n$ on all permutations of the terms of β. Hence if m_i is the number of i's in β then $U_\beta = h_{m_1} h_{m_2} \cdots$. Given $r, k, n \geq 1$, define

$$F_n^{(r,k)}(q) = \sum_{\beta} q^{s^{(r,k)}(\beta)} U_\beta,$$

where β runs over all increasing (r,k)-parking functions of length n. Write

$$\begin{aligned} \mathcal{P}^{(r,k)}(q,t) &= \sum_{n \geq 0} F_n^{(r,k)}(q) t^n \\ \mathcal{P}^{(r)}(q,t) &= \mathcal{P}^{(r,1)}(q,t). \end{aligned}$$

Thus $\mathcal{P}^{(r,k)}(1,t) = \mathcal{P}^{(r,k)}(t)$.

Theorem 3.3. *We have*

$$\mathcal{P}^{(r,k)}(q,t) = \prod_{i=0}^{k-1} \mathcal{P}^{(r)}(q, q^i t).$$

Equation (1.7) gives a relationship between a generating function $A(t)$ for all objects and $B(t)$ for prime objects. There is another basic relationship of this nature between exponential generating funcions $A(t)$ for all objects and $B(t)$ for "connected" objects, namely, the *exponential formula* $A(t) = \exp B(t)$ or $B(t) = \log A(t)$. See [6, §5.1]. Thus we can ask whether there is a combinatorial interpretation of the coefficients of $\log \mathcal{P}^{(r,k)}(t)$. Recall that $D_n^{(r,k)}$ denotes the Frobenius characteristic of the action of $\mathfrak{S}_n$ on $[rn+k]^n$ by permuting coordinates, as in the proof of Theorem 2.1. The case $k = r$ is handled by the following result.

Proposition 3.4. *We have*

$$\log \mathcal{P}^{(r,r)}(t) = \sum_{n \geq 1} D_n^{(r,r)} \frac{t^n}{n}.$$

Proof. The proof is a simple consequence of the following variant of the Lagrange inversion formula appearing in [6, Exer. 5.56]: for any power series $F(t) = a_1 t + a_2 t^2 + \cdots \in \mathbb{C}[[t]]$ with $a_1 \neq 0$ we have

$$n[t^n] \log \frac{F^{\langle -1 \rangle}(t)}{t} = [t^n] \left(\frac{t}{F(t)} \right)^n. \tag{3.2}$$

Choose $F(t) = tE(-t)^r$, where $E(t)$ is given by equation (1.4). Now

$$\frac{1}{E(-t)} = H(t) = \sum_{n\geq 0} h_n t^n.$$

Hence by equation (2.5), we see that equation (3.2) becomes

$$n[t^n] \log \mathcal{P}^{(r,r)}(t) = [t^n] H(t)^{nr}.$$

It is clear that $[t^n]H(t)^{nr} = D_n^{(r,r)}$, so the proof follows. □

4. A dual to (r,k)-parking functions

Equation (1.6) suggests looking at $\mathcal{P}^{(r)}(t)^k$ for negative integers k. We obtain an object "dual" (in the sense of combinatorial reciprocity) to (r,k)-parking functions.

We define $F_n^{(r,k)}$ for $k \leq 0$ by (2.1) (therefore all the equations in Theorem 2.1 hold for $k \leq 0$). It follows from the definition of $\mathcal{P}^{(r,k)}(t)$ and equation (1.6) that

$$\mathcal{P}^{(r)}(t)^k = \mathcal{P}^{(r,k)}(t) = \sum_{n\geq 0} F_n^{(r,k)} t^n$$

holds for all $k > 0$. Thus it also holds for all $k \leq 0$. Comparing the coefficients of t^n with those in equation (1.8), namely,

$$\mathcal{P}^{(r)}(t)^{-k} = 1 - \sum_{n\geq 1} G_n^{(r,k)} t^n, \quad \text{for all } \ k \geq 0,$$

and combining with (2.1), we see that

$$G_n^{(r,k)} = -F_n^{(r,-k)} = \frac{k}{rn-k} \sum_{\lambda \vdash n} \binom{rn-k}{d_1(\lambda), \ldots, d_n(\lambda)} h_\lambda, \quad \text{for all } k \geq 0, \ n \geq 1. \tag{4.1}$$

We then have the following combinatorial interpretation of $G_n^{(r,k)}$.

Theorem 4.1. *If $rn - k > 0$, then $G_n^{(r,k)}$ is the Frobenius characteristic of the action of $\mathfrak{S}_n$ on the set S of n-tuples whose increasing rearrangements have the following form:*

$$\big(\underbrace{w, \ldots, w}_{q(w)\ w\text{'s}}, b_{q(w)+1}, b_{q(w)+2}, \ldots, b_n\big), \tag{4.2}$$

where $w \in [k]$ and $q(w)$ is the smallest integer such that $w \leq q(w)r$, and

$$b_j \leq \min\{(j-1)r, w-1+rn-k\} \quad \textit{for} \quad j = q(w)+1, q(w)+2, \ldots, n. \tag{4.3}$$

Note that $w \leq \min\{(j-1)r, w-1+rn-k\}$ for all $j \geq q(w)+1$; therefore (4.3) is equivalent to

$$b_j \leq \min\{(j-1)r, w-1+rn-k\} \quad \textit{whenever} \quad b_j > w. \tag{4.4}$$

In other words, a weakly increasing integer sequence b is in S if and only it satisfies the following properties.

- **I.** *$b_1 = w$ for some $w \in [k]$, and $b_n - b_1 < rn - k$.*
- **II.** *$b_{q(w)} = w$.*
- **III.** *$b_j \leq (j-1)r$ for all $j \in [n]$ whenever $b_j > w$.*

Example 4.2. Let $r = 1$, $k = 2$, and $n = 5$. The coefficient of t^5 in $-\mathcal{P}^{(1)}(t)^{-2}$ is

$$2h_3h_1^2 + 2h_2^2h_1 + 4h_3h_2 + 4h_4h_1 + 2h_5.$$

This symmetric function is the Frobenius characteristic of the action of $\mathfrak{S}_5$ on all sequences $(a_1, \dots, a_5) \in \mathbb{P}^5$ whose increasing rearrangement $b_1 \geq \cdots \geq b_5$ satisfies either of the conditions (1) $b_1 = 1$, $b_2 \leq 1$ (so in fact $b_2 = 1$), $b_3 \leq 2$, $b_4 \leq 3$, $b_5 \leq 3$, or (2) $b_1 = b_2 = 2$, $b_3 \leq 2$ (so in fact $b_3 = 2$), $b_4 \leq 3$, $b_5 \leq 4$. We get the fourteen increasing sequences (orbit representatives) 11111, 11112, 11113, 11122, 11123, 11133, 11222, 11233, 11223, 22222, 22223, 22224, 22233, 22234.

A special case. When $k \in \{1, \dots, r\}$, for all $w \in [k]$ we have $q(w) = 1$ and $(n-1)r \leq rn - k \leq w - 1 + rn - k$. Therefore (4.4) becomes $b_j \leq (j-1)r$ for all $j > 1$, so b having the form (4.2) is equivalent to $b_1 \in [k]$ and $(b_2, b_3, \dots, b_n)$ is a weakly increasing (r, r)-parking functions of length $n - 1$. Thus Theorem 4.1 becomes the following result.

Theorem 4.3. *If $k \in \{1, \dots, r\}$, then $G_n^{(r,k)}$ is the Frobenius characteristic of the action of $\mathfrak{S}_n$ on the distinct n-tuples we get by adjoining $1, 2, \dots,$ or k to (r, r)-parking functions of length $n-1$; or equivalently, the n-tuples whose increasing rearrangements start with $1, 2, \dots,$ or k and followed by weakly increasing (r, r)-parking functions of length $n - 1$.*

Theorem 4.1 is a consequence of the following result, which will be proved right below Proposition 4.6.

Proposition 4.4. *Suppose that $rn - k > 0$. Given $a = (a_1, \dots, a_n) \in [rn - k]^n$, let $p \in [rn - k]$ be the smallest positive integer i such that the increasing rearrangements of a and $(a + p) \bmod rn - k$ coincide, where $a + i := (a_1 + i, \dots, a_n + i)$ and $a_j + i \bmod rn - k$ is the $a_j + i$ taken modulo $rn - k$ so that $a_j + i \in [rn - k]$; equivalently, $p = \#R_a$, where R_a is the set of increasing rearrangements of vectors $a + i \bmod rn - k$ ($i \in \mathbb{Z}$).*

Then the number of increasing vectors $b \in S$ such that the increasing rearrangement of ($b \bmod rn - k$) is in R_a is $\frac{pk}{rn-k}$.

Theorem 4.1 follows as each $b \in S$ corresponds to a unique set R_a (the vector a may not be unique).

Remark 4.5. The reason why we need the vector $b \bmod rn - k$ is that we may have $b \in S \backslash [rn - k]^n$ and $b \bmod rn - k \in S$. For instance, when $r = 2, n = 4, k = 3$, $rn - k = 5$, we have $(6, 2, 2, 4) \in S \backslash [rn - k]^n$ and $(1, 2, 2, 4) \in S$.

A special case. When $k \in \{1, \dots, r\}$, it follows from (4.3) that $b_n \leq (n-1)r \leq rn - k$ for all $b \in S$; therefore $b \bmod rn - k = b$. In other words, we only need to consider b instead of $b \bmod rn - k$. Thus, combined with Theorem 4.1, Proposition 4.4 becomes as follows.

Proposition 4.6. *If $k \in \{1, \dots, r\}$, then for any given $(a_1, \dots, a_n) \in [rn - k]^n$, there are exactly k i's ($\bmod rn - k$) such that the vector $(a_1 + i, \dots, a_n + i) \bmod rn - k$ is an (r, r)-parking function of length $n - 1$ adjoined by $1, 2, \dots,$ or k, where $a_j + i \bmod rn - k$ is the $a_j + i$ taken modulo $rn - k$ so that $a_j + i \in [rn - k]$.*

Proof of Proposition 4.4. The case $k = 0$ is trivial. Assume that $k \geq 1$. It suffices to prove the proposition for a weakly increasing sequence $a = (a_1, \dots, a_n)$ with $a_1 = 1$. For convenience, let $N := rn - k > 0$ and denote the increasing rearrangement of a sequence x by $x_\uparrow$.

We have two cases: $p < N$ and $p = N$.

Case 1. $p < N$.

Then a has the form:

$$(4.5)\qquad \big(\underbrace{1,\ldots,1}_{d\ 1\text{'s}},\underbrace{1+p,\ldots,1+p}_{d\ (1+p)\text{'s}},\underbrace{1+2p,\ldots,1+2p}_{d\ (1+2p)\text{'s}},\ldots,\underbrace{1+(\ell-1)p,\ldots,1+(\ell-1)p}_{d\ (1+(\ell-1)p)\text{'s}}\big),$$

where $d, \ell \in \mathbb{P}$ with $\ell > 1$ such that $\ell d = n$ and $\ell p = N$. Thus $k = rn - N = (rd - p)\ell$.

The following fact can be verified immediately from the definition of S and R_a.

Lemma 4.7. *If $b \in S$, then $b+i \in S$ for all $i \in \{0, -1, \ldots, -b_1+1\}$. Further, if $(b \bmod N)_\uparrow \in R_a$, then $(b + i \bmod N)_\uparrow \in R_a$.*

In particular, when $i = -b_1 + 1$, the smallest coordinate of $b + i$ is 1. According to (4.4), we have $b + i \in [N]^n$, and therefore $b + i \bmod N = b + i$. If $(b \bmod N)_\uparrow \in R_a$, then $(b + i)_\uparrow = (b + i \bmod N)_\uparrow \in R_a$.

We also need the following lemma.

Lemma 4.8. *We have $a + i \in S$ if and only if $i \in \{0, 1, \ldots, rd - p - 1\}$.*

On the strength of Lemmas 4.7 and 4.8 and the fact that $R_a = \{a + i \colon 0 \le i \le p - 1\}$, the number of vectors $b \in S$ such that $(b \bmod rn - k)_\uparrow \in R_a$ is $rd - p = \frac{pk}{N}$, as desired.

Proof of Lemma 4.8. If $a + i \in S$ with the form (4.2), then applying (4.3) to $a + i$ and $d + 1$ yields $1 + p + i \le rd$, and therefore $i \le rd - p - 1$.

On the other hand, for any $i \in \{0, 1, \ldots, rd - p - 1\}$, we have $a + i \in S$. In fact, the vector

$$a+i = \big(\underbrace{w,\ldots,w}_{d\ w\text{'s}},\underbrace{w+p,\ldots,w+p}_{d\ (w+p)\text{'s}},\underbrace{w+2p,\ldots,w+2p}_{d\ (w+2p)\text{'s}},\ldots,\underbrace{w+(\ell-1)p,\ldots,w+(\ell-1)p}_{d\ (w+(\ell-1)p)\text{'s}}\big),$$

where $w = 1 + i \le rd - p \le (rd - p)\ell = k$. Property I then follows from $(a + i)_n - (a + i)_1 = (\ell - 1)p < \ell p = N$. Property II holds since $q(\cdot)$ is weakly increasing and $q(w) \le q(rd) = d$. Finally, Property III is satisfied because $(a+i)_{jd+1} = \cdots = (a+i)_{(j+1)d} = w+jp \le j(w+p) \le j(rd - p + p) = (jd)r$ for all $j \in [\ell - 1]$. □

Case 2. $p = N$.

Namely, the vectors $(a + i \bmod N)_\uparrow$, $i \in [N]$ are distinct. We will determine explicitly the $\frac{pk}{rn-k} = k$ vectors in S desired in Proposition 4.4.

For convenience, we denote $x_j = a_{j+1}\,(\le N)$, $j = 0, \ldots, n - 1$, and consider the weakly increasing sequence $x = (x_0, \ldots, x_{n-1})$ with $x_0 = 1$. Then $x \in S$ if and only if $x_j \le rj$ for all $j \in [n - 1]$. In general, a weakly increasing integer sequence y is in S if and only if

I′. $y_0 = w$ for some $w \in [k]$, and $y_{n-1} - y_0 < N$.
II′. $y_{q(w)-1} = w$.
III′. $y_j \le jr$ for all $j \in [n - 1]$ whenever $y_j > w$.

In the rest of the proof, all variables are integers, and for a vector y, we denote by y_j its $(j + 1)$-th coordinate.

Let $\Delta_j := rj - x_j$, $j = 0, 1, \ldots, n - 1$. Then $\Delta_0 = -1$, and $x \in S$ if and only if $\Delta_j \ge 0$ for all $j \in [n - 1]$.

Lemma 4.9. *There exists $i \in \mathbb{Z}$ such that the vector $(x+i \bmod N)_\uparrow \in S$, with the smallest coordinate equal to* 1. *More precisely, if $x \in S$, then we can take $i = 0$; otherwise, take $i = 1 - x_j$, where j is the largest number in $[n-1]$ such that $\Delta_j = \min_{j' \in [n-1]} \Delta_{j'}$.*

Proof. Assume that $x \notin S$, then $\Delta_j \le -1$ and $j \in [n-1]$ for the j taken in the lemma. Taking $i = 1 - x_j$, we get

$$\begin{aligned} & x + i \bmod N \\ =& \big(2 - x_j + N, x_1 - x_j + 1 + N, \ldots, x_{j-1} - x_j + 1 + N, 1, x_{j+1} - x_j + 1, \ldots, x_{n-1} - x_j + 1\big), \end{aligned}$$

and thus

$$\begin{aligned} & \alpha := (x + i \bmod N)_\uparrow \\ =& \big(1, \underbrace{x_{j+1} - x_j + 1}_{\alpha_1}, \ldots, \underbrace{x_{n-1} - x_j + 1}_{\alpha_{n-1-j}}, \underbrace{2 - x_j + N}_{\alpha_{n-j}}, \underbrace{x_1 - x_j + 1 + N}_{\alpha_{n-j+1}}, \ldots, \underbrace{x_{j-1} - x_j + 1 + N}_{\alpha_{n-1}}\big). \end{aligned}$$

It follows from the definition of j that $x_j \ge rj+1$, and for $j' > j$ we have $\Delta_{j'} \ge \Delta_j + 1$, and therefore $x_{j'} - x_j \le r(j'-j) - 1$; for $j' < j$ we have $\Delta_{j'} \ge \Delta_j$, and therefore $x_{j'} - x_j \le r(j'-j)$. Thus

$$\begin{aligned} \alpha_u &= x_{j+u} - x_j + 1 \le r(j+u-j) - 1 + 1 = ru, \quad u \in [n-1-j], \\ \alpha_{n-j} &= 2 - x_j + rn - k \le 2 - rj - 1 + rn - 1 = r(n-j), \\ \alpha_{n-j+u} &= x_u + 1 - x_j + rn - k \le r(u - j + n), \quad u \in [j-1]. \end{aligned}$$

Hence $\alpha \in S$. □

On the strength of Lemma 4.9, we can assume that $x \in S$ with $x_0 = 1$. The following result determines the k vectors in S desired in Proposition 4.4.

Lemma 4.10. *Let $0 = j_0 < j_1 < j_2 < \cdots$ be the elements of the subset*

$$J^* := \{j \in J : \Delta_{j'} > \Delta_j, \text{ for all } n-1 \ge j' > j\} \subseteq J := \{0\} \cup \{j \in [n-1] : x_j > x_{j-1}\}$$

and m be the nonnegative integer determined by

$$-1 = \Delta_{j_0} < \Delta_{j_1} < \cdots < \Delta_{j_m} \le k - 2 < \Delta_{j_{m+1}} < \cdots$$

(if j_{m+1} does not exist, then set j_{m+1} and $\Delta_{j_{m+1}}$ to be infinity). In particular, j_1 is the largest number in $[n-1]$ such that $\Delta_{j_1} = \min_{j \in [n-1]} \Delta_j \ge 0$.

Then y is a weakly increasing sequence in S such that $(y \bmod N)_\uparrow \in R_x$ if and only if

(1) *$y = x + i$ with $0 \le i \le \Delta_{j_1} \wedge (k-1)$, where $\wedge$ represents the minimum function; or*

(2) *$y = (x + i_1 \bmod N)_\uparrow + i_2$ with*

(i) *$i_1 = 1 - x_{j_v}$ for some $v \in [m]$, and*

(ii) *$0 \le i_2 = y_0 - 1 \le \Delta_{j_{v+1}} \wedge (k-1) - \Delta_{j_v} - 1 < k - 1$.*

Further, the k vectors given in (1) and (2) are distinct.

Remark 4.11. Note that (1) is the special case of (2) with $i_1 = 0 = v$ and $i_2 = i$.

Proof. As a consequence of $p = N$, the vectors $(x + i_1 \bmod N)_\uparrow$ with i_1 given in (1) ($i_1 = i$) and (2), whose smallest coordinates are all 1, are distinct. Thus the k vectors given in (1) and (2) are distinct.

(1) If $y = x + i \in S$, then by definition we have $1 \le (x+i)_0 \le k$ and $(x+i)_{j_1} \le rj_1$. Thus $0 \le i \le \Delta_{j_1} \wedge (k-1)$.

Conversely, for any $y = x + i$ with $0 \le i \le \Delta_{j_1} \wedge (k-1)$, we have $(y \bmod N)_\uparrow \in R_x$, $y_{n-1} - y_0 = x_{n-1} - x_0 < N$, and $1 \le w := y_0 = (x+i)_0 \le (1 + \Delta_{j_1}) \wedge k \le k$, and Property I′ follows.

For Property II′, notice that for any $j \in [n-1]$ such that $x_j \ge 2$, since $rj - x_j = \Delta_j \ge \Delta_{j_1}$, we have $j \ge (2 + \Delta_{j_1})/r > w/r$, and therefore $j \ge q(w)$. Hence $y_{q(w)-1} = w$.

Finally for Property III′, for all $j \in [n-1]$, since $\Delta_{j_1} \le \Delta_j$, we get $x_j - x_{j_1} \le r(j - j_1)$, and therefore $y_j = (x+i)_j = x_j + i \le r(j - j_1) + \Delta_{j_1} = rj$.

(2) If y is a weakly increasing sequence in S such that $(y \bmod N)_\uparrow \in R_x$ but y does not have the form described in (1), then by Lemma 4.7 we get $\alpha := y - i_2 \in S$, $\alpha_0 = 1$ and $\alpha \in R_x$, where $i_2 = y_0 - 1 \ge 0$.

Since $\alpha \ne x$, we have $\alpha = (x + i_1 \bmod N)_\uparrow$ for some $i_1 \in \{-1, -2, \ldots, 1-N\}$. Recall that $\alpha_0 = 1$, and thus $i_1 = 1 - x_j$ for some $j \in [n-1]$. If there is more than one j such that $i_1 = 1 - x_j$, we choose the smallest one, i.e., the $j \in J$. Then

$$\begin{aligned} &x + i_1 \bmod N \\ =&\big(2 - x_j + N, x_1 - x_j + 1 + N, \ldots, x_{j-1} - x_j + 1 + N, 1, x_{j+1} - x_j + 1, \ldots, x_{n-1} - x_j + 1\big), \end{aligned}$$

and

$$\begin{aligned} &\alpha := (x + i_1 \bmod N)_\uparrow \\ =&\big(1, \underbrace{x_{j+1} - x_j + 1}_{\alpha_1}, \ldots, \underbrace{x_{n-1} - x_j + 1}_{\alpha_{n-1-j}}, \underbrace{2 - x_j + N}_{\alpha_{n-j}}, \underbrace{x_1 - x_j + 1 + N}_{\alpha_{n-j+1}}, \ldots, \underbrace{x_{j-1} - x_j + 1 + N}_{\alpha_{n-1}}\big). \end{aligned}$$

Recall that $\alpha \in S$ if and only if

$$\alpha_u \le ru, \quad \text{for all } \; u \in [n-1]. \tag{4.6}$$

Applying to $u = 1, \ldots, n-1-j$ leads to

$$x_{j'} - x_j + 1 \le r(j' - j), \text{ i.e., } \Delta_j < \Delta_{j'}, \quad \text{for all } j' < j \le n-1;$$

applying to $u = n - j$ leads to

$$2 - x_j + rn - k \le r(n-j), \text{ i.e., } \Delta_j \le k - 2.$$

Therefore $j = j_v$ for some $v \in [m]$.

Conversely, from the above argument we see that if $i_1 = 1 - x_j$ with $j = j_v$ for some $v \in [m]$, then we have $\alpha_u \le ru$ for all $u \in [n-j]$. Further, we have

$$\alpha_{n-j+u} = x_u - x_j + 1 + N \le ru + \Delta_j - rj + 1 + rn - k < r(n - j + u)$$

for all $u \in [j-1]$. Hence $\alpha \in S$.

It remains to show that $\alpha + i_2 \in S$ if only if i_2 satisfies the inequality in (ii).

If $\alpha + i_2 \in S$, then applying (4.6) to $\alpha' := \alpha + i_2$ and $u = j_{v+1} - j_v$ (if exists) leads to

$$x_{j_{v+1}} - x_{j_v} + 1 + i_2 \le r(j_{v+1} - j_v), \text{ i.e., } i_2 \le \Delta_{j_{v+1}} - \Delta_{j_v} - 1;$$

applying (4.6) to $\alpha' := \alpha + i_2$ and $u = n - j_v$ leads to

$$2 - x_{j_v} + rn - k + i_2 \le r(n - j_v), \text{ i.e., } i_2 \le k - 2 - \Delta_{j_v}.$$

Recall that $i_2 = y_0 - 1 \ge 0$, and thus i_2 satisfies the inequality in (ii).

Conversely, if i_2 satisfies the inequality in (ii), then $\alpha' \in S$. In fact, we have $1 \le w := 1 + i_2 \le k$ and $\alpha'_{n-1} - \alpha'_0 = \alpha_{n-1} - \alpha_0 < N$, and Property I′ then follows.

For Property II′, by the definition of j_{v+1}, we have $\Delta_u \ge \Delta_{j_{v+1}}$ for any $j_v < u \in J$, and hence for any $j_v < u \le n-1$ such that $x_u > x_{j_v}$. Thus $ru - x_u \ge \Delta_{j_{v+1}}$. It follows that

$$ru \ge \Delta_{j_{v+1}} + x_u > \Delta_{j_{v+1}} + x_{j_v} = \Delta_{j_{v+1}} + rj_v - \Delta_{j_v}$$

and

$$u - j_v > (\Delta_{j_{v+1}} - \Delta_{j_v})/r \ge w/r, \text{ i.e., } u \ge q(w) + j_v.$$

Hence $\alpha'_{q(w)-1} = x_{q(w)-1+j_v} - x_{j_v} + w = w$.

Finally for Property III′, from the above argument we see that $\alpha'_u \le ru$ for $u = j_{v+1} - j_v, n - j_v$. Further, we have

$$\alpha'_{n-j_v+u} = x_u - x_{j_v} + 1 + N + i_2 \le ru - x_{j_v} + 1 + rn - k + k - 2 - \Delta_{j_v} < r(n - j_v + u)$$

for all $u \in [j_v - 1]$. For $j_v + 1 \le u \le n-1$ such that $x_u > x_{j_v}$ and $u \in J$, we have $\Delta_u \ge \Delta_{j_{v+1}}$ by the definition of j_{v+1}, and therefore

$$\alpha'_{u-j_v} = x_u - x_{j_v} + w \le (ru - \Delta_{j_{v+1}}) - x_{j_v} + (\Delta_{j_{v+1}} - \Delta_{j_v}) = r(u - j_v).$$

Hence $\alpha' \in S$, as desired. □

□

Note. We have been unable to find a satisfactory q-analogue of Theorem 4.1, generalizing Theorem 3.3.

5. The r-parking function basis

Equation (1.6) and other considerations suggest looking at products of the symmetric functions $F_n^{(r)}$ for various values of n. Thus for any partition λ define

$$F_\lambda^{(r)} = F_{\lambda_1}^{(r)} F_{\lambda_2}^{(r)} \cdots ,$$

where $F_0 = 1$.

Recall that Λ denotes the ring of all symmetric functions that can be written as an integer linear combination of the monomial symmetric functions m_λ (or equivalently, s_λ, h_λ, or e_λ).

Proposition 5.1. *Fix $r \ge 1$. Then the symmetric functions $F_\lambda^{(r)}$, where λ ranges over all partitions of all $n \ge 0$, form an integral basis for the ring Λ.*

Proof. We need to show that for each n, the set $\{F_\lambda^{(r)} : \lambda \vdash n\}$ is an integral basis for the (additive) group Λ^n of all homogeneous symmetric functions of degree n contained in Λ. Let $\lambda^1, \lambda^2, \ldots$ be any ordering of the partitions of n that is compatible with refinement, that is, if λ^i is a refinement of λ^j then $i \le j$. Now $F_n^{(r)} = h_n + \cdots \in \Lambda^n$. Hence $F_\lambda^{(r)} = h_\lambda +$ terms involving h_μ where μ refines λ. Hence the transition matrix for expressing the $F_\lambda^{(r)}$'s in terms of the h_λ's is lower triangular with 1's on the main diagonal. Since the h_λ's form an integral basis, the same is true of the $F_\lambda^{(r)}$'s. □

Now that for each $r \ge 1$ we have this "parking function basis" $\{F_\lambda^{(r)}\}$, we can ask about its expansion in terms of other bases and vice versa. If we restrict ourselves to the six "standard" bases (where the power sums p_λ are a basis over $\mathbb{Q}$ but not $\mathbb{Z}$), we thus have twelve transition matrices to consider. We can also ask about various scalar products such as $\langle F_\lambda^{(r)}, F_\mu^{(r)}\rangle$. Moreover, we could also consider the basis $\{\tilde{F}_\lambda^{(r)}\}$ dual to $\{F_\lambda^{(r)}\}$, i.e.,

$$\langle F_\lambda^{(r)}, \tilde{F}_\mu^{(r)}\rangle = \delta_{\lambda\mu}.$$

However, these dual bases will not yield any new coefficients since the dual basis to a standard basis is also a standard basis (up to a normalizing factor in the case of p_λ). We have not systematically investigated these problems. Some miscellaneous results are below.

We first consider scalar products $\langle F_\mu^{(r,k)}, F_\lambda^{(r,k)}\rangle$. We can give an explicit formula when $\mu = (n)$. In fact, we can give a more general result where $F_\lambda^{(r,k)}$ is replaced with a "mixed" product.

Theorem 5.2. *Let $\lambda \vdash n$, and let $r, r_1, r_2, \ldots$ be positive integers. Let $k, k_1, k_2, \ldots$ be integers or even indeterminates. Then*

$$\left\langle F_n^{(r,k)}, \prod_i F_{\lambda_i}^{r_i,k_i}\right\rangle = \frac{k}{rn+k}\prod_{i\geq 1}\frac{k_i}{r_i\lambda_i+k_i}\binom{(rn+k)(r_i\lambda_i+k_i)+\lambda_i-1}{\lambda_i}.$$

First proof. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ then write $[t^\lambda]$ for the operator that takes the coefficient of $t_1^{\lambda_1}t_2^{\lambda_2}\cdots$. By equation (2.4) we have

$$\left\langle F_n^{(r,k)}, \prod_i F_{\lambda_i}^{r_i,k_i}\right\rangle = \frac{k}{rn+k}\prod\frac{k_i}{\lambda_i+k_i}[t^\lambda]\left\langle H(1)^{rn+k}, H(t_1)^{r_1\lambda_1+k_1}H(t_2)^{r_2\lambda_2+k_2}\cdots\right\rangle.$$

Writing $H(u)^b = \prod(1-x_iu)^b$, taking logarithms, expanding in terms of the power sums p_k, and then exponentiating, we get the well-known result

$$H(u)^b = \sum_\mu z_\mu^{-1}b^{\ell(\mu)}p_\mu u^{|\mu|},$$

where μ ranges over all partitions of all integers $j \geq 0$. (For the case $b = 1$, see [6, (7.22)].)

Since $\langle p_\lambda, p_\mu\rangle = z_\lambda\delta_{\lambda\mu}$, we get

$$\begin{aligned}\left\langle F_n^{(r,k)}, \prod_i F_{\lambda_i}^{(r_i,k_i)}\right\rangle &= \frac{k}{rn+k}\prod_i\left(\frac{k_i}{r_i\lambda_i+k_i}\cdot\right.\\ &\qquad [t^\lambda]\left.\left\langle\sum_{u\vdash n} z_\mu^{-1}(rn+k)^{\ell(\mu)}p_\mu, \prod_{i\geq 1}\left(\sum_{\nu\vdash\lambda_i} z_\nu^{-1}(r_i\lambda_i+k_i\right)^{\ell(\nu)} t_i^{|\nu|}p_\nu\right\rangle\right)\\ &= \frac{k}{rn+k}\prod_i\frac{k_i}{r_i\lambda_i+k_i}\cdot\prod_{i\geq 1}\left(\sum_{\nu\vdash\lambda_i} z_\nu^{-1}(rn+x)^{\ell(\nu)}(r_i\lambda_i+k_i)^{\ell(\nu)}\right).\end{aligned} \tag{5.1}$$

Now in general (equivalent for instance to [5, Prop. 1.3.7]),

$$\sum_{\nu\vdash m} z_\nu^{-1}u^{\ell(\nu)} = \binom{u+m-1}{m}.$$

Hence the proof follows immediately from equation (5.1).

Second proof. From equation (2.4) we see that

$$\left\langle F_n^{(r,k)}, \prod_i F_{\lambda_i}^{r_i,k_i} \right\rangle = \frac{k}{rn+k}\left(\prod_{i\geq 1}\frac{k_i}{r_i\lambda_i+k_i}\right)$$

$$\cdot \left\langle \sum_{a_1+\cdots+a_{rn+k}=n} h_{a_1}\cdots h_{a_{rn+k}}, \prod_i \left(\sum_{b_{i,1}+\cdots+b_{i,r_in+k_i}=\lambda_i} h_{b_{i,1}}\cdots h_{b_{i,r_in+k_i}} \right) \right\rangle,$$

where $a_i, b_{i,j} \geq 0$. Let

$$Z = \frac{k}{rn+k}\prod_{i\geq 1}\frac{k_i}{r_i\lambda_i+k_i}.$$

Now $\langle h_\lambda, h_\mu \rangle$ is equal to the number of matrices $(a_{ij})_{i,j\geq 1}$ of nonnegative integers with row sum vector λ and column sum vector μ [6, (7.31)]. Hence $\frac{1}{Z}\left\langle F_n^{(r,k)}, \prod_i F_{\lambda_i}^{r_i,k_i}\right\rangle$ is equal to the total number of $(rn+j)\times(\sum_i(r_i+n+k_i))$ matrices of nonnegative integers whose entries sum to n, such that the first $r_1\lambda_1+k_1$ columns sum to λ_1, the next $r_2\lambda_2+k_2$ columns sum to λ_2, etc. Since $\sum \lambda_i = n$, if the conditions on the columns is satisfied then the entries will automatically sum to n. By elementary and well-known reasoning, the number of ways to write λ_i as an ordered sum of $(rn+k)(r_in+k_i)$ nonnegative integers is $\binom{(rn+k)(r_i\lambda_i+k_i)+\lambda_i-1}{\lambda_i}$, and the proof follows. □

We now consider the expansion of the symmetric functions p_λ, h_λ, and e_λ in terms of the basis $F_n^{(r)}$ (for fixed r, which we may even regard as an indeterminate).

Proposition 5.3. *For $n \geq 1$ we have*

$$\begin{aligned} F_n^{(r,-rn-1)} &= (-1)^n(rn+1)e_n \\ F_n^{(r,-rn)} &= -rp_n \\ F_n^{(r,-rn+1)} &= (1-rn)h_n. \end{aligned}$$

Proof. Putting $k=-rn-1$ in equation (2.3) gives $(-1)^n(rn+1)\sum_{\lambda\vdash n} z_\lambda^{-1}(-1)^{n-\ell(\lambda)}p_\lambda$. It is well-known that this sum is just e_n, and the proof of the first equation follows. (We could also substitute $k=-rn-1$ in equation (2.2) and simplify.) The other two equations are similar. □

Now by Proposition 5.3 we have (writing $d_i = d_i(\lambda)$)

$$\begin{aligned}
(-1)^n(rn+1)e_n &= F_n^{(r,-rn-1)} \\
&= [t^n]\left(\sum_{i\geq 0} F_i^{(r)}t^i\right)^{-rn-1} \\
&= [t^n]\sum_{j\geq 0}(-1)^j\binom{rn+j}{j}\left(\sum_{i\geq 1} F_i^{(r)}t^i\right)^j \\
&= \sum_{a_1+\cdots a_j=n}(-1)^j\binom{rn+j}{j}F_{a_1}^{(r)}\cdots F_{a_j}^{(r)} \\
&= \sum_{\lambda\vdash n}(-1)^{\ell(\lambda)}\binom{rn+\ell(\lambda)}{d_1,d_2,\ldots,rn}F_\lambda^{(r)},
\end{aligned}$$

where the penultimate sum is over all 2^{n-1} compositions of n. We have therefore expressed e_n as a linear combination of $F_\lambda^{(r)}$'s. In exactly the same way we obtain

$$\begin{aligned}
-rp_n &= \sum_{\lambda\vdash n}(-1)^{\ell(\lambda)}\binom{rn+\ell(\lambda)-1}{d_1,d_2,\ldots,rn-1}F_\lambda^{(r)} \\
-(rn-1)h_n &= \sum_{\lambda\vdash n}(-1)^{\ell(\lambda)}\binom{rn+\ell(\lambda)-2}{d_1,d_2,\ldots,rn-2}F_\lambda^{(r)}.
\end{aligned}$$

(For $r = n = 1$, the last equation becomes $0 = 0$, but it is clear that $h_1 = F_1^{(r)}$.) Since $\{e_\mu\}, \{p_\mu\}, \{h_\mu\}$ and $\{F_\lambda^{(r)}\}$ are multiplicative bases, we have in principle expressed each e_μ, p_μ, and h_μ as a linear combination of $F_\lambda^{(r)}$'s. We leave open, however, whether there is some more elegant form of these expansions, e.g., a simple combinatorial interpretation of the coefficients.

Similarly, since Theorem 2.1 in the case $k = 1$ gives the expansion of $F_n^{(r)}$ in terms of the multiplicative bases p_μ, h_μ, and e_μ, we in principle also have an expansion of $F_\lambda^{(r)}$ in terms of these bases, but perhaps a better description is available. We cannot expect a simple product formula for the coefficients in general since for instance the coefficient of p_3p_6 in the power sum expansion of $F_{(3,2,1,1,1,1)}^{(1)}$ is equal to $2\cdot 7\cdot 157/3$.

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