SPROUT SYMMETRIC FUNCTIONS PART 1

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Abstract. A *sprout sequence* is a sequence $\mathfrak{R} = (R_0 = 1, R_1, R_2, \dots)$ of symmetric functions in the variables $\boldsymbol{x} = (x_1, x_2, \dots)$ over a field K generated from a power series $F(t) = 1 + a_1 t + a_2 t^2 + \cdots$ by the rule $\sum_{n \geq 0} R_n t^n = \prod_{i \geq 1} F(x_i t)$. The power series $F(t)$ is called the *seed* of $\mathfrak{R}$. This concept originated in the work of Littlewood and Richardson (though not with the name "sprout sequence"), and numerous examples of sprout sequences have appeared in the literature. They are related to chromatic Tutte polynomials of complete graphs and complete hypergraphs, binomial posets, upper homogeneous (upho) posets, topological genera, etc.

We first develop the basic theory of sprout sequences and then look at the special case $F(t) = \sec(\sqrt{t})$. We give five characterizations of sprout sequences and consider the expansion of sprout symmetric functions in terms of well-known symmetric function bases. The Schur positivity, elementary symmetric function positivity, and complete homogeneous symmetric function positivity of R_n for all n are completely characterized using the Edrei-Thoma theorem from the theory of total positivity.

The seed $F(t) = \sec(\sqrt{t})$ is especially interesting. The expansion of R_n in the power sum or monomial basis is related to alternating permutations. The Schur function expansion is related to standard Young skew tableaux. The expansion in terms of the complete symmetric functions has nonnegative integer coefficients, but we don't know a combinatorial interpretation. Finally we give a formula for R_n as a sum of chromatic symmetric functions of interval orders.

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1. Introduction

We assume knowledge of symmetric functions as may be found in Stanley [18, Ch. 7]. Let K be a field of characteristic 0, and let Λ_K denote the K-algebra of symmetric functions in the indeterminates $\boldsymbol{x} = (x_1, x_2, \dots)$. Write Λ_K^n for the space of those elements of Λ_K that are homogeneous of degree n. For any K-basis $\{b_\lambda\}$ of Λ_K and any $f \in \Lambda_K$ let $[b_\lambda]f$ denote the coefficient of b_λ in the b-expansion of f. We will deal with K-bases whose elements are homogeneous. The basis elements b_λ for Λ_K^n are indexed by partitions λ of n, denoted $\lambda \vdash n$.

Now let $F(t) = 1 + a_1 x + a_2 x^2 + \cdots$ be a formal power series over K with constant term 1. Define a sequence $\mathfrak{R} = (R_0(\boldsymbol{x}), R_1(\boldsymbol{x}), \dots)$ of symmetric functions $R_n(\boldsymbol{x})$ by

$$\sum_{n \geq 0} R_n(\boldsymbol{x}) t^n = \prod_{i \geq 1} F(x_i t). \tag{1.1}$$

Note that $R_n(\boldsymbol{x})$ is well-defined formally and is homogeneous of degree n. Moreover, $R_0(\boldsymbol{x}) = 1$ and $R_1(\boldsymbol{x}) = a_1(x_1 + x_2 + \cdots) = a_1 p_1(\boldsymbol{x})$. We call $\mathfrak{R}$ the *sprout sequence* of symmetric functions generated by the *seed* $F(t)$. We also call the $R_n(\boldsymbol{x})$'s *sprout symmetric functions* (with respect to the seed $F(t)$). We will always use the notation (1.1) in regard to a sprout sequence $\mathfrak{R}$, as well as the following:

$$\begin{aligned} F(t) &= \sum_{n \geq 0} a_n t^n, \ a_0 = 1 \\ \log F(t) &= \sum_{n \geq 1} b_n \frac{t^n}{n} \\ \mathcal{A}(t) &= \sum_{n \geq 0} R_n(\boldsymbol{x}) t^n. \end{aligned} \tag{1.2}$$

(The reason for the n in the denominator in the expansion of $\log F(t)$ is explained by Theorem 2.1(d).)

The concept of sprout symmetric functions is due to D. E. Littlewood and A. R. Richardson [10][11] (repeated in [12, pp. 99–100 and Ch. VII]), though stated in a different form. Namely, Littlewood and Richardson consider an arbitrary series $F(t) = 1 + a_1 t + \cdots$. They define the *Schur function* s_λ^F *of the series* F, where $\lambda \vdash n$, to be (in our notation) the scalar product $\langle R_n, s_\lambda \rangle$, i.e. (since the Schur functions are self-dual), the coefficient of s_λ in the Schur expansion of R_n.

For instance, if $F(t)=\prod_j(1-y_jt)^{-1}$ then by the Cauchy identity [18, Thm. 7.12.1] we have

$$F(x_1t)F(x_2t)\cdots = \prod_{i,j}(1-x_iy_jt)^{-1} \tag{1.3}$$

$$= \sum_{n\geq 0}\sum_{\lambda\vdash n} s_\lambda(\boldsymbol{x})s_\lambda(\boldsymbol{y})t^n, \tag{1.4}$$

whence $s_\lambda^F = s_\lambda(\boldsymbol{y})$. In particular, if $F_1(t)=\prod_{i=1}^n(1-q^{i-1}t)^{-1}$ then

$$\begin{aligned} s_\lambda^{F_1} &= s_\lambda(1,q,\ldots,q^{i-1}) \\ &= q^{b(\lambda)}\prod_{u\in\lambda}\frac{1-q^{n+c(u)}}{1-q^{h(u)}}, \end{aligned}$$

where $c(u)$ denotes the content and $h(u)$ the hook length of the square u (in the Young diagram of λ) and $b(\lambda)=\sum(i-1)\lambda_i$ (see [18, Thm. 7.21.2]). This formula was stated in a more complicated way by Littlewood and Richardson since hook lengths had yet to be discovered. From this formula Littlewood and Richardson obtain what is perhaps their nicest result in this area, namely, let

$$F_2(t)=\prod_{i\geq 0}\frac{1-yq^it}{1-zq^it}.$$

Then

$$s_\lambda^{F_2}=\prod_{u\in\lambda}\frac{y-zq^{c(u)}}{1-q^{h(u)}}.$$

See also [18, Exer. 7.91(c)].

We now give some very easy or already known examples of sprout symmetric functions.

Example 1.1. (a) Let $F(t)=1+t$. Then directly from the definitions we get $R_n=e_n$. Similarly if $F(t)=(1-t)^{-1}$ then $R_n=h_n$.

(b) Let $F(t)=\frac{1+t}{1-t}$. Then

$$\begin{aligned} \mathcal{A}(t) &= \prod_i\frac{1+x_it}{1-x_it} \\ &= \left(\sum_{n\geq 0}e_nt^n\right)\left(\sum_{n\geq 0}h_nt^n\right), \end{aligned}$$

whence

$$R_n = \sum_{k=0}^{n} e_k h_{n-k} = Q_n = 2P_n,\ \ n \geq 1,$$

where Q_n is the *Schur Q-function* and P_n is the *Schur P-function* indexed by the partition (n) [13, §III.8].

(c) Let $F(t) = e^t$. Then

$$\prod_i F(x_i t) = e^{t\sum x_i} = \sum_{n\geq 0} p_1^n \frac{t^n}{n!},$$

whence $R_n = \frac{p_1^n}{n!}$.

(d) Let $T_n(\boldsymbol{x}; v)$ denote the symmetric function generalization of the Tutte polynomial of the complete graph K_n, as defined in [16, §3.3]. From equation (14) of this reference it follows that the sequence $\left(1, \frac{T_1(\boldsymbol{x};v)}{1!}, \frac{T_2(\boldsymbol{x};v)}{2!}, \ldots\right)$ is a sprout sequence with seed $\sum_{n\geq 0}(1+v)^{\binom{n}{2}} \frac{t^n}{n!}$.

(e) Let $[d] = \{1, 2, \ldots, d\}$. Define

$$X_{\mathcal{H}_{d,k}}(\boldsymbol{x}) = \sum_{\kappa} x_{\kappa(1)} x_{\kappa(2)} \cdots,$$

summed over all maps $\kappa\colon [d] \to \mathbb{P}$ such that $\#\kappa^{-1}(i) < k$ for all i. $X_{\mathcal{H}_{d,k}}(\boldsymbol{x})$ is the chromatic symmetric function of the hypergraph $\mathcal{H}_{d,k}$ consisting of all k-element subsets of $[d]$. From [16, eqn. (21)] it follows that the sequence

$$\left(1, \frac{X_{\mathcal{H}_{1,k}}(\boldsymbol{x})}{1!}, \frac{X_{\mathcal{H}_{2,k}}(\boldsymbol{x})}{2!}, \ldots\right)$$

is a sprout sequence with seed $\sum_{j=0}^{k-1} \frac{t^j}{j!}$. A common generalization of this item and the previous one is given by the last displayed equation in [16].

(f) The sprout sequence with seed

$$F(t) = \frac{\frac{1}{2}\sqrt{t}}{\sinh(\frac{1}{2}\sqrt{t})}$$

is essentially the $\hat{A}$-genus of spin manifolds. More precisely, in the paper [8, p. 4] of Hitchin, replacing p_i in $\hat{A}_i$ by the elementary symmetric function e_i yields this sprout sequence. A follow-up paper [1] by Amdeberhan, Griffin, and Ono includes a further example of this nature, namely, Hirzebruch's L-genus is essentially the sprout sequence with seed

$$F(t) = \frac{\sqrt{t}}{\tanh(\sqrt{t})}.$$

For the related seed $F(t) = \sec(\sqrt{t})$, see Sections 4–7 below.

(g) Let P be a binomial poset with factorial function $B(n)$ [14][17, §§3.18–3.19]. Let Ehr_n denote the Ehrenborg quasisymmetric function [5, Def. 4.1][18, Exer. 7.48] of an n-interval of P. Then Ehr_n is in fact a symmetric function, and the sequence

$$\mathfrak{R} = \left(1, \frac{\mathrm{Ehr}_1}{B(1)}, \frac{\mathrm{Ehr}_2}{B(2)}, \dots\right)$$

is a sprout sequence with seed $\sum_{n\geq 0} \frac{t^n}{B(n)}$. Some examples will be given in Part 2 of this paper.

(h) Let P be an upho (upper homogeneous) poset, i.e., a graded poset with finitely many elements of each rank such that for all $s \in P$, the principal filter $V_s = \{u \in P : u \geq s\}$ is isomorphic to P. Let R_n be the homogeneous part of degree n of the Ehrenborg quasisymmetric function of P. Then [6, Lemma 2.2][19, (3.1)] R_n is in fact a symmetric function, and the sequence $(R_0, R_1, \dots)$ is a sprout sequence whose seed is the rank-generating function of P.

(i) Let $F(t) = 1 + \sum_{n\geq 1} a_n \frac{t^n}{n!}$, where a_n counts structures on an n-element set that are a disjoint union of their connected components (such as posets and simple graphs) so that we are in the situation of the exponential formula [18, Cor. 5.1.6]. Let $\lambda \vdash n$. Then $n!\,[p_\lambda]R_n$ is equal to the number of structures on an n-set such that the connected components have sizes $\lambda_1, \lambda_2, \dots$.

In the next section we give three basic properties of sprout sequences, namely, five equivalent conditions to be a sprout sequence, the behavior of sprout sequences under the automorphism ω, and conditions for s, e and h-positivity. The conditions for s, e, h-positivity deal with the following question: if we expand R_n in terms of the bases s_λ (Schur functions), e_λ (elementary symmetric functions), and h_λ (complete symmetric functions), then when are the coefficients nonnegative for all n? The proof of the conditions for e and h-positivity is due to Vince Vatter. In subsequent sections we discuss an example of a sprout sequence $(A_0, A_1, \dots)$ that arose originally from the problem of expressing a certain theta function of Ramanujan in terms of Eisenstein series. A continuation of the present paper is devoted to generalizations of the sprout sequence $(A_0, A_1, \dots)$.

2. Basic properties

We begin with five characterizations of sprout sequences, including the original definition. Condition (e) arose from a conversation with Jesse Kim.

Theorem 2.1. *Let* $\mathfrak{R} = (R_0(\boldsymbol{x}) = 1, R_1(\boldsymbol{x}), R_2(\boldsymbol{x}), \dots)$ *be a sequence of symmetric functions, where* $R_n(\boldsymbol{x})$ *is homogeneous of degree* n*. Let* $\mathcal{A}(t) = \sum_{n\geq 0} R_n t^n$*. The following five conditions are equivalent.*

(a) $\mathfrak{R}$ *is a sprout sequence.*

(b) *There exist elements* $b_1, b_2, \dots \in K$ *such that*

$$\log \mathcal{A}(t) = \sum b_n p_n \frac{t^n}{n}. \tag{2.1}$$

Moreover, $\log F(t) = \sum b_n \frac{t^n}{n}$.

(c) *There exist elements* $a_0 = 1, a_1, a_2, \dots \in K$ *such that for all* $n \geq 1$,

$$R_n(\boldsymbol{x}) = \sum_{\lambda \vdash n} a_{\lambda_1} a_{\lambda_2} \cdots m_\lambda(\boldsymbol{x}). \tag{2.2}$$

Moreover, $F(t) = \sum_{n\geq 0} a_n t^n$.

(d) *There exist elements* $b_0 = 1, b_1, b_2, \dots \in K$ *such that for all* $n \geq 1$,

$$R_n(\boldsymbol{x}) = \sum_{\lambda \vdash n} z_\lambda^{-1} b_{\lambda_1} b_{\lambda_2} \cdots p_\lambda(\boldsymbol{x}). \tag{2.3}$$

Moreover, $\log F(t) = \sum b_n \frac{t^n}{n}$.

(e) *For all* $f, g \in \hat{\Lambda}_K$ *(the set of all symmetric formal power series over* K *in the indeterminates* $\boldsymbol{x}$*) we have*

$$\mathcal{A}(t) * (fg) = (\mathcal{A}(t) * f)(\mathcal{A}(t) * g),$$

where $*$ *denotes internal (or Kronecker) product* [18, Exer. 7.78]. *Equivalently, the map* $\hat{\Lambda}_K \to \hat{\Lambda}_K$ *defined by* $f \mapsto f * \mathcal{A}(t)$ *is an algebra homomorphism.*

Proof. (a) $\Leftrightarrow$ (b). Assume (a). We have

$$\begin{aligned}\log \mathcal{A}(t) &= \sum_i \log F(x_i t)\\ &= \sum_i \sum_{n\geq 1} b_n x_i^n \frac{t^n}{n}\\ &= \sum_{n\geq 1} b_n p_n \frac{t^n}{n},\end{aligned}$$

so (b) holds. The steps are reversible, so (b) implies (a).

(a) $\Leftrightarrow$ (c). Assume (a). Expanding $\prod_i F(x_i t)$ gives (c). The steps are reversible, so (c) implies (a).

(b) $\Leftrightarrow$ (d). Assume (b). Thus

$$\begin{aligned}\mathcal{A}(t) &= \exp\left(\sum_{n\geq 1} b_n p_n \frac{t^n}{n}\right)\\ &= \prod_n \exp\left(b_n p_n \frac{t^n}{n}\right).\end{aligned}$$

Expand each exponential and multiply to get (d). The steps are reversible, so (d) implies (b).

(d) $\Leftrightarrow$ (e). Assume (d). By bilinearity of $*$ it suffices to prove (e) for $f = p_\lambda$ and $g = p_\mu$. From the formula [18, Exer. 7.78(d)] $p_\lambda * p_\mu = \delta_{\lambda,\mu} z_\lambda p_\lambda$ it follows that

$$\begin{aligned}[p_\nu]\mathcal{A}(t) * p_\lambda &= \delta_{\lambda,\nu} b_{\lambda_1} b_{\lambda_2} \cdots\\ [p_\nu]\mathcal{A}(t) * p_\mu &= \delta_{\mu,\nu} b_{\mu_1} b_{\mu_2} \cdots\\ [p_\nu]\mathcal{A}(t) * p_\lambda p_\mu &= \delta_{\lambda\cup\mu,\nu} b_{\lambda_1} b_{\lambda_2} \cdots b_{\mu_1} b_{\mu_2} \cdots .\end{aligned}$$

It follows that (d) $\Rightarrow$ (e). The steps are reversible, so (e) implies (d). $\square$

Define the *dimension* $\dim f$ of $f \in \Lambda_K^n$ by $\dim f = \langle f, p_1^n \rangle$. If f is the Frobenius characteristic $\operatorname{ch}\chi$ of a character χ of the symmetric group $\mathfrak{S}_n$, then $\dim f = \dim \chi$.

Corollary 2.2. *Preserve the notation of Theorem 2.1. Then* $\dim R_n = b_1^n = a_1^n$.

Proof. In equation (2.3) take the scalar product with p_1^n and use $z_{1^n} = \langle p_1^n, p_1^n \rangle = n!$. $\square$

Note that by similar reasoning, if $n = dm$ then $\langle R_n, p_d^m \rangle = b_d^m$.

From Theorem 2.1(c) we can give a formula for the expansion of R_n in terms of any homogeneous basis for Λ_K. Let $F(t) = \sum a_n t^n$ be a seed. Let $\varphi\colon \Lambda_K \to K$ be the K-algebra homomorphism defined by $\varphi(h_n) = a_n$.

Theorem 2.3. *Let $B = \{b_\lambda\}$ be a homogeneous K-basis for Λ_K such that if $\lambda \vdash n$, then* $\deg b_\lambda = n$. *Let $\{b^*_\lambda\}$ be the basis dual to B, so $\langle b_\lambda, b^*_\mu\rangle = \delta_{\lambda\mu}$. Then*

$$R_n = \sum_{\lambda\vdash n} \varphi(b^*_\lambda) b_\lambda.$$

Proof. Let $\boldsymbol{y} = (y_1, y_2, \dots)$ be a new set of indeterminates, and let φ act on $\boldsymbol{y}$-variables only. Then by Theorem 2.1(c) we have

$$\begin{aligned} R_n(\boldsymbol{x}) &= \sum_{\lambda\vdash n} a_{\lambda_1} a_{\lambda_2} \cdots m_\lambda(\boldsymbol{x}) \\ &= \sum_{\lambda\vdash n} \varphi(h_\lambda(\boldsymbol{y})) m_\lambda(\boldsymbol{x}). \end{aligned}$$

Since $\{m_\lambda\}$ and $\{h_\mu\}$ are dual bases [18, eqn. (7.30)], it follows from [18, Lemma 7.9.2] that

$$\sum_{\lambda\vdash n} h_\lambda(\boldsymbol{y}) m_\lambda(\boldsymbol{x}) = \sum_{\lambda\vdash n} b^*_\lambda(\boldsymbol{y}) b_\lambda(\boldsymbol{x}).$$

Applying φ (acting on the $\boldsymbol{y}$-variables) completes the proof. □

Corollary 2.4. *Let $\mathfrak{R} = (R_0, R_1, \dots)$ be a sprout sequence with seed $F(t) = \sum a_i t^i$, and let $\lambda \vdash n$. Then the coefficient of s_λ in R_n is given by the determinant*

$$\langle R_n, s_\lambda\rangle = \det\left[a_{\lambda_i - i + j}\right]_{i,j=1}^{\ell(\lambda)}, \tag{2.4}$$

where $\ell(\lambda)$ denotes the length (number of parts) of λ.

Proof. Apply φ to the Jacobi-Trudi identity $s_\lambda = \det[h_{\lambda_i - i + j}]$ and note that the Schur functions form a self-dual basis. □

Theorem 2.1 yields a simple proof of the following generalization of Example 1.1(d). If $\kappa\colon [d] \to \mathbb{P}$, then we define type$(\kappa)$ to be the partition $\lambda \vdash d$ whose parts are the numbers $\#\kappa^{-1}(1), \#\kappa^{-1}(2), \dots$ arranged in weakly decreasing order.

Proposition 2.5. *Let S be a nonempty subset of $\mathbb{P}$. Define $Y_{S,d} = \sum_\kappa x_{\kappa(1)} x_{\kappa(2)} \cdots$, where the sum is over all maps $\kappa\colon [d] \to \mathbb{P}$ such that the parts of the partition* type(κ) *all belong to S. Then*

$$\left(1, \frac{Y_{S,1}}{1!}, \frac{Y_{S,2}}{2!}, \dots\right)$$

is a sprout sequence with seed $\sum_{j\in S}\frac{t^j}{j!}$.

Proof. If all parts of $\lambda = \text{type}(\kappa)$ belong to S, then an elementary combinatorial argument shows that the coefficient of m_λ in the m-expansion of $Y_{S,d}$ is equal to the multinomial coefficient $\binom{d}{\lambda_1,\lambda_2,\dots} = \frac{d!}{\lambda_1!\,\lambda_2!\cdots}$. Hence the condition of Theorem 2.1(c) holds with $a_i = 1/i!$ if $i \in S$ and $a_i = 0$ if $i \notin S$, and the proof follows. $\square$

Example 1.1(d) is the special case $S = \{1, 2, \dots, k-1\}$ of Proposition 2.5.

It is easy to determine the effect of the involution ω [18, §7.6] on sprout sequences.

Theorem 2.6. *Let* $\mathfrak{R} = (1, R_1(\boldsymbol{x}), R_2(\boldsymbol{x}), \dots)$ *be a sprout sequence with seed* $F(t)$*. Then* $(1, \omega R_1(\boldsymbol{x}), \omega R_2(\boldsymbol{x}), \dots)$ *is a sprout sequence with seed* $\frac{1}{F(-t)}$.

Proof. By Theorem 2.1(b) we can write

$$\log \mathcal{A}(t) = \sum b_n p_n \frac{t^n}{n}.$$

Now $\omega p_n = (-1)^{n-1} p_n$. Since ω is a homomorphism (even an involution) we have

$$\begin{aligned} \log \omega\mathcal{A}(t) &= -\sum b_n p_n \frac{(-t)^n}{n} \\ &= \log\left(\frac{1}{\mathcal{A}(-t)}\right), \end{aligned}$$

and the proof follows.

$\square$

There are some "specializations" of sprout symmetric functions that have simple generating functions or formulas.

Theorem 2.7. (a) *We have* $[s_n]R_n = a_n$*. Equivalently,*

$$\sum_{n\geq 0} [s_n] R_n t^n = F(t).$$

Moreover, for any homogeneous symmetric function f *of degree* n*,* $[s_n]f$ *is equal to the sum of the coefficients in the* h*-expansion of* f*.*

(b) *We have* $\sum_{n\geq 0} [s_{1^n}] R_n t^n = \frac{1}{F(-t)}$.

(c) *Fix* $k \geq 1$*. Then*

$$\sum_{n\geq 0}[h_k^n]R_{kn}t^k = \frac{1}{F(-t)|_{b_i\to b_{ki},\, i\geq 1}},$$

where b_i *is defined in equation* (1.2)*. In particular,* $[h_n]R_n = b_n$*.*

(d) *If* $i, j \geq 1$ *and* $i + j = n$*, then*

$$[h_ih_j]R_n = \begin{cases} b_ib_j - b_n, & i \neq j \\ \frac{1}{2}(b_i^2 - b_n), & i = j. \end{cases}$$

(e) *We have* $\sum_{n\geq 0} R_n(1^k)t^n = F(t)^k$*, where* $R_n(1^k)$ *is short for* $R_n(x_1 = \cdots = x_k = 1, x_{k+1} = x_{k+2} = \cdots = 0)$*.*

(f) *(generalizes (a) and (b)) Let* u *be an indeterminate and write*

$$\frac{F(t)}{F(-ut)} = \sum_{n\geq 0} P_n(u)t^n.$$

Then for $n \geq 1$*,* $P_n(u)$ *is a polynomial in* u *of degree at most* n *and divisible by* $1 + u$*. Moreover,*

$$\frac{P_n(u)}{1+u} = \sum_{k=0}^{n-1}\left([s_{\langle 1^k n-k\rangle}]R_nu^k\right)t^n,$$

$$\frac{P_n(u)}{1+u} = \sum_{k=0}^{n-1}\left([s_{\langle n-k,1^k\rangle}]R_nu^k\right),$$

where the notation $\lambda = \langle 1^{m_1}, 2^{m_2}, \ldots\rangle$ *indicates that* λ *has* m_i *parts equal to* i *(so* $|\lambda| = \sum im_i$*). Equivalently,*

$$\frac{F(t)}{F(-ut)} = 1 + \sum_{n\geq 1}\left(\sum_{k=0}^{n}([s_{\langle n-k,1^k\rangle}] + [s_{\langle n-k+1,1^{k-1}\rangle}])R_nu^k\right)t^n,$$

with the understanding that when $k = 0$ *we set* $[s_{\langle n-k+1,1^{k-1}\rangle}]R_n = 0$*, and when* $k = n$ *we set* $[s_{\langle n-k,1^k\rangle}]R_n = 0$*.*

Proof. (a) It is easy to see that for any $f \in \Lambda_K^n$ we have $[s_n]f = f(x_1 = 1, x_2 = x_3 = \cdots = 0)$. Now make the substitution $x_1 = 1, x_2 = x_3 = \cdots = 0$ in equation (1.1). The second statement follows from the fact that for $\lambda \vdash n$ we have $[s_n]h_\lambda = 1$ (a simple consequence of Pieri's rule, for instance).

(b) Use (a), Theorem 2.6 and the fact that $\omega s_n = s_{1^n}$.

(c, d) It is easy to obtain, using the formula

$$\sum_{n\geq 0}\frac{p_n}{n} = \exp\left(\sum_{n\geq 0} h_n\right),$$

the well-known formula

$$\frac{p_n}{n} = \sum_{\lambda \vdash n} \frac{(-1)^{\ell(\lambda)-1}}{\ell(\lambda)} \binom{\ell(\lambda)}{m_1, m_2, \ldots} h_\lambda,$$

where $\lambda = \langle 1^{m_1}, 2^{m_2}, \ldots \rangle$. Make this substitution for $\frac{p_n}{n}$ in equation (2.1) and expand $\exp \sum b_n p_n \frac{t^n}{n}$ into a power series. It is routine to obtain the coefficients of h_n^k and $h_i h_j$, completing the proof.

NOTE. This technique can be used to obtain a formula for $[h_\lambda]R_n$ for any $\lambda \vdash n$, but it seems too complicated to be worth stating explicitly.

(e) Substitute $x_1 = x_2 = \cdots = x_k = 1, x_{k+1} = x_{k+2} = \cdots = 0$ in equation (1.1).

(f) Let $\psi\colon \hat{\Lambda}_{K[[t]]} \to K[u][[t]]$ be the continuous $K[[t]]$-algebra homomorphism defined by $\psi(p_n) = 1 - (-u)^n,\ n > 0$. ("Continuous" means that ψ preserves infinite linear combinations.) According to [18, Exer. 7.43] we have

$$\psi(s_\lambda) = \begin{cases} u^k(1+u), & \lambda = \langle n-k, 1^k \rangle,\ 0 \le k \le n-1 \\ 0, & \text{otherwise.} \end{cases}$$

Apply ψ to equation (2.1) and exponentiate to get

$$\begin{aligned} \psi(\mathcal{A}(t)) &= \exp\left(\sum_{n \ge 1} (1 - (-u)^n) b_n \frac{t^n}{n}\right) \\ &= \frac{F(t)}{F(-ut)}, \end{aligned}$$

and the proof follows.

□

Our next result concerns s-positivity (or Schur positivity). For this we need a celebrated result in the theory of total positivity, the *Edrei-Thoma theorem* [4][20]. Let $\boldsymbol{d} = (d_0 = 1, d_1, d_2, \ldots)$ be a real sequence. Set $d_n = 0$ if $n < 0$. Let $M_{\boldsymbol{d}}$ denote the infinite (Toeplitz) matrix $[d_{j-i}]_{i,j \ge 0}$.

Theorem 2.8 (Edrei-Thoma)**.** *The following two conditions are equivalent.*

(1) *Every minor of $M_{\boldsymbol{d}}$ is nonnegative, i.e., $M_{\boldsymbol{d}}$ is a* totally nonnegative *matrix.*

(2) *We can write*

$$\sum_{n\geq 0} d_n t^n = e^{\gamma t}\prod_{i\geq 1}\frac{1+\alpha_j t}{1-\beta_j t}, \tag{2.5}$$

where $\gamma \geq 0$ *and the* α_j*'s and* β_j*'s are nonnegative real numbers such that the sum* $\sum_j(\alpha_j+\beta_j)$ *is convergent.*

Note that this is an analytic, not combinatorial, result. The product in equation (2.5) is not defined formally; it depends on the notion of a convergent sum.

Theorem 2.9. *Let* $\mathfrak{R} = (1, R_1, R_2, \dots)$ *be a sprout sequence over* $\mathbb{R}$ *with seed* $F(t)$*. Then each* R_n *is s-positive if and only if we can write*

$$F(t) = e^{\gamma t}\prod_{j\geq 1}\frac{1+\alpha_j t}{1-\beta_j t}, \tag{2.6}$$

where $\gamma \geq 0$ *and the* α_j*'s and* β_j*'s are nonnegative real numbers such that* $\sum_j(\alpha_j+\beta_j)$ *is convergent.*

Proof. Let Γ be the algebra of symmetric functions in the variables $\boldsymbol{y} = (y_1, y_2, \dots)$ whose coefficients lie in the $\mathbb{R}$-algebra $\Lambda_{\mathbb{R}}(\boldsymbol{x})$ of symmetric functions with real coefficients in the variables $\boldsymbol{x}$. Define a ring homomorphism $\psi\colon \Gamma \to \Lambda_{\mathbb{R}}(\boldsymbol{x})$ by $\psi(h_n(\boldsymbol{y})) = a_n$ and $\psi(f) = f$ for $f \in \Lambda_{\mathbb{R}}(\boldsymbol{x})$.

It is a basic fact [18, Prop. 7.5.3 and Thm. 7.12.1] from the theory of symmetric functions that

$$\sum_{\lambda} m_\lambda(\boldsymbol{x})h_\lambda(\boldsymbol{y})t^{|\lambda|} = \sum_{\lambda} s_\lambda(\boldsymbol{x})s_\lambda(\boldsymbol{y})t^{|\lambda|}, \tag{2.7}$$

where λ ranges over all partitions of all $n \geq 0$. Write $a_\lambda = a_{\lambda_1}a_{\lambda_2}\cdots$. Apply ψ to equation (2.7). The left-hand side becomes

$$\sum_{\lambda} a_\lambda m_\lambda(\boldsymbol{x})t^{|\lambda|} = \prod_i \left(\sum_{n\geq 0} a_n x_i^n t^n\right) = \mathcal{A}(t).$$

Thus for $\lambda \vdash n$, $\psi(s_\lambda(\boldsymbol{y})) = \langle R_n, s_\lambda\rangle$.

Write $\boldsymbol{a} = (a_0, a_1, \dots)$, so $M_{\boldsymbol{a}} = [a_{j-i}]_{i,j\geq 0}$ (with $a_n = 0$ for $n < 0$). If we replace each a_n by h_n in $M_{\boldsymbol{a}}$, obtaining the matrix $M_{\boldsymbol{h}}$, then every minor of $M_{\boldsymbol{h}}$ is either 0 or the Jacobi-Trudi matrix of a skew Schur function $s_{\lambda/\mu}$, and conversely the Jacobi-Trudi matrix of every skew Schur function occurs as a minor. Thus if N is a minor of $M_{\boldsymbol{h}}$ equal to s_λ, where $\lambda \vdash n$, then the corresponding minor of $M_{\boldsymbol{a}}$ is $\langle R_n, s_\lambda\rangle$. Moreover, every skew Schur function is a nonnegative linear combination of ordinary Schur functions [18, eqn. (7.64) and

Cor. 7.18.6]. It follows that R_n is s-positive for all n if and only if every minor of $M_{\boldsymbol{a}}$ is nonnegative. The proof now follows from Theorem 2.8. $\square$

We turn to the question of the e-positivity and h-positivity of sprout symmetric functions. The proof of the "if" part of Theorem 2.11 below was provided by Vince Vatter, private communication, March 2026. First we need a simple lemma.

Lemma 2.10. *Let $\lambda \vdash n$. Then*

$$[e_1^n]m_\lambda = \begin{cases} 1, & \lambda = (n) \\ 0, & \text{otherwise,} \end{cases}$$

and

$$[e_2 e_1^{n-2}]m_\lambda = \begin{cases} -n, & \lambda = (n) \\ 1, & \lambda = (n-1,1)\rangle \\ 0. & \text{otherwise.} \end{cases}$$

Proof. There are many proofs. One straightforward one is the following. It can be seen by inspection that

$$[m_n]e_\mu = \begin{cases} 1, & \mu = \langle 1^n \rangle \\ 0, & \text{otherwise,} \end{cases}$$

and

$$[m_{n-1,1}]e_\mu = \begin{cases} n, & \mu = \langle 1^n \rangle \\ 1, & \mu = \langle 2, 1^{n-2} \rangle \\ 0, & \text{otherwise,} \end{cases}$$

from which the proof follows easily. $\square$

Theorem 2.11. (a) *Every $\beta_j = 0$ in equation* (2.6) *if and only if each R_n is e-positive.*

(b) *Every $\alpha_j = 0$ in equation* (2.6) *if and only if each R_n is h-positive.*

Proof. (a) Suppose that each $\beta_j = 0$. Thus

$$\begin{aligned} \mathcal{A}(t) &= \prod_i e^{\gamma x_i t} \prod_j (1+\alpha_j x_i t) \\ &= e^{\gamma e_1 t} \prod_j \prod_i (1+\alpha_j x_i t) \\ &= e^{\gamma e_1 t} \prod_j \sum_{n\geq 0} \alpha_j^n e_n t^n. \end{aligned}$$

When this is expanded as a power series in t, it is clear (since $\gamma \geq 0$ and each $\alpha_j \geq 0$) that for $\lambda \vdash n$ the coefficient of $e_\lambda t^n$ is nonnegative.

Conversely, by (2.2) and the previous lemma we get

$$\begin{aligned} R_n &= a_n m_n + a_1 a_{n-1} m_{n-1,1} + \cdots \\ &= a_n(e_1^n - n e_2 e_1^{n-2} + \cdots) + a_1 a_{n-1}(e_2 e_1^{n-2} + \cdots) + \cdots \\ &= a_n e_1^n + (a_{n-1} - n a_n) e_2 e_1^{n-2} + \cdots . \end{aligned}$$

Hence if R_n is e-positive then $a_n \leq \frac{a_{n-1}}{n}$. Therefore

$$a_n \leq \frac{a_{n-1}}{n} \leq \frac{a_{n-2}}{n(n-1)} \leq \cdots \leq \frac{a_1}{n!}.$$

It follows that $F(t) = \sum a_n t^n$ is an entire function. In order for the product (2.6) to be entire we must have $\beta_j = 0$ for all j. This completes the proof of (a).

(b) Apply (a) to the seed $1/F(-t)$ and use Theorem 2.6.

□

We say that a sprout sequence $\mathfrak{R}$ is *Schur positive* if every R_n is Schur positive. There are many known results (stated in the language of Pólya frequency sequences) about operations preserving the Schur positivity of sprout sequences. One that will be useful to us in Part 2 (under preparation) is the following.

Corollary 2.12. *Let d be a positive integer. If the sprout sequence $\mathfrak{R}$ with seed $F(t) = \sum a_n t^n$ is Schur positive, then the sprout sequence $\mathfrak{R}_d$ with seed $\sum a_{dn} t^n$ is Schur positive.*

Proof. The matrix $[a_{d(j-i)}]$ is a submatrix of $[a_{j-i}]$. Hence every minor of $[a_{d(j-i)}]$ is also a minor of $[a_{j-i}]$. The proof follows from Theorems 2.8 and 2.9. □

3. A symmetric function arising from the work of Amdeberhan-Ono-Singh

In 2024 Amdeberhan, Ono, and Singh [2] considered the function ϕ defined on partitions λ of n by

$$\phi(\lambda) = (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{4^k(4^k-1)B_{2k}}{(2k)(2k)!} \right)^{m_k},$$

where $\lambda = \langle 1^{m_1}, \ldots, n^{m_n} \rangle$ and B_{2k} is a Bernoulli number. The original motivation was to express a certain theta function of Ramanujan in terms of Eisenstein series.

It is not hard to see that $\phi(\lambda) \in \mathbb{Z}$ and

$$\sum_{\lambda \vdash n} |\phi(\lambda)| = E_{2n}, \tag{3.1}$$

an Euler number. The Euler numbers E_k are defined by

$$\sec t + \tan t = \sum_{k\geq 0} E_k \frac{t^k}{k!}. \tag{3.2}$$

Moreover, E_k is the number of *alternating permutations*[1] $w \in \mathfrak{S}_k$, i.e., $w = a_1a_2\cdots a_k$ where $a_1 > a_2 < a_3 > a_4 < \cdots$. Equation (3.1) suggests the question: what "nice" class of alternating permutations in $\mathfrak{S}_{2n}$ does $|\phi(\lambda)|$ count? This question was the original impetus for the present paper.

We can find a combinatorial interpretation of $|\phi(\lambda)|$ without any appeal to the theory of symmetric functions. After we do this, we will define a symmetric function A_n associated with the $\phi(\lambda)$'s for $\lambda \vdash n$ and show that $(A_0, A_1, \dots)$ is a sprout sequence with seed $\sec(\sqrt{t})$. We will then investigate further properties of A_n.

Define $\mathfrak{A}_n$ to be the set of alternating permutations in the group $\mathfrak{S}_n$, so $\#\mathfrak{A}_n = E_n$. If $w = a_1a_2\cdots a_{2n} \in \mathfrak{A}_{2n}$, then define $\hat{w}$ to be the sequence $a_1a_3a_5\cdots a_{2n-1}$. For notational convenience set $b_i = a_{2i-1}$, so $\hat{w} = b_1b_2\cdots b_n$. The *record set* $\mathrm{rec}(\hat{w})$ is the set of indices $1 \leq i \leq n$ for which b_i is a left-to-right maximum (or *record*) in $\hat{w}$. In other words, $b_i > b_j$ for all $1 \leq j < i$. Thus always $1 \in \mathrm{rec}(\hat{w})$. Suppose that the elements of $\mathrm{rec}(\hat{w})$ are $r_1 < r_2 < \cdots < r_j$. Define the *record partition* $\mathrm{rp}(\hat{w})$ of $\hat{w}$ to be the partition of n with parts $r_2 - r_1, r_3 - r_2, r_4 - r_3, \dots, n + 1 - r_j$ arranged in weakly decreasing order. Note that $\mathrm{rp}(\hat{w}) \vdash n$.

Example 3.1. Let $w = 7, 2, 5, 4, 8, 3, 10, 6, 9, 5 \in \mathfrak{A}_{10}$. Then $\hat{w} = 7, 5, 8, 10, 9$, $\mathrm{rec}(\hat{w}) = \{1, 3, 4\}$, and $\mathrm{rp}(\hat{w}) = (2, 2, 1)$.

We can now give a combinatorial interpretation of $|\phi(\lambda)|$.

Theorem 3.2. *We have* $|\phi(\lambda)| = \#\{w \in \mathfrak{A}_{2n} : \mathrm{rp}(\hat{w}) = \lambda\}$.

Proof. It is known (see for example [9, (17), p. 259]) that

$$E_{2k-1} = 4^k(4^k - 1)\frac{|B_{2k}|}{2k}.$$

Therefore, if $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots \rangle$ then

$$|\phi(\lambda)| = (2n)! \cdot \prod_{k=1}^{n} \frac{1}{m_k!}\left(\frac{E_{2k-1}}{(2k)!}\right)^{m_k}. \tag{3.3}$$

[1] Do not confuse alternating permutations with even permutations, which are permutations in the alternating group.

The theorem will follow once we show that the right side of equation (3.3) counts the number of alternating permutations w of length $2n$ with record partition λ. So, we complete the proof by observing that each such w is obtained exactly once through the following process.

First, choose a set partition

$$\pi = \{P_1, \dots, P_\ell\}$$

of $[2n]$ such that if $\lambda = (\lambda_1, \dots, \lambda_\ell)$ then $|P_j| = 2\lambda_j$. There are

$$\frac{\binom{2n}{2\lambda_1,\dots,2\lambda_\ell}}{\prod_{k=1}^n m_k!} = \frac{(2n)!}{\prod_{k=1}^n m_k!(2k)!^{m_k}}$$

ways to choose π.

Second, for each $j \in [\ell]$ let p_j be the largest element of P_j and reorder the P_j if necessary so that $p_1 < p_2 < \cdots < p_\ell$.

Third, for each $j \in [\ell]$ choose an alternating permutation w^j of P_j in which p_j appears first. Since one obtains such a permutation by choosing a permutation $q_1 \dots q_{\#P_j - 1}$ of $P_j \setminus \{p_j\}$ such that $q_1 < q_2 > q_3 < \cdots$, the number of ways to choose all of the w^j is $\prod_{j=1}^\ell E_{2\lambda_j - 1}$.

Finally, we obtain w by concatenating the w^j so that w^j appears before w^{j+1} for each $j \in [\ell - 1]$. □

Open problem. We defined the record partition $\mathrm{rp}(\hat{w})$ to be certain numbers $r_2 - r_1, r_3 - r_2, \dots, n+1-r_j$ arranged in decreasing order. If we keep them in their given order, then we obtain the *record composition* $(r_2 - r_1, r_3 - r_2, \dots, n+1-r_j)$. Can we refine Theorem 3.2 by replacing record partitions with record compositions? In the theory of noncommutative symmetric functions [7] the role of partitions is replaced by compositions, so perhaps noncommutative symmetric functions arise in the putative refinement of Theorem 3.2.

4. The seed $\sec(\sqrt{t})$

The form of equation (3.3) suggests to someone sufficiently versed in the theory of symmetric functions that it might be worthwhile to define the symmetric function

$$A_n = A_n(\boldsymbol{x}) = \frac{1}{(2n)!} \sum_{\lambda \vdash n} |\phi(\lambda)| \cdot p_\lambda,$$

where p_λ is a power sum symmetric function.

Theorem 4.1. *The sequence $\mathfrak{A} = (A_0, A_1, \dots)$ is a sprout sequence with seed $F(t) = \sec(\sqrt{t})$.*

Proof. Comparing equations (2.3) and (3.3), and using the definition $z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots$, shows that $\mathfrak{A}$ is a sprout sequence with seed

$$F(t) = \exp\left(\sum_{n\geq 1} \frac{E_{2n-1}t^n}{(2n)!}\right).$$

By equation (3.2), we have

$$\sum_{n\geq 1} \frac{E_{2n-1}x^{2n-1}}{(2n-1)!} = \tan x.$$

Integrating both sides from 0 to t gives

$$\sum_{n\geq 1} \frac{E_{2n-1}t^{2n}}{(2n)!} = \log\sec(t). \tag{4.1}$$

Applying exp to both sides and substituting $\sqrt{t}$ for t gives $F(t) = \sec(\sqrt{t})$. □

We can ask about the expansion of A_n in terms of bases for $\Lambda_\mathbb{R}$ other than the power sums. Theorem 2.1(c) and equation (3.2) immediately give the monomial expansion:

$$(2n)!\, A_n = \sum_{\lambda\vdash n} \binom{2n}{2\lambda_1, 2\lambda_2, \ldots} E_{2\lambda_1}E_{2\lambda_2}\cdots m_\lambda.$$

We can interpret this monomial expansion of $(2n)!A_n$ combinatorially as follows.

Theorem 4.2. *Let $\lambda \vdash n$. Then $(2n)!\,[m_\lambda]A_n$ is equal to the number of permutations $c_1, c_2, \ldots, c_{2n} \in \mathfrak{S}_{2n}$ such that the first $2\lambda_1$ terms are alternating, i.e., $c_1 > c_2 < c_3 > \cdots > c_{2\lambda_1}$, then the next $2\lambda_2$ terms are alternating, then the next $2\lambda_3$ terms are alternating, etc.*

Proof. First choose the sets $\{c_1, c_2, \ldots, c_{2\lambda_1}\}$, $\{c_{2\lambda_1+1}, c_{2\lambda_1+2}, \ldots, c_{2\lambda_1+2\lambda_2}\}$, etc., in $\binom{2n}{2\lambda_1,2\lambda_2,\ldots}$ ways. Then arrange the elements of each set to be an alternating permutation in $E_{2\lambda_1}E_{2\lambda_2}, \ldots$ ways. Thus the number of permutations satisfying the conditions of the theorem is

$$\binom{2n}{2\lambda_1, 2\lambda_2, \ldots} E_{2\lambda_1}E_{2\lambda_2}\cdots,$$

which by Theorem 2.1(c) is equal to $(2n)!\,[m_\lambda]A_n$.

□

Example 4.3. The coefficient of m_{311} in $10!A_5$ is the number of permutations $c_1, c_2, \ldots, c_{10} \in \mathfrak{S}_{10}$ satisfying $c_1 > c_2 < c_3 > c_4 < c_5 > c_6$, $c_7 > c_8$, and $c_9 > c_{10}$, namely, 76860.

5. The h-expansion of $(2n)!A_n$

We turn to the h-expansion of A_n.

Theorem 5.1. *The symmetric function* $(2n)!A_n$ *is* h*-integral and* h*-positive.*

Proof. First, h-integrality is clear since $(2n)!A_n$ is m-integral, and the m-basis and h-basis are both integral bases (with respect to the lattice $\Lambda_{\mathbb{Z}}$) for $\Lambda_{\mathbb{R}}$.

The Weierstrass product formula for $\cos(t)$ asserts that

$$\cos(t) = \prod_{k\geq 1}\left(1 - \frac{4t^2}{\pi^2(2k-1)^2}\right).$$

Hence

$$F(t) = \sec(\sqrt{t}) = \prod_{j\geq 1}\left(1 - \frac{4t}{\pi^2(2j-1)^2}\right)^{-1}.$$

It follows from Theorem 2.11(a) that A_n is h-positive. □

Although Theorem 5.1 shows that $(2n)!A_n$ is h-integral and h-positive, it gives no idea what the coefficients are *as integers* in the h-expansion. Here is a table of these expansions for $1 \leq n \leq 5$.

$$\begin{aligned}
2!A_1 &= h_1\\
4!A_2 &= h_1^2 + 4h_2\\
6!A_3 &= h_1^3 + 12h_2h_1 + 48h_3\\
8!A_4 &= h_1^4 + 24h_2h_1^2 + 256h_3h_1 + 16h_2^2 + 1088h_4\\
10!A_5 &= h_1^5 + 40h_2h_1^3 + 800h_3h_1^2 + 80h_2^2h_1 + 9280h_4h_1\\
&\quad + 640h_3h_2 + 39680h_5.
\end{aligned}$$

Problem. Find a combinatorial interpretation of the coefficients in the h-expansion of $(2n)!\,A_n$. (Theorem 5.2(b) below suggests that the coefficients should count some property of alternating permutations in $\mathfrak{S}_{2n}$. This property should be indexed by partitions λ of n.) Toward this end we have the following result.

Theorem 5.2. *Regarding the* h*-expansion of* $(2n)!\,A_n$*, we have:*

- (a) *The coefficient of* h_1^n *is* 1.
- (b) *The sum of the coefficients is* E_{2n}.
- (c) *The coefficient of* h_n *is* nE_{2n-1}*, the number of* cyclically alternating *permutations* $w \in \mathfrak{A}_{2n}$*, i.e.,* $w = a_1a_2\cdots a_{2n} \in \mathfrak{A}_{2n}$ *and* $a_{2n} < a_1$.

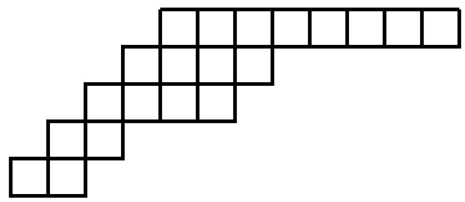

FIGURE 1. The skew shape $\rho(5,3,1,1)$

(d) *Write* $E'_{2n} = nE_{2n-1}$. *Then for* $i,j \geq 1$ *and* $n = i+j$,

$$[h_ih_j](2n)!A_n = \begin{cases} \binom{2n}{2i}E'_{2i}E'_{2j} - E'_{2n}, & i \neq j \\ \frac{1}{2}\left(\binom{2n}{n}(E'_n)^2 - E'_{2n}\right), & i = j = n/2. \end{cases}$$

Proof. (a) Immediate from the case $k=1$ of Theorem 2.7 and the formula $\frac{1}{F(-t)} = \sum_{n\geq 0}\frac{t^n}{(2n)!}$.

(b) Use Theorem 2.7(a).

(c) Use Theorem 2.7(c). To see that nE_{2n-1} is the number of cyclically alternating permutations of $[2n]$, take a reverse alternating permutation $w = c_1, c_2, \dots, c_{2n-1}$ in $\mathfrak{S}_{2n-1}$ (i.e., $c_1 < c_2 > c_3 < \cdots > c_{2n-1}$) and adjoin $2n$ at the end. The resulting word has n cyclic shifts that are cyclically alternating.

(d) Follows from Theorem 2.7(d).

□

6. The Schur expansion of $(2n)!A_n$

We now turn to the Schur expansion (or s-expansion) of A_n. Given $\lambda \vdash n$, let $\mu = 2\lambda'$, i.e., take the conjugate partition λ' to λ and double each part. Let $\rho(\lambda)$ be the skew partition (or skew shape) obtained from λ as follows: the row lengths of $\rho(\lambda)$ are the parts of μ, and each row of $\rho(\lambda)$ begins one square to the left of the row above. In symbols, $\rho(\lambda) = \sigma/\tau$, where $\sigma_i = 2\lambda' + \ell(\lambda') - i$ and $\tau_j = \ell(\lambda') - j$.

Example 6.1. Let $\lambda = (5,3,1,1) \vdash 10$. Then $\lambda' = (4,2,2,1,1)$, $\mu = (8,4,4,2,2)$, and $\rho(5,3,1,1) = (12,7,6,3,2)/(4,3,2,1)$, as illustrated in Figure 1.

Theorem 6.2. *For any* $\lambda \vdash n$ *we have* $\langle (2n)!A_n, s_\lambda\rangle = f^{\rho(\lambda)}$, *the number of standard Young tableaux of the skew shape* $\rho(\lambda)$.

Proof. By Theorem 2.6 we have that $(1, 2!\omega(A_1), 4!\omega(A_2), 6!\omega(A_3), \dots)$ is a sprout sequence with seed

$$\frac{1}{\sec\sqrt{-t}} = \sum_{n\geq 0}\frac{t^n}{(2n)!}.$$

By Corollary 2.4,

$$\langle (2n)!\omega(A_n), s_\lambda\rangle = (2n)! \det \left[\frac{1}{(2\lambda_i - 2i + 2j)!}\right]_{i,j=1}^{n}.$$

Hence

$$\begin{aligned}\langle (2n)! A_n, s_\lambda\rangle &= (2n)! \det \left[\frac{1}{(2\lambda'_i - 2i + 2j)!}\right]_{i,j=1}^{\ell(\lambda')} \\ &= (2n)! \det \left[\frac{1}{(\sigma_i - \tau_j - i + j)!}\right]_{i,j=1}^{\ell(\lambda')}. \qquad (6.1)\end{aligned}$$

By [18, Cor. 7.16.3], the right-hand side of equation (6.1) is equal to $f^{\rho(\lambda)}$, and the proof follows. □

7. A connection with chromatic symmetric functions and interval orders

To each perfect matching M on vertex set $[2n]$ we associate a partially ordered set P_M by declaring that $\{a,b\} <_{P_M} \{c,d\}$ if $\max\{a,b\} < \min\{c,d\}$. The posets P_M are interval orders: identify each edge $\{a,b\}$ ($a < b$) with the closed interval $[a,b]$ on the real line, and declare $\{a,b\} < \{c,d\}$ if $[a,b]$ lies entirely to the left of $[c,d]$. We write $\mathsf{Inc}(P)$ for the incomparability graph of a poset P and X_G for the chromatic symmetric function of a graph G (see [15]).

Theorem 7.1. *Given a positive integer n, let $\mathcal{M}(2n)$ be the set of all perfect matchings on the set $[2n]$. Then*

$$(2n)! A_n = \sum_{M \in \mathcal{M}(2n)} \omega X_{\mathsf{Inc}(P_M)}.$$

Proof. Given $M \in \mathcal{M}(2n)$, let S_M be the set of ordered lists $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ of the elements of P_M. Each $\sigma \in S_M$ has descent set

$$\mathbf{DES}(\sigma) := \{i \in [n-1] : \sigma_i \not<_{P_M} \sigma_{i+1}\}$$

and ascent set $\mathbf{ASC}(\sigma) := [n-1] \setminus \mathbf{DES}(\sigma)$. Given $T \subseteq [n-1]$, we write $L_{T,n}$ for the associated fundamental quasisymmetric function and $M_{T,n}$ for the associated monomial quasisymmetric function (see for example [18, Section 7.19] for definitions). By [3, Corollary 2],

$$\begin{aligned} X_{\mathsf{Inc}(P_M)} &= \sum_{\sigma \in S_M} L_{\mathbf{DES}(\sigma),n} \\ &= \sum_{\sigma \in S_M} \sum_{\mathbf{DES}(\sigma) \subseteq T \subseteq [n-1]} M_{T,n}, \end{aligned}$$

the second equality following from [18, Theorem 7.19.1].

By [18, Exercise 7.94],

$$\omega X_{\mathsf{Inc}(P_M)} = \sum_{\sigma \in S_M} \sum_{\mathbf{ASC}(\sigma) \subseteq T \subseteq [n-1]} M_{T,n}.$$

We define

$$\iota : \bigcup_{M \in \mathcal{M}(2n)} S_M \to \mathfrak{S}_{2n}$$

as follows. If $\sigma = \sigma_1 \dots \sigma_n \in S_M$ with $\sigma_i = \{a_i, b_i\}$ and $a_i < b_i$ for each $i \in [n]$, then

$$\iota(\sigma) := a_1 b_1 a_2 b_2 \dots a_n b_n.$$

Writing $\mathbf{ASC}(w)$ for the usual ascent set of $w \in \mathfrak{S}_{2n}$, we observe that

$$\mathbf{ASC}(\iota(\sigma)) = \{1, 3, 5, \dots, 2n-1\} \cup \{2j : j \in \mathbf{ASC}(\sigma)\}.$$

Moreover, ι is injective with image

$$\iota\left(\bigcup_{M \in \mathcal{M}(2n)} S_M\right) = \{w \in \mathfrak{S}_{2n} : \{1, 3, 5, \dots, 2n-1\} \subseteq \mathbf{ASC}(w)\}.$$

For a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n and $j \in [\ell - 1]$, we set

$$v_j(\lambda) := \sum_{i=1}^{j} \lambda_i$$

and define

$$T(\lambda) := \{v_j(\lambda) : j \in [\ell - 1]\}.$$

We observe now that the coefficient of m_λ in the monomial symmetric expansion of $\sum_{M \in \mathcal{M}(2n)} \omega X_{\mathsf{Inc}(P_M)}$ is the same as the coefficient of $M_{T(\lambda),n}$ in the monomial quasisymmetric expansion. We have shown that this second coefficient is the number of $w \in \mathfrak{S}_{2n}$ satisfying

$$\{1, 3, 5, \dots, 2n-1\} \subseteq \mathbf{ASC}(w) \subseteq \{1, 3, 5, \dots, 2n-1\} \cup \{2v_j(\lambda) : j \in [\ell-1]\},$$

which is equal to the number of $w \in \mathfrak{S}_{2n}$ satisfying

$$\{1, 3, 5, \dots, 2n-1\} \subseteq \mathbf{DES}(w) \subseteq \{1, 3, 5, \dots, 2n-1\} \cup \{2v_j(\lambda) : j \in [\ell-1]\}.$$

The proof now follows from Theorem 4.2. □

We remark that even though $\sum_{M \in \mathcal{M}(2n)} \omega X_{\mathsf{Inc}(P_M)}$ is h-positive, there are M such that $X_{\mathsf{Inc}(P_M)}$ is not Schur positive. One example is $M = \{\{1, 8\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}$, for which P_M is the complete bipartite graph $K_{1,3}$. See [15, p. 186].

Theorem 7.1 suggests the following general question: are there other "interesting" sums (or more generally, linear combinations) of chromatic symmetric functions that are e-positive?

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