Some Combinatorial Properties of Hook Lengths, Contents, and Parts of Partitions[1]

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Dedicated to George Andrews for his 70th birthday

Abstract

The main result of this paper is a generalization of a conjecture of Guoniu Han, originally inspired by an identity of Nekrasov and Okounkov. Our result states that if F is any symmetric function (say over $\mathbb{Q}$) and if

$$\Phi_n(F) = \frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2\, F(h_u^2\ :\ u\in\lambda),$$

where h_u denotes the hook length of the square u of the partition λ of n and f_λ is the number of standard Young tableaux of shape λ, then $\Phi_n(F)$ is a polynomial function of n. A similar result is obtained when $F(h_u^2\ :\ u\in\lambda)$ is replaced with a function that is symmetric separately in the contents c_u of λ and the shifted parts λ_i+n-i of λ.

1 Introduction.

We assume basic knowledge of symmetric functions such as given in [13, Ch. 7]. Let f_λ denote the number of standard Young tableaux (SYT) of shape $\lambda\vdash n$. Recall the hook length formula of Frame, Robinson, and Thrall [3][13, Cor. 7.21.6]:

$$f_\lambda = \frac{n!}{\prod_{u\in\lambda} h_u},$$

where u ranges over all squares in the (Young) diagram of λ, and h_u denotes the hook length at u. A basic property of the numbers f_λ is the formula

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!,$$

which has an elegant bijective proof (the RSK algorithm). We will be interested in generalizing this formula by weighting the sum on the left by various functions of λ. Our primary interest is the sum

$$\Phi_n(F) = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \, F(h_u^2 \,:\, u \in \lambda),$$

where $F = F(x_1, x_2, \dots)$ is a symmetric function, say over $\mathbb{Q}$ (denoted $F \in \Lambda_{\mathbb{Q}}$). The notation $F(h_u^2 \,:\, u \in \lambda)$ means that we are substituting for n of the variables in F the quantities h_u^2 for $u \in \lambda$, and setting all other variables equal to 0. For instance, if $F = p_k := \sum x_i^k$, then

$$F(h_u^2 \,:\, u \in \lambda) = \sum_{u \in \lambda} h_u^{2k}.$$

This paper is motivated by the conjecture [7, Conj. 3.1] of Guoniu Han that for all $k \in \mathbb{P} = \{1, 2, \dots\}$, we have that $\Phi_n(p_k) \in \mathbb{Q}[n]$, i.e.,

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \sum_{u \in \lambda} h_u^{2k}$$

is a polynomial function of n. This conjecture in turn was inspired by the remarkable identity of Nekrasov and Okounkov [10] (later given a more elementary proof by Han [6])

$$\sum_{n \geq 0} \left(\sum_{\lambda \vdash n} f_\lambda^2 \prod_{u \in \lambda} (t + h_u^2) \right) \frac{x^n}{n!^2} = \prod_{i \geq 1} (1 - x^i)^{-1-t}. \qquad (1)$$

(We have stated this identity in a slightly different form than given in [6][10].) Our main result (Theorem 4.3) states that $\Phi_n(F) \in \mathbb{Q}[n]$ for any $F \in \Lambda_{\mathbb{Q}}$, i.e., for fixed F, $\Phi_n(F)$ is a polynomial function of n. In the course of the proof we also show that

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \, G(\{c_u \,:\, u \in \lambda\}; \{\lambda_i + n - i \,:\, 1 \leq i \leq n\}) \in \mathbb{Q}[n].$$

Here $G = G(x; y)$ is any formal power series of bounded degree over $\mathbb{Q}$ that is symmetric in the x and y variables separately. Moreover, c_u denotes the content of $u \in \lambda$ [13, p. 373]; and we write $\lambda = (\lambda_1, \dots, \lambda_n)$, adding 0's at the end so that there are exactly n parts.

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2 Contents.

In the next section we will obtain a stronger result than the main result of this section (Theorem 2.1). Since Theorem 2.1 may be of independent interest and may be helpful for understanding the next section, we treat it separately.

If $t \in \mathbb{P}$ and F is a symmetric function in the variables $x_1, x_2, \dots$, then we write $F(1^t)$ for the result of setting $x_1 = x_2 = \cdots = x_t = 1$ and all other $x_j = 0$ in F. For instance, $p_\lambda(1^t) = t^{\ell(\lambda)}$, where $\ell(\lambda)$ is the number of (positive) parts of λ. The *hook-content* formula for the case $q = 1$ [13, Cor. 7.21.4] asserts that

$$s_\lambda(1^t) = \frac{\prod_{u \in \lambda}(t + c_u)}{H_\lambda}, \tag{2}$$

where s_λ is a Schur function and

$$H_\lambda = \prod_{u \in \lambda} h_u,$$

the product of the hook lengths of λ (so $f_\lambda = n!/H_\lambda$).

Theorem 2.1. *For any $F \in \Lambda_{\mathbb{Q}}$ we have*

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \, F(c_u \, : \, u \in \lambda) \in \mathbb{Q}[n].$$

Proof. By linearity it suffices to take $F = e_\mu$, the elementary symmetric function indexed by μ. Let $k \in \mathbb{P}$, and for $1 \le i \le k$ let $x^{(i)}$

denote the set of variables $x_1^{(i)}, x_2^{(i)}, \dots$. Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $\{1, \dots, n\}$. For $w \in \mathfrak{S}_n$ write $\rho(w)$ for the cycle type of w, i.e., $\rho(w)$ is the partition of n whose parts are the cycle lengths of w. We use the identity [5, Prop. 2.2][13, Exer. 7.70]

$$\sum_{\lambda \vdash n} H_\lambda^{k-2} s_\lambda(x^{(1)}) \cdots s_\lambda(x^{(k)}) = \frac{1}{n!} \sum_{\substack{w_1 w_2 \cdots w_k = 1 \\ \text{in } \mathfrak{S}_n}} p_{\rho(w_1)}(x^{(1)}) \cdots p_{\rho(w_k)}(x^{(k)}). \tag{3}$$

Make the substitution $x^{(i)} = 1^{t_i}$ as explained above. Letting $c(w)$ denote the number of cycles of $w \in \mathfrak{S}_n$, we obtain

$$\sum_{\lambda \vdash n} H_\lambda^{-2} \prod_{u \in \lambda} \prod_{i=1}^{k} (t_i + c_u) = \frac{1}{n!} \sum_{\substack{w_1 w_2 \cdots w_k = 1 \\ \text{in } \mathfrak{S}_n}} t_1^{c(w_1)} \cdots t_k^{c(w_k)}. \tag{4}$$

For any $n \geq \mu_1$ let $\mu = (\mu_1, \dots, \mu_k)$ be a partition with k parts, and take the coefficient of $t_1^{n-\mu_1} \cdots t_k^{n-\mu_k}$ on both sides of equation (4). Using $f_\lambda = n!/H_\lambda$, we obtain

$$\begin{aligned} &\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \, e_\mu(c_u \, : \, u \in \lambda) \\ &= \#\{(w_1, \cdots, w_k) \in \mathfrak{S}_n^k \, : \, w_1 \cdots w_k = 1, \; c(w_i) = n - \mu_i\}. \end{aligned} \tag{5}$$

We therefore need to show that the right-hand side of equation (5) is a polynomial function of n.

Suppose that $c(w_i) = n - \mu_i$ and that the union F of the non-fixed points of all the w_i's has r elements. Then

$$1 + \mu_1 \leq r \leq 2 \sum \mu_i. \tag{6}$$

We can choose the set F in $\binom{n}{r}$ ways. Once we make this choice there is a certain number of ways (depending on r but independent of n) that we can have $w_1 \cdots w_k = 1$. (In more algebraic terms, $\mathfrak{S}_n$ acts on S_μ by conjugation, where S_μ is the set on the right-hand side of (5), and the number of orbits of this action is independent of n.) Hence for $n \geq 1 + \mu_1$, $\#S_\mu$ is a finite linear combination (over $\mathbb{N} = \{0, 1, 2, \dots\}$) of polynomials $\binom{n}{r}$, and is thus a polynomial $N_\mu(n)$ as desired.

If $n < 1+\mu_1$, then it is clear from the previous paragraph that the polynomial N_μ satisfies $N_\mu(n) = 0$. On the other hand, if $\lambda \vdash n$ then we also have $e_\mu(c_u : u \in \lambda) = 0$. Hence the two sides of equation (5) agree for $0 \le n < 1 + \max \mu_i$, and the proof is complete. $\square$

Note that the proof of Theorem 2.1 shows that $N_\mu(n)$ is a *nonnegative* integer linear combination of the polynomials $\binom{n}{r}$. It can be shown that either $N_\mu = 0$ or $\deg N_\mu = \sum \mu_i$. Moreover $N_\mu \ne 0$ if and only $\sum \mu_i$ is even, say $2r$, and $\mu_1 \le r$. The nonzero polynomials $N_\mu(n)$ for $|\mu| \le 6$ are given by

$$\begin{aligned}
N_{1,1}(n) &= \frac{n(n-1)}{2} \\
N_{2,2}(n) &= \frac{n(n-1)(n-2)(3n-1)}{24} \\
N_{2,1,1}(n) &= \frac{n(n-1)(n-2)(n+1)}{4} \\
N_{1,1,1,1}(n) &= \frac{n(n-1)(3n^2+n-12)}{4} \\
N_{3,3}(n) &= \frac{n^2(n-1)^2(n-2)(n-3)}{48} \\
N_{3,2,1}(n) &= \frac{n(n-1)(n-2)(n-3)(3n^2+5n+4)}{48} \\
N_{3,1,1,1}(n) &= \frac{n(n-1)(n-2)(n-3)(n^2+3n+4)}{8} \\
N_{2,2,2}(n) &= \frac{n(n-1)(n-2)(3n^3-9n-46)}{24} \\
N_{2,2,1,1}(n) &= \frac{n(n-1)(n-2)(15n^3+20n^2-59n-312)}{48}
\end{aligned}$$

$$N_{2,1,1,1,1}(n) = \frac{n(n-1)(n-2)(3n^3+8n^2-7n-96)}{4}$$

$$N_{1,1,1,1,1,1}(n) = \frac{n(n-1)(15n^4+30n^3-105n^2-700n+1344)}{8}.$$

A slight modification of the proof of a special case of Theorem 2.1 leads to a "content Nekrasov-Okounkov formula."

Theorem 2.2. *We have*

$$\sum_{n\geq 0}\left(\sum_{\lambda\vdash n} f_\lambda^2 \prod_{u\in\lambda}(t+c_u^2)\right)\frac{x^n}{n!^2} = (1-x)^{-t}.$$

Proof. By the "dual Cauchy identity" [13, Thm. 7.14.3] we have

$$\sum_{\lambda\vdash n} s_\lambda(x)s_{\lambda'}(y) = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\varepsilon_w p_{\rho(w)}(x)p_{\rho(w)}(y),$$

where $\varepsilon(w)$ is given by equation (15), and where λ' denotes the conjugate partition to λ. Substitute $x=1^t$ and $y=1^t$. Since the contents of λ' are the negative of those of λ, we obtain

$$\sum_{\lambda\vdash n} H_\lambda^{-2}\prod_{u\in\lambda}(t^2-c_u^2) = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\varepsilon_w t^{2c(w)}.$$

It is a well-known and basic fact that the sum on the right is $\binom{t^2}{n}$. Put $-t$ for t^2, multiply by $(-x)^n$ and sum on $n\geq 0$ to get the stated formula. □

A simple variant of Theorem 2.2 follows from considering the usual Cauchy identity (the case $k=2$ of equation (3)) instead of the dual one:

$$\sum_{n\geq 0}\left(\sum_{\lambda\vdash n} f_\lambda^2 \prod_{u\in\lambda}(t+c_u)(v+c_u)\right)\frac{x^n}{n!^2} = (1-x)^{-tv}.$$

A related identity is due to Fujii *et al.* [4, Appendix], namely, for any $r\geq 0$ we have

$$\frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2\sum_{u\in\lambda}\prod_{i=0}^{r-1}(c_u^2-i^2) = \frac{(2r)!}{(r+1)!^2}\langle n\rangle_{r+1}, \tag{7}$$

where $\langle n\rangle_{r+1} = n(n-1)\cdots(n-r)$. It follows from this formula that

$$\frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2\sum_{u\in\lambda}c_u^{2k} = \sum_{j=1}^{k}T(k,j)\frac{(2j)!}{(j+1)!^2}\langle n\rangle_{j+1}, \tag{8}$$

where $T(k,j)$ is a *central factorial number* [13, Exer. 5.8]. One of several equivalent definitions of $T(k,j)$ is the explicit formula

$$T(k,j) = 2\sum_{i=1}^{j}\frac{(-1)^{j-i}i^{2k}}{(j-i)!(j+i)!}.$$

Another definition is the generating function

$$\sum_{k\geq 0}T(k,j)x^k = \frac{x^j}{(1-1^2x)(1-2^2x)\cdots(1-j^2x)}. \tag{9}$$

The equivalence of equations (7) and (8) is a simple consequence of (9). For "hook length analogues" of equations (7) and (8), see the Note at the end of Section 4.

3 Shifted parts.

In this section we write partitions λ of n as $(\lambda_1,\ldots,\lambda_n)$, placing as many 0's at the end as necessary. Thus for instance the three partitions of 3 are $(3,0,0)$, $(2,1,0)$, and $(1,1,1)$. Let $G(x;y)$ be a formal power series over $\mathbb{Q}$ of bounded degree that is symmetric in the variables $x=(x_1,x_2,\ldots)$ and $y=(y_1,y_2,\ldots)$ separately; in symbols, $G\in\Lambda_{\mathbb{Q}}[x]\otimes\Lambda_{\mathbb{Q}}[y]$. We are interested in the quantity

$$\Psi_n(G) = \frac{1}{n!}\sum_{\lambda\vdash n}f_\lambda^2\,G(\{c_u\,:\,u\in\lambda\};\{\lambda_i+n-i\,:\,1\leq i\leq n\}). \tag{10}$$

The case $y_i=0$ for all i reduces to what was considered in the previous section. We will show that $\Psi_n(G)$ is a polynomial in n by an argument similar to the proof of Theorem 2.1. In addition to the substitution $x^{(i)}=1^{t_i}$ we use a certain linear transformation φ which we now define.

Let $x^{(1)}, \dots, x^{(j)}$ and $y^{(1)}, \dots, y^{(k)}$ be disjoint sets of variables. We will work in the ring R of all bounded formal power series over $\mathbb{Q}$ that are symmetric in each set of variables separately. Define a map $\varphi\colon R \to \mathbb{Q}[v_1, \dots, v_k]$ by the conditions:

- The map φ is linear over $\Lambda_{\mathbb{Q}}[x^{(1)}] \otimes \cdots \otimes \Lambda_{\mathbb{Q}}[x^{(j)}]$, i.e, the $x^{(i)}$-variables are treated as scalars.

- We have

$$\varphi\left(s_\lambda(y^{(h)})\right) = \frac{\prod_{i=1}^n (v_h + \lambda_i + n - i)}{H_\lambda},$$

  where $\lambda \vdash n$.

- We have

$$\varphi\left(G_1(y^{(1)}) \cdots G_k(y^{(k)})\right) = \varphi\left(G_1(y^{(1)})\right) \cdots \varphi\left(G_k(y^{(k)})\right),$$

  where $G_h \in \Lambda_{\mathbb{Q}}[x^{(1)}, \dots, x^{(j)}, y^{(h)}]$.

More algebraically, let $\Psi = \Lambda_{\mathbb{Q}}[x^{(1)}] \otimes \cdots \otimes \Lambda_{\mathbb{Q}}[x^{(j)}]$, and let $\varphi_h\colon \Psi[y^{(h)}] \to \mathbb{Q}[v_h]$ be the Ψ-linear transformation defined by

$$\varphi_h(s_\lambda(y^{(h)})) = H_\lambda^{-1} \prod_{i=1}^n (v_h + \lambda_i + n - i).$$

Then $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_k$ (tensor product over Ψ).

Write for simplicity f for $f(y^{(1)})$ and v for v_1. We would like to evaluate $\varphi(p_\mu)$, where p_μ is a power-sum symmetric function. We first need the following lemma. Define

$$A_\lambda(v) = H_\lambda^{-1}(v + \lambda_1 + n - 1)(v + \lambda_2 + n - 2) \cdots (v + \lambda_n).$$

Lemma 3.1. *For all $n \geq 0$ we have*

$$\sum_{i=0}^n \binom{v+i-1}{i} p_1^i e_{n-i} = \sum_{\lambda \vdash n} A_\lambda(v) s_\lambda. \tag{11}$$

Equivalently, we have

$$(1 - p_1)^{-v} \sum_{n \geq 0} e_n = \sum_{n \geq 0} \sum_{\lambda \vdash n} A_\lambda(v) s_\lambda.$$

First proof (sketch). I am grateful to Guoniu Han for providing the following proof. Complete details may be found in his paper [8]. Denote the left-hand side of equation (11) by $L_n(v)$ and the right-hand side by $R_n(v)$. It is easy to see that $L_n(v) = L_n(v-1) + p_1 L_{n-1}(v)$, $L_n(0) = R_n(0)$, and $L_0(v) = R_0(v)$. Hence we need to show that

$$R_n(v) = R_n(v-1) + p_1 R_{n-1}(v). \tag{12}$$

Now for $\lambda \vdash n$ let

$$E_\lambda(v) = A_\lambda(v+n+1) - A_\lambda(v+n) - \sum_{\mu \in \lambda \backslash 1} A_\mu(v+n+1),$$

where $\lambda\backslash 1$ denotes the set of all partitions μ obtained from λ by removing one corner. Clearly $E_\lambda(v)$ is a polynomial in v of degree at most n, and it is not difficult to check that the degree in fact is at most $n-2$. The core of the proof (which we omit) is to show that $E_\lambda(i - \lambda_i) = 0$ for $i = 1, 2, \ldots, n-1$. Since $E_\lambda(v)$ has degree at most $n-2$ and vanishes at $n-1$ distinct integers, we conclude that $E_\lambda(v) = 0$. It is now straightforward to verify that equation (12) holds. □

Second proof. I am grateful to Tewodros Amdeberhan for helpful discussions. A formula of Andrews, Goulden, and Jackson [2] asserts that

$$\sum_\lambda s_\lambda(y_1,\ldots,y_n)s_\lambda(z_1,\ldots,z_m)\prod_{i=1}^n(v-\lambda_i-n+i)$$

$$= \prod_{j=1}^n\prod_{k=1}^m \frac{1}{1-y_jz_k}\cdot[t_1\cdots t_n](1+t_1+\cdots+t_n)^v \prod_{k=1}^m\left(1-\sum_{j=1}^n \frac{t_jy_jz_k}{1-y_jz_k}\right),$$

where the sum is over all partitions λ satisfying $\ell(\lambda) \leq n$, and where $[t_1\cdots t_n]X$ denotes the coefficient of $t_1\cdots t_n$ in X. Change v to $-v$ and multiply by $(-1)^n$ to get

$$\sum_\lambda s_\lambda(y_1,\ldots,y_n)s_\lambda(z_1,\ldots,z_m)\prod_{i=1}^n(v+\lambda_i+n-i)$$

$$= (-1)^n \prod_{j=1}^{n} \prod_{k=1}^{m} \frac{1}{1 - y_j z_k} \cdot$$

$$[t_1 \cdots t_n](1 + t_1 + \cdots + t_n)^{-v} \prod_{k=1}^{m} \left(1 - \sum_{j=1}^{n} \frac{t_j y_j z_k}{1 - y_j z_k}\right).$$

Let $m = n$, and take the coefficient of $z_1 \cdots z_n$ on both sides. The left-hand side becomes

$$\sum_{\lambda \vdash n} f_\lambda s_\lambda(y) \prod_{i=1}^{n} (v + \lambda_i + n - i).$$

Consider the coefficient of $z_1 \cdots z_n$ on the right-hand side. A term from this coefficient is obtained as follows. Pick a subset S of $[n] = \{1, 2, \ldots, n\}$, say $\#S = r$. Choose the coefficient of $\prod_{i \in S} z_i$ from $\prod_{j=1}^{n} \prod_{k=1}^{n} (1 - y_j z_k)^{-1}$. This coefficient is $p_1(y)^r$, and there are $\binom{n}{r}$ choices for S. We now must choose the coefficient $\prod_{i \in [n]-S} z_i$ from $\prod_{k=1}^{n} \left(1 - \sum_{j=1}^{n} \frac{t_j y_j z_k}{1 - y_j z_k}\right)$. This coefficient is $(-1)^{n-r}(t_1 y_1 + \cdots + t_n y_n)^{n-r}$. Hence

$$\sum_{\lambda \vdash n} f_\lambda s_\lambda(y) \prod_{i=1}^{n} (t + \lambda_i + n - i)$$

$$= (-1)^n \sum_{r=0}^{n} \binom{n}{r} p_1(y)^r [t_1 \cdots t_n] \frac{(-1)^{n-r}(t_1 y_1 + \cdots + t_n y_n)^{n-r}}{(1 + t_1 + \cdots + t_n)^{-v}}.$$

Let $\{i_1, \ldots, i_{n-r}\}$ be an $(n-r)$-element subset of $[n]$, and let $\{j_1, \ldots, j_r\}$ be its complement. Then

$$[t_{i_1} \cdots t_{i_{n-r}}](t_1 y_1 + \cdots + t_n y_n)^{n-r} = (n-r)! y_{i_1} \cdots y_{i_{n-r}}$$

$$[t_{j_1} \cdots t_{j_r}](1 + t_1 + \cdots + t_n)^{-v} = \binom{-v}{r} r!.$$

Hence

$$\sum_{\lambda \vdash n} f_\lambda s_\lambda(y) \prod_{i=1}^{n} (t + \lambda_i + n - i)$$

$$= \sum_{r=0}^{n} r!(n-r)! \binom{n}{r} p_1(y)^r (-1)^r \binom{-v}{r} e_{n-r}(y). \qquad (13)$$

Write $(-1)^r \binom{-v}{r} = \binom{v+r-1}{r}$ and divide both sides of equation (13) by $n!$ to complete the proof. □

NOTE. (a) Amdeberhan [1] has simplified the second proof of Lemma 3.1; in particular, he avoids the use of the Andrews-Goulden-Jackson formula.

(b) Since the left-hand side of equation (11) is an *integral* linear combination of Schur functions when $v \in \mathbb{Z}$ (e.g., by Pieri's rule), it follows that for every $v \in \mathbb{Z}$ we have $A_\lambda(v) \in \mathbb{Z}$. By expanding the left-hand side of (11) in terms of Schur functions, we in fact obtain the following combinatorial expression for $A_\lambda(v)$:

$$A_\lambda(v) = \sum_{i=0}^{n} \binom{v+i-1}{i} f_{\lambda/1^{n-i}},$$

where $f_{\lambda/1^{n-i}}$ denotes the number of SYT of the skew shape $\lambda/1^{n-i}$.

We now turn to the evaluation of $\varphi(p_\mu)$.

Lemma 3.2. *For any partition $\mu \vdash n$ with $\ell = \ell(\mu)$ nonzero parts, we have*

$$\varphi(p_\mu) = (-1)^{n-\ell} \sum_{i=0}^{m} \binom{m}{i} (v)_i,$$

where $m = m_1(\mu)$, the number of parts of μ equal to 1, and $(v)_i = v(v+1)\cdots(v+i-1)$.

Proof. We will work with two sets of variables $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. Recall that φ acts on symmetric functions in y only, regarding symmetric function in x as scalars. Thus using Lemma 3.1 we have

$$\begin{aligned} \varphi \sum_{\lambda \vdash n} s_\lambda(x) s_\lambda(y) &= \sum_{\lambda \vdash n} A_\lambda(v) s_\lambda(x). \\ &= \sum_{i=0}^{n} \binom{v+i-1}{i} p_1^i e_{n-i}. \qquad (14) \end{aligned}$$

A standard symmetric function identity [13, (7.23)] states that

$$e_{n-i} = \sum_{\rho \vdash n-i} \varepsilon_\rho z_\rho^{-1} p_\rho,$$

where

$$\varepsilon_\rho = (-1)^{|\rho|-\ell(\rho)}, \tag{15}$$

and if ρ has m_i parts equal to i then $z_\rho = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$. Let ν be the partition obtained from μ by removing all parts equal to 1. Write $(\nu, 1^j)$ for the partition obtained from ν by adjoining j 1's, so $\mu = (\nu, 1^m)$. Note that

$$\varepsilon_{(\nu,1^{m-i})} = (-1)^{|\nu|+m-i-\ell(\nu)-(m-i)} = (-1)^{|\nu|-\ell(\nu)} = (-1)^{n-\ell(\mu)}.$$

Note also that

$$z_{(\nu,1^{m-i})} = \frac{(m-i)!}{m!} z_\mu.$$

Hence if we expand the right-hand side of equation (14) in terms of power sum symmetric functions, then the coefficient of p_μ is

$$\sum_{i=0}^{m} \binom{v+i-1}{i} \varepsilon_{(\nu,1^{m-i})} z_{(\nu,1^{m-i})}^{-1}$$

$$= (-1)^{n-\ell} \sum_{i=0}^{m} \binom{m}{i} (v)_i z_\mu^{-1}. \tag{16}$$

It follows from the Cauchy identity [13, Thm. 7.12.1] (and is also the special case $k = 2$ of equation (3)) that

$$\sum_{\lambda \vdash n} s_\lambda(x) s_\lambda(y) = \sum_{\mu \vdash n} z_\mu^{-1} p_\mu(x) p_\mu(y). \tag{17}$$

Thus when we apply φ (acting on the y variables) to equation (17) and use (16), then we obtain

$$\sum_{\mu \vdash n} \varphi(p_\mu(y)) p_\mu(x)$$

$$= \sum_{\mu \vdash n} \left((-1)^{n-\ell(\mu)} \sum_{i=0}^{m} \binom{m}{i} (v)_i \right) z_\mu^{-1} p_\mu(x).$$

Since the p_μ's are linearly independent, the proof follows. ☐

Theorem 3.3. *For any $G \in \Lambda_{\mathbb{Q}}[x] \otimes \Lambda_{\mathbb{Q}}[y]$ we have*

$$\Psi_n(G) \in \mathbb{Q}[n],$$

where $\Psi_n(G)$ is given by equation (10).

Proof. By linearity it suffices to take $G = e_\mu(x)e_\nu(y)$. Apply φ to the identity (3) in the variables $x^{(1)}, \dots, x^{(j)}$, $y^{(1)}, \dots, y^{(k)}$. Then make the substitution $x^{(h)} = 1^{t_h}$ and multiply by $n!$. By equation (2) and Lemma 3.2 we obtain

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \prod_{h=1}^{j} \prod_{u \in \lambda} (t_h + c_u) \cdot \prod_{h=1}^{k} \prod_{i=1}^{n} (v_h + \lambda_i + n - i)$$

$$= \sum_{\substack{w_1 \cdots w_j w'_1 \cdots w'_k = 1 \\ \text{in } \mathfrak{S}_n}} \prod_{h=1}^{j} t_h^{c(w_h)}$$

$$\cdot \prod_{h=1}^{k} \left((-1)^{n-\ell(\rho(w'_h))} \sum_{i=0}^{m_1(\rho(w'_h))} \binom{m_1(\rho(w'_h))}{i} (v_h)_i \right). \tag{18}$$

The remainder of the proof is a straightforward generalization of that of Theorem 2.1. Take the coefficient of $t_1^{n-\mu_1} \cdots t_j^{n-\mu_j} v_1^{n-\nu_1} \cdots v_k^{n-\nu_k}$. The left-hand side becomes $\Psi_n(e_\mu(x)e_\nu(y))$, so we need to show that the coefficient of $t_1^{n-\mu_1} \cdots t_j^{n-\mu_j} v_1^{n-\nu_1} \cdots v_k^{n-\nu_k}$ on the right-hand side of equation (18) is a polynomial in n. Suppose that $n \geq \mu_1$ and $n \geq \nu_1$. The coefficient of $v_h^{n-\nu_h}$ in $v_h(v_h+1)\cdots(v_h+n-i-1)$ is the signless Stirling number $c(n-i, n-\nu_h)$. The coefficient of $v_h^{n-\nu_h}$ in (18) is 0 unless $n - m_1(\rho(w'_h)) \leq i \leq \nu_h$. For each choice of $0 \leq i_h \leq i$ $(1 \leq h \leq k)$, there are only finitely many orbits of the action of $\mathfrak{S}_n$ by (coordinatewise) conjugation on the set of $(w_1, \dots, w_j, w'_1, \cdots, w'_k) \in \mathfrak{S}_n^{j+k}$ for which $w_1 \cdots w_j w'_1 \cdots w'_k = 1$, w_h has $n - \mu_h$ cycles, and w'_h has $n - i_h$ fixed points. The size of each of these orbits is a polynomial in n, as in the proof of Theorem 2.1. Moreover, the Stirling number $c(n-i, n-\nu_h)$ is a polynomial in n for fixed i and ν_h, and similarly for the binomial coefficient $\binom{n-i_h}{n-i}$, so $\Psi_n(e_\mu(x)e_\nu(y))$ is a polynomial $N_{\mu,\nu}(n)$ for $n \geq \max\{\mu_1, \nu_1\}$. If $0 \leq$

$n < \max\{\mu_1, \nu_1\}$, then both $N_{\mu,\nu}(n)$ and $\Psi_n(e_\mu(x)e_\nu(y))$ are equal to 0 (as in the proof of Theorem 2.1), so the proof is complete. □

NOTE. Since n is a polynomial in n, it is easy to see that Theorem 3.3 still holds if we replace $\Psi_n(G)$ with

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2\, G(\{c_u\,:\,u\in\lambda\};\{\lambda_i - i\,:\,1\le i\le n\}).$$

On the other hand, Theorem 3.3 becomes *false* if we replace $\Psi_n(G)$ with

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2\, G(\{c_u\,:\,u\in\lambda\};\{\lambda_i\,:\,1\le i\le n\}).$$

For instance,

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2)$$

is not a polynomial function of n, nor is it integer valued.

4 Hook lengths squared.

The connection between contents, hook lengths, and the shifted parts $\lambda_i + n - i$ is given by the following result, an immediate consequence [13, Lemma 7.21.1].

Lemma 4.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \vdash n$. Then we have the multiset equality*

$$\begin{aligned}\{h_u\,:\,u\in\lambda\}&\cup\{\lambda_i-\lambda_j-i+j\,:\,1\le i<j\le n\}\\ &=\{n+c_u\,:\,u\in\lambda\}\cup\{1^{n-1},2^{n-2},\ldots,n-1\}.\end{aligned}$$

For example, when $\lambda = (3,1)$ Lemma 4.1 asserts that

$$\{4,2,1,1\}\cup\{3,5,6,2,3,1\}=\{3,4,5,6\}\cup\{1,1,1,2,2,3\}$$

as multisets.

Lemma 4.2. *For any $F \in \Lambda_{\mathbb{Q}}$, we have*

$$F(1^{n-1},2^{n-2},\ldots,n-1)\in\mathbb{Q}[n],$$

where the exponents denote multiplicity.

Proof. It suffices to take $F = p_j$ since the polynomials in n form a ring. Thus we want to show that

$$\sum_{i=1}^{n-1} (n-i)i^j \in \mathbb{Q}[n],$$

which is routine. □

We come to the main result of this paper. Recall the definition

$$\Phi_n(F) = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \, F(h_u^2 \, : \, u \in \lambda).$$

Theorem 4.3. *For any symmetric function $F \in \Lambda_{\mathbb{Q}}$ we have $\Phi_n(F) \in \mathbb{Q}[n]$.*

Proof. As usual it suffices to take $F = e_\mu$, where $\mu = (\mu_1, \ldots, \mu_k)$. Define the multisets (or *alphabets*)

$$\begin{aligned} A_\lambda &= \{h_u^2 \, : \, u \in \lambda\} \\ B_\lambda &= \{(\lambda_i - \lambda_j - i + j)^2 \, : \, 1 \le i < j \le n\} \\ C_\lambda &= \{(n + c_u)^2 \, : \, u \in \lambda\} \\ D_n &= \{b_1^{n-1}, b_2^{n-2}, \ldots, b_{n-1}\}, \end{aligned}$$

where $b_i = i^2 \in \mathbb{Z}$ (so for instance $D_4 = \{1, 1, 1, 4, 4, 9\}$). Write $\Omega(a, b, c) = (-1)^c e_a(C_\lambda) e_b(D_n) h_c(B_\lambda)$. Using standard λ-ring notation and manipulations (see e.g. Lascoux [9, Ch. 2]), we have from Lemma 4.1 that

$$\begin{aligned} \Phi_n(e_\mu) &= \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \, e_\mu(A_\lambda) \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \, e_\mu(C_\lambda + D_n - B_\lambda) \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \prod_{i=1}^{k} \left(\sum_{\substack{a,b,c \ge 0 \\ a+b+c=\mu_i}} \Omega(a, b, c) \right) \\ &= \sum_{\substack{a_1,b_1,c_1 \ge 0 \\ a_1+b_1+c_1=\mu_1}} \cdots \sum_{\substack{a_k,b_k,c_k \ge 0 \\ a_k+b_k+c_k=\mu_k}} \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \prod_{r=1}^{k} \Omega(a, b, c). \end{aligned}$$

Consider the inner sum over λ, together with the factor $1/n!$. By Lemma 4.2 each $e_{b_r}(D_n)$ is a polynomial in n which we can factor out of the sum. Note that $h_{c_r}(B_\lambda)$ is a symmetric function of the numbers $\rho_i = \lambda_i + n - i$ since $(\rho_i - \rho_j)^2$ is symmetric in i and j. (This is the one point in the proof that requires the use of the alphabet $\{h_u^2 : u \in \lambda\}$ rather than the more general $\{h_u : u \in \lambda\}$.) What remains after factoring out each $e_{b_r}(D_n)$ is therefore a polynomial in n by Theorem 3.3, and the proof follows. □

NOTE. (a) The λ-ring computations in the proof of Theorem 4.3 can easily be replaced with more "naive" techniques such as generating functions. The λ-ring approach, however, makes the computation more routine.

(b) An interesting feature of the proofs of Theorems 2.1, 3.3, and 4.3 is that they don't involve just "formal" properties of symmetric functions; use of representation theory is required. This is because the only known proof of the crucial equation (3) involves representation theory, viz., the determination of the primitive orthogonal idempotents in the center of the group algebra of $\mathfrak{S}_n$. Is there a proof of (3) or of Theorems 2.1, 3.3, and 4.3 that doesn't involve representation theory?

Here is a small table of the polynomials $\Phi_n(e_\mu)$:

$$\begin{aligned}
\Phi_n(e_1) &= \frac{1}{2}n(3n-1) \\
\Phi_n(e_2) &= \frac{1}{24}n(n-1)(27n^2-67n+74) \\
\Phi_n(e_1^2) &= \frac{1}{12}n(27n^3-14n^2-9n+8) \\
\Phi_n(e_3) &= \frac{1}{48}n(n-1)(n-2)(27n^3-174n^2+511n-552) \\
\Phi_n(e_2e_1) &= \frac{1}{48}n(n-1)(81n^4-204n^3+137n^2+390n-512) \\
\Phi_n(e_1^3) &= \frac{1}{24}n(81n^5-45n^4-69n^3-31n^2+216n-128).
\end{aligned}$$

NOTE. Soichi Okada has conjectured [11] the following "hook analogue" of equation (7):

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda}\prod_{i=1}^{r}(h_u^2-i^2) = \frac{1}{2(r+1)^2}\binom{2r}{r}\binom{2r+2}{r+1}\langle n\rangle_{r+1}. \qquad (19)$$

This conjecture has been proved by Greta Panova [12] using Theorem 4.3. From this result we get the following analogue of equation (8):

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda} h_u^{2k} = \sum_{j=1}^{k+1} T(k+1,j)\frac{1}{2j^2}\binom{2(j-1)}{j-1}\binom{2j}{j}\langle n\rangle_j.$$

NOTE. Using Theorem 3.3 and the method of the proof of Theorem 4.3 to reduce hook lengths squared to contents and shifted parts, it is clear that we have the following "master theorem" subsuming both Theorems 3.3 and 4.3.

Theorem 4.4. *For any $K \in \Lambda_{\mathbb{Q}}[x]\otimes\Lambda_{\mathbb{Q}}[y]\otimes\Lambda_q[z]$, we have*

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 K_\lambda \in \mathbb{Q}[n],$$

where

$$K_\lambda = K(\{c_u\,:\,u\in\lambda\};\{\lambda_i+n-i\,:\,1\le i\le n\};\{h_u^2\,:\,u\in\lambda\}).$$

5 Some questions.

1. Can the Nekrasov-Okounkov formula (1) be proved using the techniques we have used to prove Theorem 4.3?

2. Can the Nekrasov-Okounkov formula (1) be generalized with the left-hand side replaced with the following expression (or some simple modification thereof)?

$$\sum_{n\ge 0}\left(\sum_{\lambda\vdash n} f_\lambda^{2k}\prod_{i=1}^{k}\prod_{u\in\lambda}(t_i+h_u^2)\right)\frac{x^n}{n!^{2k}}$$

Note that if we put each $t_i = 0$ then we obtain the partition generating function $\prod_{i\geq 1}(1-x^i)^{-1}$. The same question can be asked with h_u^2 replaced with c_u^2 or c_u.

3. Define a linear transformation $\psi\colon \Lambda_{\mathbb{Q}} \to \mathbb{Q}[t]$ by

$$\psi(s_\lambda) = H_\lambda^{-1} \prod_{u\in\lambda} (t + h_u^2).$$

Is there a nice description of $\psi(p_\mu)$?

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Footnotes

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