A Conjectured Combinatorial Interpretation of the Normalized Irreducible Character Values of the Symmetric Group

Richard P. Stanley
Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA 02139, USA
rstan@math.mit.edu

version of 26 July 2006

The irreducible characters χ^λ of the symmetric group $\mathfrak{S}_n$ are indexed by partitions λ of n (denoted $\lambda \vdash n$ or $|\lambda| = n$), as discussed e.g. in [2, §1.7] or [4, §7.18]. If $w \in \mathfrak{S}_n$ has cycle type $\nu \vdash n$ then we write $\chi^\lambda(\nu)$ for $\chi^\lambda(w)$.

Let μ be a partition of $k \leq n$, and let $(\mu, 1^{n-k})$ be the partition obtained by adding $n-k$ 1's to μ. Thus $(\mu, 1^{n-k}) \vdash n$. Regarding k as given, define the *normalized character* $\widehat{\chi}^\lambda(\mu, 1^{n-k})$ by

$$\widehat{\chi}^\lambda(\mu, 1^{n-k}) = \frac{(n)_k \chi^\lambda(\mu, 1^{n-k})}{\chi^\lambda(1^n)},$$

where $\chi^\lambda(1^n)$ denotes the dimension of the character χ^λ and $(n)_k = n(n-1)\cdots(n-k+1)$. Thus [2, (7.6)(ii)][4, p. 349] $\chi^\lambda(1^n)$ is the number f^λ of standard Young tableaux of shape λ.

Suppose that (the diagram of) the partition λ is a union of m rectangles of sizes $p_i \times q_i$, where $q_1 \geq q_2 \geq \cdots \geq q_m$, as shown in Figure 1. The following result was proved in [5, Prop. 1] for $\mu = (k)$ and attributed to J. Katriel (private communication) for arbitrary μ.

Proposition 1. *Let λ be the shape in Figure 1, and fix $k \geq 1$. Let $\mu \vdash k$. Set $n = |\lambda|$ and*

$$F_\mu(\boldsymbol{p}; \boldsymbol{q}) = F_\mu(p_1, \ldots, p_m; q_1, \ldots, q_m) = \widehat{\chi}^\lambda(\mu, 1^{n-k}).$$

Then $F_\mu(\boldsymbol{p}; \boldsymbol{q})$ is a polynomial function of the p_i's and q_i's with integer coefficients, satisfying

$$(-1)^k F_\mu(1, \ldots, 1; -1, \ldots, -1) = (k+m-1)_k.$$

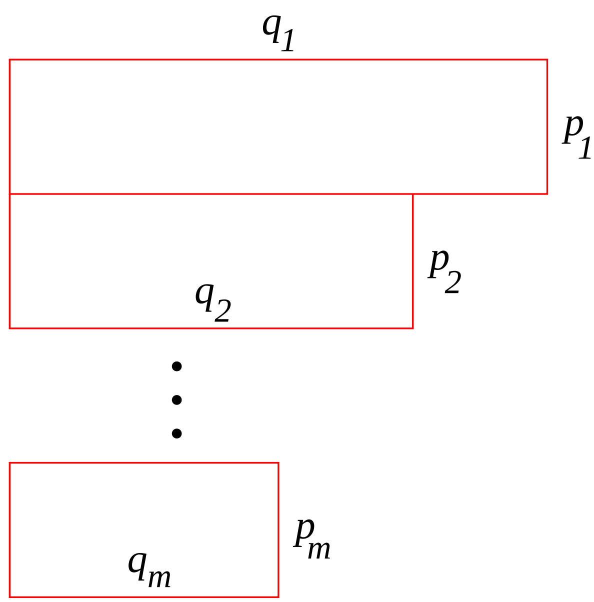

Figure 1: A union of m rectangles

Note. When $\mu = (k)$, the partition with a single part k, we write F_k for $F_{(k)}$. A formula was given in [5, (9)] for $F_k(\boldsymbol{p};\boldsymbol{q})$, viz.,

$$F_k(\boldsymbol{p};\boldsymbol{q}) = -\frac{1}{k}[x^{-1}]\frac{(x)_k \prod_{i=1}^{m}(x-(q_i+p_i+p_{i+1}+\cdots+p_m))_k}{\prod_{i=1}^{m}(x-(q_i+p_{i+1}+p_{i+2}+\cdots+p_m))_k},$$

where $[x^{-1}]f(x)$ denotes the coefficient of x^{-1} in the expansion of $f(x)$ in *descending* powers of x (i.e., as a Taylor series at $x=\infty$).

It was conjectured in [5] that the coefficients of the polynomial $(-1)^k F_\mu(\boldsymbol{p};-\boldsymbol{q})$ are *nonnegative*, where $-\boldsymbol{q} = (-q_1,\dots,-q_m)$. This conjecture was proved in [5] for the case $m=1$, i.e., when λ is a $p\times q$ rectangle, denoted $\lambda = p\times q$. For $w\in\mathfrak{S}_n$ let $\kappa(w)$ denote the number of cycles of w (in the disjoint cycle decomposition of w). The main result of [5] was the following (stated slightly differently but clearly equivalent).

Theorem 2. *Let $\mu \vdash k$ and fix a permutation $w_\mu \in \mathfrak{S}_k$ of cycle type μ. Then*

$$F_\mu(p;q) = (-1)^k \sum_{uw_\mu = v} p^{\kappa(u)}(-q)^{\kappa(v)},$$

where the sum ranges over all $k!$ pairs $(u,v) \in \mathfrak{S}_k \times \mathfrak{S}_k$ satisfying $uw_\mu = v$.

To state our conjectured generalization of Theorem 2, let $\mathfrak{S}_k^{(m)}$ denote the set of permutations $u \in \mathfrak{S}_k$ whose cycles are colored with $1, 2, \ldots, m$. More formally, if $C(u)$ denotes the set of cycles of u, then an element of $\mathfrak{S}_k^{(m)}$ is a pair (u, φ), where $u \in \mathfrak{S}_k$ and $\varphi : C(u) \to [m]$. (We use the standard notation $[m] = \{1, 2, \ldots, m\}$.) If $\alpha = (u, \varphi) \in \mathfrak{S}_k^{(m)}$ and $v \in \mathfrak{S}_k$, then define a "product" $\alpha v = (w, \psi) \in \mathfrak{S}_k^{(m)}$ as follows. First let $w = uv$. Let $\tau = (a_1, a_2, \ldots, a_j)$ be a cycle of w, and let ρ_i be the cycle of u containing a_i. Set

$$\psi(\tau) = \max\{\varphi(\rho_1), \ldots, \varphi(\rho_j)\}.$$

For instance (multiplying permutations from left to right),

$$(\overbrace{1,2,3}^{1})(\overbrace{4,5}^{2})(\overbrace{6,7}^{3})(\overbrace{\ 8\ }^{2})\cdot(1,7)(2,4,8,5)(3,5) = (\overbrace{1,4,2,6}^{3})(\overbrace{3,7}^{3})(\overbrace{5,8}^{2}).$$

Note that it an immediate consequence of the well-known formula

$$\sum_{w \in \mathfrak{S}_k} x^{\kappa(w)} = x(x+1)\cdots(x+k-1)$$

that $\#\mathfrak{S}_k^{(m)} = (k+m-1)_k$.

Note. The product αv does not seem to have nice algebraic properties. In particular, it does not define an action of $\mathfrak{S}_k$ on $\mathfrak{S}_k^{(m)}$, i.e., it is not necessarily true that $(\alpha u)v = \alpha(uv)$. For instance (denoting a cycle colored 1 by leaving it as it is, and a cycle colored 2 by an overbar), we have

$$\begin{aligned} [(\bar{1})(2)\cdot(1,2)]\cdot(1,2) &= (\bar{1})(\bar{2}) \\ (\bar{1})(2)\cdot[(1,2)\cdot(1,2)] &= (\bar{1})(2). \end{aligned}$$

Given $\alpha = (u, \varphi) \in \mathfrak{S}_k^{(m)}$, let $\boldsymbol{p}^{\kappa(\alpha)} = \prod_i p_i^{\kappa_i(\alpha)}$, where $\kappa_i(\alpha)$ denotes the number of cycles of u colored i, and similarly $\boldsymbol{q}^{\kappa(\beta)}$, so $(-\boldsymbol{q})^{\kappa(\beta)} = \prod_i (-q_i)^{\kappa_i(\beta)}$ We can now state our conjecture.

Conjecture 3. *Let λ be the partition of n given by Figure 1. Let $\mu \vdash k$ and fix a permutation $w_\mu \in \mathfrak{S}_k$ of cycle type μ. Then*

$$F_\mu(\boldsymbol{p}; \boldsymbol{q}) = (-1)^k \sum_{\alpha w_\mu = \beta} \boldsymbol{p}^{\kappa(\alpha)} (-\boldsymbol{q})^{\kappa(\beta)},$$

where the sum ranges over all $(k+m-1)_k$ pairs $(\alpha, \beta) \in \mathfrak{S}_k^{(m)} \times \mathfrak{S}_k^{(m)}$ satisfying $\alpha w_\mu = \beta$.

Example 1. Let $m = 2$ and $\mu = (2)$, so $w_\mu = (1, 2)$. There are six pairs $(\alpha, \beta) \in \mathfrak{S}_n^{(2)}$ for which $\alpha(1, 2) = \beta$, viz. (where as in the above Note an unmarked cycle is colored 1 and a barred cycle 2),

α	β	$\boldsymbol{p}^{\kappa(\alpha)} \boldsymbol{q}^{\kappa(\beta)}$
$(1)(2)$	$(1, 2)$	$p_1^2 q_1$
$(\overline{1})(2)$	$(\overline{1, 2})$	$p_1 p_2 q_2$
$(1)(\overline{2})$	$(\overline{1, 2})$	$p_1 p_2 q_2$
$(\overline{1})(\overline{2})$	$(\overline{1, 2})$	$p_2^2 q_2$
$(1, 2)$	$(1)(2)$	$p_1 q_1^2$
$(\overline{1, 2})$	$(\overline{1})(\overline{2})$	$p_2 q_2^2.$

It follows (since the conjecture is true in this case) that

$$F_2(p_1, p_2; q_1, q_2) = -p_1^2 q_1 - 2p_1 p_2 q_2 - p_2^2 q_2 + p_1 q_1^2 + p_2 q_2^2.$$

We can reduce Conjecture 3 to the case $p_1 = \cdots = p_m = 1$; i.e., $\lambda = (q_1, , q_2, \ldots, q_m)$. Let

$$G_\mu(\boldsymbol{p}, \boldsymbol{q}) = \sum_{\alpha w_\mu = \beta} \boldsymbol{p}^{\kappa(\alpha)} \boldsymbol{q}^{\kappa(\beta)},$$

so that Conjecture 3 asserts that $F_\mu(\boldsymbol{p}; \boldsymbol{q}) = (-1)^k G_\mu(\boldsymbol{p}, -\boldsymbol{q})$.

Proposition 4. *We have*

$$G_\mu(\boldsymbol{p},\boldsymbol{q})|_{q_{i+1}=q_i} = G_\mu(p_1,\ldots,p_{i-1},p_i+p_{i+1},p_{i+2},\ldots,p_m;$$
$$q_1,\ldots,q_{i-1},q_i,q_{i+2},\ldots,q_m). \qquad (1)$$

Proof. Let $\alpha w_\mu = \beta$, where $\alpha,\beta \in \mathfrak{S}_k^{(m)}$ and $\mu \vdash k$. If τ is a cycle of β colored $i+1$ then change the color to i, giving a new colored permutation β'. We can also get the pair (α,β') by changing all the cycles in α colored $i+1$ to i, producing a new colored permutation α' for which $\alpha' w_\mu = \beta'$, and then changing back the colors of the recolored cycles of α to $i+1$. Equation (1) is simply a restatement of this result in terms of generating functions. □

It is clear, on the other hand, that

$$F_\mu(\boldsymbol{p},\boldsymbol{q})|_{q_{i+1}=q_i} = F_\mu(p_1,\ldots,p_{i-1},p_i+p_{i+1},p_{i+2},\ldots,p_m;$$
$$q_1,\ldots,q_{i-1},q_i,q_{i+2},\ldots,q_m),$$

because the parameters $p_1,\ldots,p_m;q_1,\ldots,q_{i-1},q_i,q_i,q_{i+2},\ldots,q_m$ and $p_1,\ldots,p_{i-1},p_i+p_{i+1},p_{i+2},\ldots,p_m;q_1,\ldots,q_{i-1},q_i,q_{i+2},\ldots,q_m$ specify the same shape λ. (Note that Proposition 1 requires only $q_1 \geq q_2 \geq \cdots \geq q_m$, not $q_1 > q_2 > \cdots > q_m$.) Hence if Conjecture 3 is true when $p_1 = \cdots = p_m = 1$, then it is true in general by iteration of equation (1).

Remarks. 1. Conjecture 3 has been proved by Amarpreet Rattan [3] for the terms of highest degree of F_k, i.e., the terms of $F_k(\boldsymbol{p};\boldsymbol{q})$ of total degree $k+1$.

2. Kerov's character polynomials (e.g., [1]) are related to $F_k(\boldsymbol{p};\boldsymbol{q})$ and are also conjectured to have nonnegative (integral) coefficients. Is there a combinatorial interpretation of the coefficients similar to that of Conjecture 3?

References

[1] I. P. Goulden and A. Rattan, An explicit form for Kerov's character polynomials, *Trans. Amer. Math. Soc.*, to appear; `math.CO/0505317`.

[2] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Oxford University Press, Oxford, 1995.

[3] A. Rattan, in preparation.

[4] R. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, New York/Cambridge, 1999.

[5] R. Stanley, Irreducible symmetric group characters of rectangular shape, *Sém. Lotharingien de Combinatoire* (electronic) **50** (2003), B50d.