SUPPLEMENTARY PROBLEMS

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Here we provide some exercises but not their solutions, culled from a variety of sources. We have retained the difficulty rating scale used in our previous exercises.

CHAPTER 1

1. [2] Let $\bar{c}(n,m)$ denote the number of compositions of n with largest part $\leq m$. Show that
$$\sum_{n\geq 0} \bar{c}(n,m)x^n = \frac{1-x}{1-2x+x^{m+1}}.$$

2. [2] Fix $n \in \mathbb{P}$. In how many ways (as a function of n) can one choose a composition α of n, and then choose a composition of each part of α? (Give an elegant combinatorial proof.)

3. [2] Let $\kappa(n,j,k)$ be the number of weak compositions of n into k parts, each $< j$. Give a generating function proof that
$$\kappa(n,j,k) = \sum_{r+sj=n} (-1)^s \binom{k+r-1}{r}\binom{k}{s},$$
where the sum is over all pairs $(r,s) \in \mathbb{N}^2$ satisfying $r+sj=n$.

4. [2] How many n-element multisets on $[2m]$ are there satisfying: (i) $1,2,\ldots,m$ must appear at most once each, and (ii) $m+1, m+2, \ldots, 2m$ must appear an even number of times each?

5. [2] Show that
$$\sum_{n_1,\ldots,n_k\geq 0} \min(n_1,\ldots,n_k)x_1^{n_1}\cdots x_k^{n_k} = \frac{x_1\cdots x_k}{(1-x_1)\cdots(1-x_k)(1-x_1x_2\cdots x_k)}.$$

6. [2+] Find a *simple* description of all $n \in \mathbb{P}$ with the following property: There exists $k \in [n]$ such that $\binom{n}{k-1}, \binom{n}{k}, \binom{n}{k+1}$ are in arithmetic progression.

7. [2] How many permutations $a_1 a_2 \cdots a_n$ of $[n]$ satisfy the following property: if $2 \leq j \leq n$, then $|a_i - a_j| = 1$ for some $1 \leq i \leq j$. (E.g., for $n = 3$ there are the four permutations $123, 213, 231, 321$.) More generally, for any $S \subseteq [n]$, find the number of such permutations with descent set S.

8. [2] Fix $k, n \in \mathbb{P}$. Show that

$$\sum c_1 \cdots c_k = \binom{n+k-1}{2k-1},$$

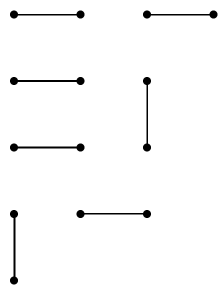
where the sum ranges over all compositions $c_1 + c_2 + \cdots + c_k$ of n into k parts.

9. [2–] Fix $k, n \in \mathbb{P}$. How many sequences $(T_1, T_2, \ldots, T_k)$ of subsets T_i of $[n]$ are there such that the *nonempty* T_i form a partition of $[n]$?

10. [2+] Choose n points on the circumference of a circle in "general position". Draw all $\binom{n}{2}$ chords connecting two of the points. ("General position" means that no three chords intersect in a point.) Into how many regions will the interior of the circle be divided? Try to give an elegant proof avoiding induction, finite differences, generating functions, summations, etc.

11. [2+] How many pairs (μ, λ) of partitions of integers are there such that $\mu \vdash n$, and the Young diagram of λ is obtained from the Young diagram of μ by adding a single square? Express your answer in terms of the partition function $p(k)$ for $k \leq n$. Give a simple combinatorial proof.

12. [3] Let $f(n)$ be the number of partitions of $2n$ whose Ferrers diagram can be covered by n edges, each connecting two adjacent dots. For instance, $(4, 3, 3, 3, 1)$ can be covered as follows:



Show that
$$\sum_{n\geq 0} f(n)x^n = \prod_{i\geq 1}(1-x^i)^{-2}.$$

13. [2] Show combinatorially that the number of partitions of n into distinct odd parts equals the number $c(n)$ of self-conjugate partitions of n. Use this to express $\sum_{n\geq 0} c(n)x^n$ as an infinite product.

14. [2+] Fix $j, k \in \mathbb{Z}$. Show that
$$\sum_{n\geq 0} \frac{(2n-j-k)!x^n}{(n-j)!(n-k)!(n-j-k)!n!} = \left[\sum_{n\geq 0} \frac{x^n}{n!(n-j)!}\right]\left[\sum_{n\geq 0} \frac{x^n}{n!(n-k)!}\right].$$
Any term with $(-r)!$ in the denominator, where $r > 0$, is set equal to zero.

15. [2+] An n-dimensional cube K_n has $2n$ facets (or $(n-1)$-dimensional faces), which come in n antipodal pairs. A *shelling* of K_n is equivalent to a linear ordering $F_1, \ldots, F_{2n}$ of its facets such that for all $1 \leq i \leq n-1$, the set $\{F_1, \ldots, F_{2i}\}$ does not consist of i antipodal pairs. Let $f(n)$ be the number of shellings of K_n. Show that
$$\sum_{n\geq 1} f(n)\frac{x^n}{n!} = 1 - \left(\sum_{n\geq 0}(2n)!\frac{x^n}{n!}\right)^{-1}.$$

16. (a) [2] Let $f(n)$ denote the number of fixed point free involutions $\pi \in \mathfrak{S}_{2n}$ (i.e., $\pi^2 = 1$, and $\pi(i) \neq i$ for all $i \in [2n]$). Find a simple expression for $\sum_{n \geq 0} f(n) \frac{x^n}{n!}$. (Set $f(0) = 1$.)

(b) [2–] If $X \subseteq \mathbb{P}$, then write $-X = \{-n : n \in X\}$. Let $g(n)$ be the number of ways to choose a subset X of $[n]$, and then choose fixed point free involutions π on $X \cup (-X)$ and $\bar{\pi}$ on $\bar{X} \cup (-\bar{X})$, where $\bar{X} = \{i \in [n] : i \notin X\}$. Use (a) to find a simple expression for $g(n)$.

(c) [3–] Find a combinatorial proof of the formula for $g(n)$ obtained in (b).

17. Let D be the operator $\frac{d}{dx}$.

(a) [2] Show that $(xD)^n = \sum_{i=0}^{n} S(n,i) x^i D^i$.

(b) [2] Show that $x^n D^n = xD(xD-1)(xD-2)\cdots(xD-n+1)$.

18. [2+] A *perfect partition* of $n \in \mathbb{P}$ is a partition λ of n which "contains" precisely one partition of each positive integer $\ell \leq n$. In other words, if λ has m_i parts equal to i, then for each $1 \leq \ell \leq n$ there is exactly one solution in nonnegative integers r_i to the equations $\sum i r_i = \ell$ and $0 \leq r_i \leq m_i$. Show that the number of perfect partitions of n is equal to the number of *ordered* factorizations of $n+1$ into integers ≥ 2.

Example. The perfect partitions of 5 are $(1,1,1,1,1)$, $(3,1,1)$, and $(2,2,1)$. The ordered factorizations of 6 are $6 = 3 \cdot 2 = 2 \cdot 3$.

19. [2–] Give a combinatorial proof that $2\binom{2n-1}{n} = \binom{2n}{n}$.

20. [2+] Find a combinatorial proof of the identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{j=0}^{n} \binom{n}{j} \binom{2n-j}{n} (x-1)^j.$$

21. [3–] Fix $n \in \mathbb{N}$. Prove the identity

$$\sum_{i+j+k=n} \binom{i+j}{i} \binom{j+k}{j} \binom{k+i}{k} = \sum_{r=0}^{n} \binom{2r}{r},$$

where the left sum ranges over all $(i, j, k) \in \mathbb{N}^3$ satisfying $i+j+k = n$. Is there a combinatorial proof?

22. [3–] Let $k, n \in \mathbb{P}$ with $k \leq n$. Let $V(n, k)$ be the volume of the region in $\mathbb{R}^n$ defined by

$$0 \leq x_i \leq 1, \text{for } \ 1 \leq i \leq n$$

$$k-1 \leq x_1 + x_2 + \cdots + x_n \leq k.$$

Show that $V(n, k) = A(n, k)/n!$, where $A(n, k)$ is an Eulerian number.

23. [2–] Let $\pi \in \mathfrak{S}_n$. Show that $i(\pi) = i(\pi^{-1})$, where i denotes number of inversions.

24. [2+] Show that

$$\sum_{\pi} q^{i(\pi)} = q^n \prod_{j=0}^{n-1} (1 + q^2 + q^4 + \cdots + q^{4j}),$$

where π ranges over all fixed-point free involutions in $\mathfrak{S}_{2n}$ (i.e., π has n 2-cycles), and where $i(\pi)$ denotes the number of inversions of π. Give a simple combinatorial proof analogous to the proof of Corollary 1.3.10.

25. [3+] Let $n \in \mathbb{P}$ and $r, s, t \in \mathbb{N}$. Let x, y_k, z_k, and a_{ij} be indeterminates, with $1 \leq k \leq n$ and $1 \leq i < j \leq n$. Let M be the multiset with n occurences of x, r occurences of each y_k, s occurences of each z_k, and t occurences of each a_{ij}. Let $f(n, r, s, t)$ be the number of permutations π of M such that (a) all y_k's appear before the kth x (reading the x's from left to right in π), (b) all z_k's appear after the kth x, and (c) all a_{ij}'s appear between the ith and jth x. Show that

$$f(n, r, s, t) = \frac{[(r+s+1)n + tn(n-1)]!}{n! r!^n s!^n t!^n (2t)!^{\binom{n}{2}}} \prod_{j=1}^{n} \frac{(r+(j-1)t)!(s+(j-1)t)!(jt)!}{(r+s+1+(n+j-2)t)!}.$$

26. (a) [2] Let x and y be variables satisfying the commutation relation $yx = qxy$, where q commutes with x and y. Show that

$$(x+y)^n = \sum_{k=0}^{n} \binom{\mathbf{n}}{\mathbf{k}} x^k y^{n-k}.$$

(b) [2] Generalize to $(x_1 + x_2 + \cdots + x_m)^n$, where $x_i x_j = q x_j x_i$ for $i > j$.

(c) [2+] Generalize further to $(x_1+x_2+\cdots+x_m)^n$, where $x_i x_j = q_j x_j x_i$ for $i > j$, and where the q_k's are variables commuting with all the x_i's and with each other.

27. [2–] How many $m \times n$ matrices of 0's and 1's have an odd number of ones in every row and column? Generalize to an arbitrary specification of the parity of the number of ones in every row and column.

28. [2+] Let

$$J_n(2x) = \sum_{k \geq 0} \frac{(-1)^k x^{n+2k}}{k!(n+k)!}.$$

Show that

$$e^x = \sum_{n \geq 0} L_{n+1} J_n(2x),$$

where $L_0 = 1,\ L_1 = 1,\ L_2 = 3,\ L_{n+1} = L_n + L_{n-1}$ for $n \geq 2$. (For $n \geq 1$, L_n is a Lucas number.)

29. Define

$$x \prod_{n \geq 1} (1 - x^n)^{24} = \sum_{n \geq 1} \tau(n) x^n.$$

(a) [3+] Show that $\tau(mn) = \tau(m)\tau(n)$ if m and n are relatively prime.

(b) [3+] Show that if p is prime then

$$\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}).$$

(c) [4] Show that if p is prime then $|\tau(p)| < 2p^{11/2}$.

(d) [5] Show that $\tau(n) \neq 0$ for all $n \geq 1$.

CHAPTER 2

1. (a) [2] Let $f(m,n)$ be the number of $m \times n$ matrices of 0's and 1's with at least one 1 in every row and column. Use the Principle of Inclusion-Exclusion to show that

$$f(m,n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^m.$$

(b) [2] (b) Show that

$$\sum_{m\geq 0}\sum_{n\geq 0} f(m,n)\frac{x^m y^n}{m!n!} = e^{-x-y}\sum_{i\geq 0}\sum_{j\geq 0}\frac{2^{ij}x^i y^j}{i!j!}.$$

2. [2+] Let $\pi \in \Pi_n$, the lattice of partitions of $[n]$. Let $S(\pi, r)$ denote the number of $\sigma \in \Pi_n$ such that $|\sigma| = r$ and $\#(A \cap B) \leq 1$ for all $A \in \pi$ and $B \in \sigma$. (This last condition is equivalent to $\pi \wedge \sigma = \hat{0}$ in the lattice structure on Π_n defined in Example 3.10.4.) Show that

$$\begin{aligned} S(\pi, r) &= \tfrac{1}{r!}\textstyle\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\prod_{A\in\pi}(i)_{\#A}. \\ &= \tfrac{1}{r!}\Delta^r \textstyle\prod_{A\in\pi}(n)_{\#A}\,|_{n=0}\ . \end{aligned}$$

3. [3–] Given $a \in \mathbb{P}$, let $C(a)$ denote the number of lattice paths in $\mathbb{Z}^2$ from $(0,0)$ to $(2a,0)$ with steps $(1,1)$ or $(1,-1)$ which never pass below the x-axis. (We will see in Chapter 6 that $C(a)$ is the Catalan number $\frac{1}{a+1}\binom{2a}{a}$, but this fact is irrelevant here.) Now given $\{a_1, a_2, \ldots, a_{2n}\}_< \subset \mathbb{Z}$, let $C(a_1, a_2, \ldots, a_{2n})$ denote the number of ways to connect the points $(2a_1, 0), (2a_2, 0), \ldots, (2a_{2n}, 0)$ with n pairwise disjoint lattice paths $L_1, \ldots, L_n$ of the type just described. (Thus each L_i connects some $(2a_j, 0)$ to some $(2a_k, 0)$. If $i \neq j$ then L_i and L_j have no points in common, including endpoints, so each $(2a_i, 0)$ is an endpoint of exactly one L_j.)

   Now given a triangular array $A = (a_{ij})$, with $1 \leq i < j \leq 2n$, recall that the *pfaffian* of A is defined by

$$\mathrm{Pf}(A) = \sum \varepsilon(i_1, j_1, \ldots, i_n, j_n) a_{i_1 j_1} \cdots a_{i_n j_n},$$

   where the summation is over all partitions $\{\{i_1, j_1\}_<, \ldots, \{i_n, j_n\}_<\}$ of $[2n]$ into 2-element blocks, and where $\varepsilon(i_1, j_1, \ldots, i_n, j_n)$ denotes the sign of the permutation

$$\left(\begin{array}{ccccc} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{array}\right).$$

   (It is easy to see that $\varepsilon(i_1, j_1, \ldots, i_n, j_n)$ does not depend on the order of the n blocks.) Give a proof analogous to that of Theorem 2.7.1 of the formula

$$C(a_1, a_2, \ldots, a_{2n}) = \mathrm{Pf}(C(a_j - a_i)).$$

For instance,

$$\begin{aligned}
C(0,3,5,6) &= \mathrm{Pf}\begin{vmatrix} C(3) & C(5) & C(6) \\ & C(2) & C(3) \\ & & C(1) \end{vmatrix} \\
&= \mathrm{Pf}\begin{vmatrix} 5 & 42 & 132 \\ & 2 & 5 \\ & & 1 \end{vmatrix} \\
&= 5\cdot 1 + 132\cdot 2 - 42\cdot 5 \\
&= 59.
\end{aligned}$$

4. (a) [2+] Let $A_k(n)$ denote the number of k-element antichains in the boolean algebra B_n. Show that

$$A_1(n) = 2^n$$

$$A_2(n) = \frac{1}{2}\left(4^n - 2\cdot 3^n + 2^n\right)$$

$$A_3(n) = \frac{1}{6}\left(8^n - 6\cdot 6^n + 6\cdot 5^n + 3\cdot 4^n - 6\cdot 3^n + 2\cdot 2^n\right)$$

$$A_4(n) = \frac{1}{24}\left(16^n - 12\cdot 12^n + 24\cdot 10^n + 4\cdot 9^n - 18\cdot 8^n \right.$$
$$\left. +6\cdot 7^n - 36\cdot 6^n + 36\cdot 5^n + 11\cdot 4^n - 22\cdot 3^n + 6\cdot 2^n\right).$$

(b) [2+] Show that for fixed $k\in\mathbb{P}$, there exist integers $a_{k2},\dots,a_{k2^k}$ such that

$$A_k(n) = \frac{1}{k!}\sum_{i=2}^{2^k} a_{ki} i^n.$$

Show in particular that $a_{k2^k} = 1, a_{ki} = 0$ if $3\cdot 2^{k-2} < i < 2^k$, and $a_{k,3\cdot 2^{k-2}} = k(k-1)$.

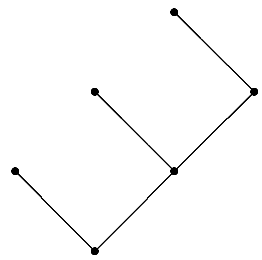
CHAPTER 3

1. (a) [2] Let P be an n-element poset. If $x\in P$, then set $\lambda_x = \#\{y\in P : y\le x\}$. Show that

$$e(P) \ge \frac{n!}{\prod_{x\in P}\lambda_x} \qquad (*).$$

(b) [2+] Show that equality holds in $(*)$ if and only if every component of P is a rooted tree (with root $\hat{1}$).

2. (a) [2] Let P_n be the poset with elements x_i, y_i for $i \in [n]$, and cover relations $x_1 < x_2 < \cdots < x_n$ and $y_i > x_i$ for all $i \in [n]$. E.g., P_3 has the Hasse diagram



Find a "nice" expression for the rank-generating function $F(J(P_n), q)$.

(b) [2–] Let $P = \lim_{n\to\infty} P_n$. Find the rank-generating function $F(J_f(P), q)$.

(c) [2] Find a simple formula for $e(P_n)$.

(d) [2] Let $\Omega(P_n, m)$ denote the order polynomial of P_n. For $m \in \mathbb{P}$ express $\Omega(P_n, m)$ in terms of Stirling numbers of the second kind.

(e) [2] For $m \in \mathbb{N}$ express $\Omega(P_n, -m)$ in terms of Stirling numbers of the first kind.

3. (a) [2–] How many maximal chains does Π_n have?

(b) [3–] The symmetric group $\mathfrak{S}_n$ acts on Π_n in an obvious way. This action induces an action on the set M of maximal chains of Π_n. Show that the number $\#(M/\mathfrak{S}_n)$ of $\mathfrak{S}_n$-orbits on M is equal to the Euler number E_{n-1}. For instance, when $n = 5$ a set of orbit representatives is given by (omitting $\hat{0}$ and $\hat{1}$ from each chain, and writing e.g. 12-34 for the partition whose non-singleton blocks are $\{1,2\}$ and $\{3,4\}$): $12 < 123 < 1234$, $12 < 123 < 123\text{-}45$, $12 < 12\text{-}34 < 125\text{-}34$, $12 < 12\text{-}34 < 12\text{-}345$, $12 < 12\text{-}34 < 1234$.

4. [2+] Let P be a finite poset, and let μ be the Möbius function of $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$. Suppose that P has a fixed point free automorphism $\sigma : P \to P$ of prime order p (i.e., $\sigma(x) \neq x$ and $\sigma^p(x) = x$ for all $x \in P$). Show that
$$\mu(\hat{0}, \hat{1}) \equiv -1 \pmod{p}.$$

What does this say in the case $\hat{P} = \Pi_p$?

5. (a) [3–] Let L be a finite lattice with n atoms. Show that

$$|\mu(\hat{0}, \hat{1})| \le \binom{n-1}{\lfloor (n-1)/2 \rfloor},$$

and that this result is best possible.

(b) [3+] Assume also that the longest chain of L has length $\le \ell$. Show that

$$|\mu(\hat{0}, \hat{1})| \le \binom{n-1}{k},$$

where $k = \min(\ell - 1, \lfloor (n-1)/2 \rfloor)$, and that this result is best possible.

6. (a) [2–] Let E_n denote the poset (ordered by inclusion) of all subsets of $[n]$ whose elements have even sum. Find $\#E_n$.

(b) [2+] Compute $\mu(S, T)$ for all $S \le T$ in E_n.

(c) [5–] Let $k \ge 3$. Do (b) for the poset of all subsets of $[n]$ whose elements have sum divisible by k.

7. [2+] Let $f(n)$ be the total number of chains containing $\hat{0}$ and $\hat{1}$ in the lattice of faces of the n-dimensional cube. Show that

$$\sum_{n \ge 0} f(n) \frac{x^n}{n!} = \frac{e^x}{2 - e^{2x}}.$$

(See Exercise 3.71 for information on this lattice.)

8. [2+] Let L be a finite geometric lattice, and let x be a coatom of L. Let $\eta(x)$ be the number of atoms $y \in L$ satisfying $y \not\le x$. Show that

$$|\mu(\hat{0}, \hat{1})| \le |\mu(\hat{0}, x)| \cdot \eta(x).$$

9. [2] Let B_k denote a boolean algebra of rank k, and $\Omega(B_k, m)$ its order polynomial. Show that $\Omega(B_{n+1}, 2) = \Omega(B_n, 3)$.

10. (a) [2+] Let P be a p-element poset, with every maximal chain of length ℓ. Let e_s (respectively, $\bar{e}_s$) denote the number of surjective (respectively, strict surjective) order-preserving maps $P \to \mathbf{s}$. Show that:

    (i) $2e_{p-1} = (p+\ell-1)e(P)$

    (ii) $2\bar{e}_{p-1} = (p-\ell-1)e(P)$

    (iii) $\displaystyle\sum_{s=1}^{p} e_s = 2^\ell \sum_{s=1}^{p} \bar{e}_s$.

    Hint: Use Corollary 4.5.17.

    (b) [1+] With P as above, show that if $p \equiv \ell \pmod 2$ then $e(P)$ is even.

    (c) [2] With P as above, suppose that $\ell = p-4$. Let $j(P)$ denote the number of order ideals of P. Show that

$$e(P) = 2(j(P) - p).$$

11. [3–] Let $\pi \in \Pi_n$ have type $(a_1, a_2, \ldots)$, with $|\pi| = \sum a_i = m$. Let $f(\pi)$ be the number of $\sigma \in \Pi_n$ satisfying $\pi \vee \sigma = \hat{1}$, $\pi \wedge \sigma = \hat{0}$, and $|\sigma| = n + 1 - |\pi|$. Show that

$$f(\pi) = 1^{a_1} 2^{a_2} \cdots n^{a_n} (n-m+1)^{m-2}.$$

12. [2+] Let L be a finite geometric lattice of rank n. Let $L' = L - \{\hat{0}\}$, and let $f : L' \to A$ be a function from L' to the set A of atoms of L satisfying $f(x) \le x$ for all $x \in L'$. Let $\alpha(L, f)$ be the number of maximal chains $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ of L such that

$$f(x_1) \vee \cdots \vee f(x_n) = \hat{1}.$$

    Show that $\alpha(L, f) = (-1)^n \mu(\hat{0}, \hat{1})$.

13. [2+] Let P be a finite poset. Show that the following two conditions are equivalent:

    (i) For all $x < y$, the interval $[x, y]$ has an odd number of atoms.

    (ii) For all $x < y$, the interval $[x, y]$ has an odd number of coatoms.

    Hint: Consider $\mu(x, y)$ modulo 2.

14. [3–] Let P be a finite poset, and let $f(P)$ denote the number of ways to partition the elements of P into (non-empty) disjoint saturated chains. For instance, $f(\mathbf{n}) = 2^{n-1}$. Suppose that every element of P covers at most two elements and is covered by at most two elements. Show that $f(P)$ is a product of Fibonacci and Lucas numbers. In particular, compute $f(\mathbf{m} \times \mathbf{n})$. (The Lucas numbers are defined by $L_1 = 1, L_2 = 3, L_{n+1} = L_n + L_{n-1}$.)

15. [3–] Let P be a finite poset. Let A be an antichain of P which intersects every maximal chain. Show that

$$e(P) = \sum_{x \in A} e(P - x).$$

Try to give an elegant bijective proof.

16. [3–] Let A be a finite alphabet and A^* the free monoid generated by A. Partially order A^* by $u \leq v$ if u is a subword of v. Let μ be the Möbius function of A^*. Given $v = a_1 a_2 \cdots a_n$ where $a_i \in A$, call the letter a_i *special* if $a_i = a_{i-1}$. Show that

$$\mu(u, v) = (-1)^{\ell(v) - \ell(u)} s(u, v),$$

where $s(u, v)$ is the number of subwords of v isomorphic to u which use every special letter of v. For instance,

$$\mu(aba, aba\underline{a}ba) = -2$$

(where we have underlined the only special letter.)

17. [2+] Let P be an Eulerian poset of rank $d+1$ with d atoms, such that $P - \{\hat{1}\}$ is a simplicial poset. Show that if d is even, then P has an even number of coatoms.

18. (a) [2] Let $L_k(n)$ denote the number of k-element order ideals of the boolean algebra B_n. Show that for fixed k, $L_k(n)$ is a polynomial function of n of degree $k - 1$ and leading coefficient $1/(k-1)!$. Moreover, the differences $\Delta^i L_k(0)$ are all nonnegative integers.

(b) [3–] Show that

$$\begin{aligned}
L_0(n) &= L_1(n) = 1\\
L_2(n) &= \tbinom{n}{1}\\
L_3(n) &= \tbinom{n}{2}\\
L_4(n) &= \tbinom{n}{2} + \tbinom{n}{3}\\
L_5(n) &= 3\tbinom{n}{3} + \tbinom{n}{4}\\
L_6(n) &= 3\tbinom{n}{3} + 6\tbinom{n}{4} + \tbinom{n}{5}\\
L_7(n) &= \tbinom{n}{3} + 15\tbinom{n}{4} + 10\tbinom{n}{5} + \tbinom{n}{6}\\
L_8(n) &= \tbinom{n}{3} + 20\tbinom{n}{4} + 45\tbinom{n}{5} + 15\tbinom{n}{6} + \tbinom{n}{7}\\
L_9(n) &= 19\tbinom{n}{4} + 120\tbinom{n}{5} + 105\tbinom{n}{6} + 21\tbinom{n}{7} + \tbinom{n}{8}\\
L_{10}(n) &= 18\tbinom{n}{4} + 220\tbinom{n}{5} + 455\tbinom{n}{6} + 210\tbinom{n}{7} + 28\tbinom{n}{8} + \tbinom{n}{9}\\
L_{11}(n) &= 13\tbinom{n}{4} + 322\tbinom{n}{5} + 1385\tbinom{n}{6} + 1330\tbinom{n}{7} + 378\tbinom{n}{8} + 36\tbinom{n}{9} + \tbinom{n}{10}.
\end{aligned}$$

NOTE: It was conjectured that $L_k(n)$ has real zeros. This conjecture fails, however, for $k = 11$.

19. (a) [3–] Let $\pi = a_1a_2\cdots a_n \in \mathfrak{S}_n$. Let $P_\pi = \{(i, a_i) : i \in [n]\}$, regarded as a subposet of $\mathbb{P}\times\mathbb{P}$. (In other words, define $(i, a_i) \leq (k, a_k)$ if $i \leq k$ and $a_i \leq a_k$.) Let $j(P)$ denote the number of order ideals of the poset P. Show that

$$\sum_{\pi\in\mathfrak{S}_n} j(P_\pi) = \sum_{i=0}^{n} \frac{n!}{i!}\binom{n}{i}.$$

(b) [3] Let π be as in (a), and let $Q_\pi = \{(i, j) : 1 \leq i < j \leq n,\ a_i < a_j\}$. Partially order Q_π by $(i, j) \leq (r, s)$ if $r \leq i < j \leq s$. Show that

$$\sum_{\pi\in\mathfrak{S}_n} j(Q_\pi) = (n+1)^{n-1}.$$

CHAPTER 4

1. [2+] Let M be a diagonalizable $p \times p$ matrix over a field K. Let $\lambda_1, \dots, \lambda_r$ be the distinct nonzero eigenvalues of M. Fix $(i,j) \in [p] \times [p]$. Show that there exist constants $a_1, \dots, a_r \in K$ such that for all $n \in \mathbb{P}$,

$$(M^n)_{ij} = a_1 \lambda_1^n + \cdots + a_r \lambda_r^n.$$

2. (a) [2] By combinatorial reasoning, find the number $f(r,n)$ of sequences $\phi = S_0, S_1, \dots, S_{2n} = \phi$ of subsets of $[r]$ such that for each $1 \le i \le 2n$, either $S_{i-1} \subset S_i$ and $|S_i - S_{i-1}| = 1$, or $S_i \subset S_{i-1}$ and $|S_{i-1} - S_i| = 1$.

   (b) [2] Let $A(r)$ be the adjacency matrix of the Hasse diagram of the boolean algebra B_r. Thus the rows and columns of $A(r)$ are indexed by $S \in B_r$, with

$$A(r)_{S,T} = \begin{cases} 1, & \text{if } S \text{ covers } T \text{ or } T \text{ covers } S \text{ in } B_r \\ 0, & \text{otherwise.} \end{cases}$$

   Use (a) to find the eigenvalues of $A(r)$. (It is more customary to use (b) to solve (a).)

3. [3] Fix $k \in \mathbb{P}$, $n \in \mathbb{N}$. Let $f(n,k)$ denote the number of sequences $\lambda^0, \lambda^1, \dots, \lambda^{2k} = \lambda^0$, satisfying: (i) λ^{2i} is a partition of n, (ii) λ^{2i-1} is a partition of $n-1$, (iii) $\lambda^{2i} \supset \lambda^{2i-1}$ and $\lambda^{2i} \supset \lambda^{2i+1}$ (inclusion of Young diagrams).

   E.g., $f(3,2) = 10$, given by (omitting λ^4)

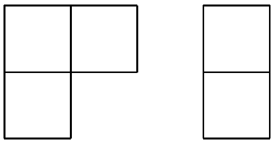
$$\begin{array}{ll} 3,2,3,2 & 21,11,111,11 \\ 3,2,21,2 & 21,11,21,11 \\ 21,2,3,2 & 21,11,21,2 \\ 21,2,21,2 & 111,11,111,11 \\ 21,2,21,11 & 111,11,21,11. \end{array}$$

   Show that

$$f(n,k) = \sum_{i=0}^{n} \left(p(i) - p(i-1)\right)(n-i)^k,$$

   where $p(j)$ denotes the number of partitions of j (with $p(-1) = 0$).

4. (a) [2] Let $f(n)$ be the number of ways a $2 \times n$ chessboard can be partitioned into copies of the following two pieces:



(Any rotation or reflection of the pieces is allowed.) For instance, $f(0) = 1$, $f(1) = 1$, $f(2) = 2$, $f(3) = 5$. Find $F(x) = \sum_{n \geq 0} f(n)x^n$.

(b) [2] Let $g(n)$ be the number of ways if we also allow the piece $\square$. Thus $g(0) = 1$, $g(1) = 2$, $g(2) = 11$. Find $G(x) = \sum_{n \geq 0} g(n)x^n$.

5. (a) [2+] Let A_k be the set of all permutations $w = a_1 a_2 \cdots a_{2k}$ of the multiset $M_k = \{1^2, 2^2, \ldots, k^2\}$ with the following property: If $r < s < t$ and $a_r = a_t$, then $a_s > a_r$. (E.g., $A_2 = \{1122, 1221, 2211\}$.) Let B_k be the set of all permutations $w = a_1 a_2 \cdots a_{2k}$ of M_k with the following property: If $r < s$ and $a_r = a_s < a_t$, then $r < t$. (E.g., $B_2 = \{1122, 1212, 1221\}$.) Let

$$F_k(x) = \sum_{w \in A_k} x^{d(w)},$$

$$G_k(x) = \sum_{w \in B_k} x^{d(w)},$$

where $d(w)$ denotes the number of descents of w. (E.g., $F_2(x) = G_2(x) = 1 + 2x$.) Show that $F_k(x) = G_k(x)$.

(b) [2+] Show that

$$\sum_{n \geq 0} S(n+k, n)x^n = \frac{xF_k(x)}{(1-x)^{2k+1}},$$

where $S(n+k, n)$ denotes a Stirling number of the second kind.

6. [2+] Define polynomials $p_n(u)$ by

$$\sum_{n \geq 0} p_n(u)x^n = \frac{1}{1 - ux - x^2}.$$

Use combinatorial reasoning to find

$$\sum_{n\geq 0} p_n(u)p_n(v)x^n.$$

7. [2] Let $a, b \in \mathbb{R}$. Define a function $f : \mathbb{N} \to \mathbb{R}$ by $f(0) = a, f(1) = b$, and

$$f(n+2) = |f(n+1)| - f(n), \ n \geq 0.$$

Find $F(x) = \sum_{n\geq 0} f(n)x^n$. (If you prefer not to look at a large number of cases, then assume that $0 \leq a \leq b$.)

8. (a) [3] Let S be a finite sequence of positive integers, say 2224211. We can describe this sequence as "three two's, one four, one two, two one's," yielding the *derived sequence* $32141221 = \delta(S)$. Suppose we start with $S = 1$ and form successive derived sequences $\delta(S) = 11$, $\delta^2(S) = 21$, $\delta^3(S) = 1211$, $\delta^4(S) = 111221$, $\delta^5(S) = 312211$, etc. Let $f(n)$ be the length of $\delta^n(S)$, and set

$$F(x) = \sum_{n\geq 0} f(n)x^n = 1 + 2x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + \cdots.$$

Show that $F(x)$ is a rational function which, when reduced to lowest terms, has denominator $D(x)$ of degree 92. Moreover, the largest reciprocal zero $\lambda = 1.30357726903\cdots$ of $D(x)$ is an algebraic integer of degree 71.

(b) [3] Compute the (integer) polynomial $x^{71} - x^{69} - 2x^{68} + \cdots$ of degree 71 for which λ is a zero.

(c) [3+] What if we start with a sequence other than $S = 1$?

9. [2+] Let $\alpha \in \mathbb{C}$, and define for $n \in \mathbb{N}$,

$$f_\alpha(n) = \sum_{k=0}^{n} \binom{n-k}{k} \alpha^k.$$

Show that $F_\alpha(x) := \sum_{n\geq 0} f_\alpha(n)x^n$ is a rational function, and compute it explicitly. Find an explicit formula for $f_\alpha(n)$. What value of α requires special treatment?

10. [2+] Given a sequence $\mathbf{a} = (a_0, a_1, \ldots)$ with entries in a field, and given $n \in \mathbb{N}$, the *Hankel determinant* $H_n(\mathbf{a})$ is defined by

$$H(\mathbf{a}) = \det(a_{i+j})_{0 \le i,j \le n}.$$

Show that the power series $\sum_{n\ge 0} a_n x^n$ is rational if and only if $H_n(\mathbf{a}) = 0$ for all sufficiently large n.

11. (a) [1] Let $f_k(n)$ denote the number of sequences $a_1a_2\cdots a_n \in [k]^n$ of length n such that $a_i \ne a_{i+1}$ for $1 \le i \le n-1$. Show that $f_k(n) = k(k-1)^{n-1}$.

(b) [2+] Let $g_k(n)$ denote the number of sequences as above with additional condition $a_n \ne a_1$. Use the transfer matrix method to show

$$\sum_{n\ge 1} g_k(n)x^n = \frac{k(k-1)x^2}{(1-(k-1)x)(1+x)}.$$

Deduce that

$$g_k(n) = (k-1)^n + (k-1)(-1)^n.$$

(c) [2–] Give a simple combinatorial proof that $g_k(n-1) + g_k(n) = k(k-1)^{n-1}$, and deduce from this the above formula for $g_k(n)$.

12. (a) [2] Let $\Phi = \Phi_n$ be the $1 \times (n+1)$ matrix

$$\Phi = [1, 2, 3, \ldots, n, -n].$$

Show that the number of generators of the monoid E_Φ, as a function of n, is superpolynomial (i.e., grows faster than any polynomial in n).

(b) Compute the generating function

$$E_{\Phi_3}(\mathbf{x}) = \sum_{\alpha \in \Phi_3} \mathbf{x}^\alpha.$$

Express your answer as a rational function reduced to lowest terms.

13. Let $f(n)$ denote the number of 2×3 $\mathbb{N}$-matrices such that every row sums to $3n$ and every column to $2n$. Find an explicit formula for $f(n)$ and compute (as a rational function reduced to lowest terms) the generating function $\sum_{n\ge 0} f(n)x^n$.