ADDITIONAL EXERCISES

for *Enumerative Combinatorics*

by Richard P. Stanley

version of 15 July 2024

Here are some extra exercises for EC1 and EC2. Possibly some of them inadvertently already appear in EC1 and EC2. A few solutions, hints, references, etc., are included at the end of this manuscript. Some of these problems deserve an attribution to whoever submitted or solved them, but I have been unable to find this information.

CHAPTER 1

1. Call two permutations $u, v \in \mathfrak{S}_n$ *equivalent* if v can be obtained from u by sequentially interchanging adjacent elements that differ by 1 (clearly an equivalence relation). For instance, the equivalence classes for $n = 3$ are $\{123, 213, 312\}$ and $\{231, 321, 312\}$.

   (a) [3–] Let $f(n)$ be the number of equivalence classes in $\mathfrak{S}_n$, with $f(0) = 1$. Find a simple formula for $f(n)$ as a finite sum. Use this to express the generating function $F(x) = \sum_{n\geq 0} f(n)x^n$ in terms of the power series $G(x) = \sum_{n\geq 0} n!\, x^n$.

   (b) [2+] Show that the size of every equivalence class is a product of Fibonacci numbers.

   (c) [3–] Let $N(n)$ be the number of one-element equivalance classes in $\mathfrak{S}_n$. Express the generating function $\sum_{n\geq 0} N(n)x^n$ in terms of $G(x)$.

2. (a) [2] Let $0 \leq k \leq 2$. Show that for $n \geq 3$, the number of permutations $w \in \mathfrak{S}_n$ whose number of inversions is congruent to k modulo 3 is independent of k. For instance, when $n = 3$ there are two permutations with 0 or 3 inversions, two with one inversion, and two with two inversions.

   (b) [2+] Let M be the multiset $\{1^{a_1}, \ldots, k^{a_k}\}$, where each $a_i \in \mathbb{P}$. Find a simple characterization of those sequences $(a_1, \ldots, a_k)$ for which the number of permutations of M with an even number of inversions is equal to the number with an odd number of inversions. Your condition should not involve any sums.

3. [3] Let $\sigma(n,k)$ be the number of surjections $[n] \to [k]$. Regarding n as fixed, let c_n be the value of k that maximizes $\sigma(n,k)$. Show that $c_n \sim n/(2\log 2) \approx 0.7213475n$.

4. [2+] Show by simple combinatorial reasoning that the Bell number $B(n)$ is even if and only if $n \equiv 2 \pmod 3$.

5. (a) [2+] For $\mathrm{id} \neq w \in \mathfrak{S}_n$, let $m_1(w)$ be the smallest element of the descent set $D(w)$. Set $m_1(\mathrm{id}) = 0$. Find the expected value $E_1(n)$ of $m_1(w)$ over all $w \in \mathfrak{S}_n$. Express your answer as a simple sum. Find $\delta_1 := \lim_{n\to\infty} E_1(n)$.

   (b) [3] Let $m_k(w)$ denote the kth smallest element of $D(w)$. Set $m_k(w) = 0$ if $\mathrm{des}(w) < k$. Let $\delta_k := \lim_{n\to\infty} E_k(n)$, where $E_k(n)$ is the expected value of $m_k(w)$ for $w \in \mathfrak{S}_n$. Find an explicit formula for the power series $\sum_{k\geq 1} \delta_k x^k$ and an asymptotic formula for δ_k as $k \to \infty$.

6. [2+] Let $f(n)$ be the number of ways to choose a permutation $w \in \mathfrak{S}_n$ and then choose an element of each cycle of w. (Set $f(0) = 1$.) Find a simple formula (no infinite sums, in particular) for $\sum_{n\geq 0} f(n)\frac{x^n}{n!}$. (You don't need to find a formula for $f(n)$.)

7. (a) [2+] Let $1 \leq a \leq b \leq c \leq d$ with $ad = bc$. Show that $\binom{a+d}{a} \leq \binom{b+c}{b}$.

   (b) [5–] Show that the polynomial $\binom{\boldsymbol{b+c}}{\boldsymbol{b}} - \binom{\boldsymbol{a+d}}{\boldsymbol{a}}$ has nonnegative coefficients. Here $\binom{\boldsymbol{n}}{\boldsymbol{k}}$ denotes a q-binomial coefficient.

   (c) [5–] Show in fact that the coefficients of $\binom{\boldsymbol{b+c}}{\boldsymbol{b}} - \binom{\boldsymbol{a+d}}{\boldsymbol{a}}$ are unimodal.

8. [3–] Let p be a prime. Find a simple description of all positive integers d with the following property:

$$A(d,k) \equiv (-1)^{k-1}\binom{d-1}{k-1} \pmod p, \text{ for all } 1 \leq k \leq d,$$

where $A(d,k)$ is an Eulerian number.

9. [3] For $S, T \subseteq [n-1]$, define $\beta_n(S, T)$ to be the number of permutations $w \in \mathfrak{S}_n$ satisfying $D(w) = S$ and $D(w^{-1}) = T$. Let A_n be the $2^{n-1} \times 2^{n-1}$ matrix whose rows and columns are indexed by subsets of $[n-1]$ (in some order), and whose (S, T)-entry is $\beta_n(S, T)$. Show that $\operatorname{rank}(A_n) = p(n)$, the number of partitions of n.

10. (a) [2+] Fix $n \geq 1$. Let $\operatorname{oa}(n)$ be the number of sets $S \subseteq [n-1]$ for which $\alpha_n(S)$ is odd. Find a simple formula for $\operatorname{oa}(n)$ involving the number $f(m)$ of ordered set partitions of an m-element set. NOTE. Though irrelevant here, we have by Example 3.18.10 that $\sum_{m \geq 0} f(m) \frac{x^m}{m!} = 1/(2 - e^x)$.

    (b) [3] Fix $n \geq 1$. Let $\operatorname{ob}(n)$ be the number of sets $S \subseteq [n-1]$ for which $\beta_n(S)$ is odd. Is $\operatorname{ob}(n)$ always a power of 2?

    NOTE. For some analogous problems, see Exercises 1.14(b), 1.15, 4.25, and 7.15.

11. [2] Let $f(n)$ be the number of partitions of n for which each part occurs at most twice. For instance, $f(5) = 5$, the partitions being 5, 41, 32, 311, 221. Let $g(n)$ be the number of partitions of n whose parts are not divisible by three, Show that $f(n) = g(n)$ for all $n \geq 0$.

12. (a) [2+] What curious property does the following power series possess?

$$F(x) = x - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^5 + \frac{13}{64}x^7 - \frac{145}{256}x^9 + \frac{291}{128}x^{11}$$
$$- \frac{6223}{512}x^{13} + \frac{1358965}{16384}x^{15} + \cdots$$

    (b) [5–] Is there a "reasonable" formula for the coefficients?

13. [3–] Let $f_n(q)$ be the number of pairs (A, B) of $n \times n$ matrices over $\mathbb{F}_q$ satisfying $AB = BA$. Show that (using notation from Section 1.10)

$$\sum_{n \geq 0} f_n(q) \frac{x^n}{\gamma_n(q)} = \prod_{i \geq 1} \prod_{j \geq 0} (1 - q^{1-j} x^i)^{-1}.$$

CHAPTER 2

1. [2+] Let $f(n)$ be the number of permutations $w \in \mathfrak{S}_{2n}$ such that we never have $w(i) = i$ for $1 \leq i \leq n$ (*not* $1 \leq i \leq 2n$). Find a formula for $f(n)$ involving a single summation symbol, and find a simple expression (no summations) for $\lim_{n\to\infty} f(n)/(2n)!$.

CHAPTER 3

1. [2] Find a finite poset P with the following property, or show that no such P exists. The longest chain in P has m elements. P can be written as a union of two chains C_1 and C_2, but cannot be written in this way where $\#C_1 = m$.

2. (a) [3–] Find a finite poset P with the following property. The automorphism group $\mathrm{Aut}(P)$ acts transitively on the set M of minimal elements of P. Moreover, the restriction of $\mathrm{Aut}(P)$ to M does not contain a full cycle of the elements of M.

   (b) [5–] Does such a poset exist if all maximal chains have two elements?

3. [2+] Let $w = t_1, \ldots, t_p$ be a permutation of the elements of a finite poset P. Call a permutation w' a *permissible swap* of w if it is obtained from w by interchanging some t_i and t_{i+1} where $t_i < t_{i+1}$. Clearly a sequence of permissible swaps must eventually terminate in a permutation v that has no permissible swaps. Show that v is independent of the sequence of permissible swaps.

4. [2+] Let $0 \leq p \leq 1$, and let P be a finite n-element poset with $\hat{0}$ and $\hat{1}$. Let $\sigma \colon P[n]$ be a linear extension of P. Define a random digraph D on the vertex set $[n]$ as follows. For each $s < t$ in P, choose the edge $s \to t$ of D with probability p.

   Now start at the vertex $\hat{0}$ of D. If there is an arrow from $\hat{0}$, then move to the vertex t for which $\hat{0} \to t$ is an edge of D and $\sigma(t)$ is as small as possible; otherwise stop. Continue this procedure (always moving from a vertex u to a vertex v for which $u \to v$ is an edge of D and $\sigma(v)$ is as small as possible) until unable to continue. What is the probability that we end at vertex $\hat{1}$? Try to give an elegant proof avoiding recurrence relations, linear algebra, etc.

5. (a) [2+] Let $f(n)$ be the average value of $\mu_P(\hat{0},\hat{1})$, where P ranges over all (induced) subposets of the boolean algebra B_n containing $\hat{0}$ and $\hat{1}$. (The number of such P is 2^{2^n-2}.) Define the *Genocchi number* G_n by

$$\sum_{n\geq 0} G_n \frac{x^n}{n!} = \frac{2x}{1+e^x},$$

as in Exercise 5.8(d). Show that $f(n) = 2G_{n+1}/(n+1)$.

(b) [2] It follows from (a) that $f(n) = 0$ when n is even. Give a noncomputational proof.

6. (a) [2] How many nonisomorphic n-element posets contain an $(n-1)$-element chain?

(b) [2] How many nonisomorphic n-element posets contain an $(n-1)$-element antichain?

(c) [2–] How many nonisomorphic n-element posets contain both an $(n-1)$-elements antichain and an $(n-1)$-element chains?

7. [3] Let $f(n)$ denote the number of partial orderings of $[n]$, so for instance $f(1) = 1$, $f(2) = 3$, $f(3) = 19$. Show that for any $N \in \mathbb{P}$, the sequence $\{f(n) \pmod N\}_{n\geq 1}$ is eventually periodic.

8. [2+] Find the number $f(n)$ of pairs (π, σ) of partitions of $[n]$ such that σ covers π in the lattice Π_n of partitions of $[n]$. Express your answer in terms of Bell numbers. It should not involve any summation symbols or implied summations like $B(0) + B(1) + \cdots + B(n)$.

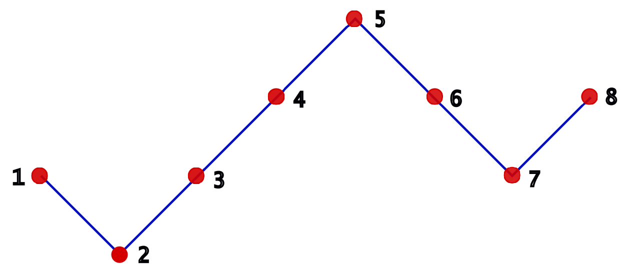
9. [2+] Evaluate the sum

$$F_n := \sum (-1)^{\lfloor \frac{n-1}{2} \rfloor - k},$$

where the sum is over all chains $\emptyset \subset S_1 \subset \cdots \subset S_k \subset [n]$ of subsets of $[n]$ such that $\#S_i$ is even for $1 \leq i \leq k$. The chain $\emptyset \subset [n]$ (the case $k = 0$) contributes $(-1)^{\lfloor \frac{n-1}{2} \rfloor}$ to the sum.

10. (a) [2] Let U_n be the set of all lattice paths λ of length $n-1$ (i.e., with $n-1$ steps), starting at $(0,0)$, with steps $(1,1)$ and $(1,-1)$. Thus $\#U_n = 2^{n-1}$. Regard the n integer points on the path λ as the elements of a poset P_λ, such that λ is the Hasse diagram of P_λ. Find $\sum_{\lambda \in U_\lambda} e(P_\lambda)$.

(b) [2+] Give P_λ the labeling ω_λ by writing the numbers $1, 2, \ldots, n$ on the vertices along the path from left-to-right. For example, when $n = 8$ one possible pair $(P_\lambda, \omega_\lambda)$ is given by



Find $\sum_{\lambda \in U_\lambda} \Omega_{P_\lambda, \omega_\lambda}(m)$ and $\sum_{\lambda \in U_\lambda} W_{P_\lambda, \omega_\lambda}(q)$.

(c) [3–] Let V_n consist of those $\lambda \in U_n$ which never fall below the x-axis. It is well-known that $\#V_n = \binom{n-1}{\lfloor (n-1)/2 \rfloor}$. Show that $\sum_{\lambda \in V_\lambda} e(P_\lambda)$ is equal to the number of permutations $w \in \mathfrak{S}_n$ of odd order. A formula for this number is given in EC2, Exercise 5.10(c) (the case $k = 2$).

(d) [5–] Is there a nice bijective proof or "conceptual proof" of (c)?

(e) [5–] Are there nice expressions for $\sum_{\lambda \in V_\lambda} \Omega_{P_\lambda, \omega_\lambda}(m)$ and/or $\sum_{\lambda \in V_\lambda} W_{P_\lambda, \omega_\lambda}(q)$?

(f) [3–] Now let W_n consist of all $\lambda \in V_{2n+1}$ that end on the x-axis. It is well known that $\#W_n = C_{n-1}$ (a Catalan number). Show that $\sum_{\lambda \in W_n} e(P_\lambda)$ is equal to the Eulerian-Catalan number $\mathrm{EC}_n = A(2n+1, n+1)/(n+1)$ of EC1, Exercise 1.53.

11. [2+] For each permutation $w \in \mathfrak{S}_n$, let σ_w be the simplex in $\mathbb{R}^n$ defined by

$$\sigma_w = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ 0 \le x_{w(1)} \le x_{w(2)} \le \cdots \le x_{w(n)} \le 1.$$

For any nonempty subset $S \subseteq \mathfrak{S}_n$ define

$$X_S = \bigcup_{w \in S} \sigma_w \subset \mathbb{R}^n.$$

Show that X_S is convex if and only if S is the set of linear extensions of some partial ordering of $[n]$.

12. [2+] Let P be a finite poset with $\hat{0}$ and $\hat{1}$. For each $t \in P$ define a polynomial $f_t(x)$ with coefficients in $\mathbb{Z}[y]$ as follows:

$$\begin{aligned} f_{\hat{0}}(x) &= y \\ f_t(x+y) &= \sum_{s \le t} f_s(x). \end{aligned}$$

Express $f_{\hat{1}}(x)$ in terms of the zeta polynomial $Z_P(n)$.

CHAPTER 4

1. (a) [3–] For $0 \le k \le d$ define a polynomial $P_{d,k}(n)$ by

$$\sum_{n \ge 0} P_{d,k}(n) x^n = \frac{(1+x)^k}{(1-x)^{d+1}}.$$

Show that $P_{d,k}(n)$ has positive coefficients.

(b) [5–] Is there a nice combinatorial interpretation of the coefficients of $d!\,P_{d,k}(n)$? The case $k = d$ is especially interesting.

2. [2+] Let $g(n)$ be the number of ways to tile a $2 \times n$ rectangle with $a \times b$ rectangles for any integers $a, b \ge 1$. (Set $g(0) = 1$.) Show that

$$\sum_{m \ge 0} g(n) x^n = \frac{(1-x)(1-3x)}{1-6x+7x^2}.$$

3. (a) [2+] Let $f_k(n)$ be the middle coefficient (i.e., the coefficient of $q^{\lfloor kn/2 \rfloor}$) of the q-binomial coefficient $\binom{\boldsymbol{n+k}}{\boldsymbol{k}}$. Find a simple formula for the generating function $\sum_{n \ge 0} f_3(n) x^n$.

(b) [3–] Show that for any $k \in \mathbb{P}$, $f_k(n)$ is a quasipolynomial.

4. Let $f_k(n)$ denote the number of odd coefficients in the q-binomial coefficient $\binom{\boldsymbol{n}}{\boldsymbol{k}}$.

(a) [2+] Show that

$$\sum_{n \ge 2} f_2(n) x^n = \frac{x^2(1+x)}{(1-x)^2(1+x^2)}.$$

(b) [5–] Show that

$$\sum_{n\geq 3} f_3(n)x^n = \frac{P_3(x)}{\phi_1^2\phi_2^2\phi_3\phi_4^2\phi_6\phi_{12}},$$

where $P_3(x)$ has coefficients (beginning with the coefficient of x^3)

$$1, 4, 4, 8, 6, 4, 8, 4, 6, 8, 4, 4, 1,$$

and where ϕ_k is the kth cyclotomic polynomial.

(c) [3] Show that $f_k(n)$ is a quasipolynomial for fixed k. More generally, if $f_{k,p,j}(n)$ is the number of coefficients of $\binom{\boldsymbol{n}}{\boldsymbol{k}}$ congruent to j modulo the prime p, then for fixed k, p, j the function $f_{k,p,j}(n)$ is a quasipolynomial in n.

5. [2+] Let $f(n)$ be the number of sequences $a_1a_2\cdots a_n$ with terms $1, 2, 3$ such that no two "cyclically consecutive" elements are equal, i.e., we cannot have $a_i = a_{i+1}$ (subscripts taken modulo n), and such that we cannot have 3 cyclically followed by 1. Give a simple formula for $f(n)$ in terms of the Lucas numbers L_n. Use the transfer-matrix method.

6. [3–] Fix integers $d \geq 0$ and $N \geq 1$. Let $f(n)$ be an *integer-valued* quasipolynomial of degree d and quasiperiod N. Suppose that $f(n) = cn^d + O(n^{d-1})$ for some constant $c > 0$. Write

$$\sum_{n\geq 0} f(n)x^n = \frac{P(x)}{Q(x)},$$

where P and Q are relatively prime polynomials. What is the smallest possible value of c? What is the least possible degree of $Q(x)$ for which this value of c is achieved?

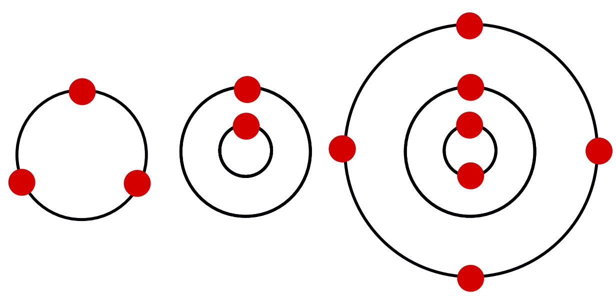
CHAPTER 5

1. [2] Let $h(n)$ be the number of ways to choose a partition π of $[n]$ and then arrange the blocks of π in a cycle. (Set $f(0) = 0$.) Show that

$$E_f(x) = -\log(2 - e^x).$$

2. [2] Let $h(n)$ be the number of ways n children can divide into groups, where each group consists of a nonempty subset of children standing in a circle, with some children (at least one) inside the circle. This is just like Example 5.2.3, except that the circles can contain any positive number of children, not just one (perhaps not very physically realistic). As usual set $h(0) = 1$. For instance, $h(1) = 0$, $h(2) = 2$, $h(3) = 6$, $h(4) = 30$. Find $E_h(x) = \sum_{n\geq 0} \frac{x^n}{n!}$. Your answer should not involve logarithms.

3. (a) [2+] Let $h(n)$ be the number of ways n children can form a collection of concentric circles by holding hands. Any two of the circles either have the same center (i.e., they are concentric) or their interiors are disjoint, an (unlabeled) example with twelve children being



Show that

$$\begin{aligned} E_h(x) &= (1-x)^{-1/(1+\log(1-x))} \\ &= 1 + x + 4\frac{x^2}{2!} + 24\frac{x^3}{3!} + 190\frac{x^4}{4!} + 1860\frac{x^5}{5!} + \cdots. \end{aligned}$$

(b) [2+] Now the children can arrange themselves into circles in *any* way. That is, inside any circle C is a disjoint union (possibly empty) of circles with a similar structure inside each of them. Show that

$$\begin{aligned} E_h(x) &= 1 + (1-(1+x)^{1/(1+x)})^{\langle -1\rangle} \\ &= 1 + x + 4\frac{x^2}{2!} + 27\frac{x^3}{3!} + 260\frac{x^4}{4!} + 3280\frac{x^5}{5!} + \cdots. \end{aligned}$$

4. [2+] Let $f(n)$ be the number of distinct graphs G (allowing multiple edges) on the vertex set $[2n]$ such that the edges of G can be partitioned

into two complete matchings. Thus G has $2n$ edges. Find a simple formula for the generating function

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{(2n)!} = 1 + \frac{x}{2!} + 6\frac{x^2}{4!} + \cdots .$$

5. [2+] Let $H(x) = \sum_{n\geq 0} h(n)\frac{x^n}{n!}$, where $h(n)$ is the number of certain structures that can be put on an n-set as in Section 5.1. (Thus each structure is uniquely a disjoint union of connected structures.) Let $r(n)$ (respectively, $s(n)$) be the number of ways of putting a structure counted by $h(n)$ on an n-set and then putting connected components into a cycle (respectively, linearly ordering them). Express $E_r(x)$ and $E_s(x)$ in terms of $H(x)$. Use this to express $E_s(x)$ in terms of $E_r(x)$. Then give a simple explanation for this last formula.

6. [3–] Let $T(x) = \sum_{n\geq 0} n^{n-1}\frac{x^n}{n!}$ and $U(x) = \sum_{n\geq 0} n^n\frac{x^n}{n!}$. Show that

$$U(x)^3 - U(x)^2 = \frac{T(x)}{(1-T(x))^3} = \sum_{n\geq 0} n^{n+1}\frac{x^n}{n!}.$$

7. [2+] Fix $k \geq 1$. Choose an unrooted tree T on the vertex set $[n]$ uniformly at random (so a given T has probability $n^{-(n-2)}$ of being chosen). What is the probability $p_k(n)$ that vertex 1 has degree k (i.e., has exactly k neighbors)? Find $\lim_{n\to\infty} p_k(n)$.

8. (a) [2+] Let $f(n)$ be the number of ways to choose a rooted tree T on $[n]$ and then for each vertex v of T, either do nothing or choose a child of v. (Thus if v is an endpoint then we have only one choice—do nothing.) For instance, $f(1) = 1$, $f(2) = 4$, $f(3) = 33$. Find a formula for $f(n)$ as a simple sum.

   (b) [3–] Give a simple combinatorial proof.

9. (a) [2] Let $f(n)$ denote the number of rooted trees on the vertex set $[n]$ whose endpoints (leaves) are colored either red or blue. Find a functional equation (analogous to the equation $y = xe^y$ satisfied by the exponential generating function for rooted trees) satisfied by the exponential generating function

$$y = E_f(x) = 2x + 4\frac{x^2}{2!} + 24\frac{x^3}{3!} + 224\frac{x^4}{4!} + \cdots .$$

(b) [2] Use the Lagrange inversion formula and (a) to deduce that

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} k^{n-1}.$$

(When $n = 1$ and $k = 0$ in the above sum, set $0^0 = 1$.)

(c) [2+] Give a direct combinatorial proof of (b), analogous to either the first or second proof of Proposition 5.3.2.

10. (a) [2+] Let L_n denote the complete graph K_n with an n-cycle removed (so L_n has $\binom{n}{2} - n$ edges). Let $f(n)$ denote the number of spanning trees of L_n, so $f(3) = f(4) = 0$, $f(5) = 5$, $f(6) = 75$, etc. Use the Matrix-Tree theorem to show that

$$f(n) = \frac{1}{n} \prod_{k=1}^{n-1} \left(n - 2 + 2\cos\frac{2\pi k}{n} \right).$$

You will need to use standard results on eigenvalues of circulant matrices.

(b) [3–] Define polynomials $V_n(x)$ by $V_1(x) = 1$, $V_2(x) = x - 2$, and

$$V_n(x) = xV_{n-1}(x) - V_{n-2}(x) - 2(-1)^n \text{ if } n \geq 3.$$

Show that $f(n) = \frac{1}{n} V_n(n-2)$.

(c) [2] Deduce from (b) (using formulas for solving linear recurrences with constant coefficients) the formula

$$f(n) = \frac{1}{n^2} \left[2(-1)^{n+1} + \left(\frac{n}{2} - 1 + \sqrt{\frac{n^2}{4} - n} \right)^n + \left(\frac{n}{2} - 1 - \sqrt{\frac{n^2}{4} - n} \right)^n \right]$$

(d) [2+] Deduce from (c) the asymptotic formula

$$f(n) \sim e^{-2} n^{n-2},$$

i.e., $\lim_{n\to\infty} f(n)/e^{-2}n^{n-2} = 1$. Obtain further terms of the asymptotic expansion of $f(n)$.

11. [2] Let E_n be the cube graph C_n (Example 5.6.10) with an additional edge between each antipodal pair of vertices, i.e., between α and $\alpha + (1,1,\ldots,1)$ for all $\alpha \in (\mathbb{Z}/2\mathbb{Z})^n$. Thus every vertex of E_n has degree $n+1$. Show that the number $c(E_n)$ of spanning trees of E_n is given by

$$c(E_n) = 2^{2^{n+1}-n-2} \prod_{k=1}^{\lfloor (n+1)/2 \rfloor} k^{\binom{n+1}{2k}}.$$

12. [3–]

(a) Let $f(n)$ for $n \geq 2$ be the number of Eulerian digraphs on the vertex set $[n]$ with no loops and with exactly one Eulerian tour (up to cyclic shift). For instance, $f(3) = 5$; two such digraphs are triangles, and three consist of two 2-cycles with a common vertex. Show that $f(n) = \frac{1}{2}(n-1)!C_n = (2n-1)_{n-2}$, where C_n denotes a Catalan number.

(b) Now suppose that loops are allowed, and let $g(n)$ be the number of digraphs on $[n]$ with exactly one Eulerian tour. Show that $g(n) = (n-1)!(s_{n-1} + s_n)$, where s_n denotes a Schröder number as on page 178 of EC2. For instance, $g(3) = 2!(3+11) = 28$. Equivalently,

$$\sum_{n\geq 1} g(n)\frac{x^n}{(n-1)!} = \frac{(1-x)^2 - (1+x)\sqrt{1-6x+x^2}}{4x}.$$

CHAPTER 6

1. [3+] Show that the four power series

$$\begin{aligned}
F_1(x) &= \sum_{n\geq 0} \frac{(6n)!\, n!}{(3n)!\, (2n)!^2} x^n \\
F_3(x) &= \sum_{n\geq 0} \frac{(10n)!\, n!}{(5n)!\, (4n)!\, (2n)!} x^n \\
F_3(x) &= \sum_{n\geq 0} \frac{(20n)!\, n!}{(10n)!\, (7n)!\, (4n)!} x^n \\
F_4(x) &= \sum_{n\geq 0} \frac{(30n)!\, n!}{(15n)!\, (10n)!\, (6n)!} x^n
\end{aligned}$$

are algebraic. What are their (minimal) degrees?

2. (a) [2+] Let $f(n)$ be the number of plane trees with n vertices such that if a vertex u has exactly one child v, then v is an endpoint. Let

$$F(x) = \sum_{n \geq 1} f(n)x^n = x + x^2 + x^3 + 3x^4 + \cdots .$$

Find an explicit formula for $F(x)$ and $F(x)^{\langle -1 \rangle}$.

   (b) [5–] Is there some simple combinatorial explanation for the relationship between these two generating functions? Can this phenomenon be generalized?

3. (a) [2+] Let t be an indeterminate. Find the coefficients of the generating function $F(x,y) = 1/(1-x-y)^t$.

   (b) [3–] Find a simple formula (involving a single finite sum) for the diagonal $D(z) = \mathcal{D}F(x,y)$ when $t \in \mathbb{P}$.

4. [5–] Let $f(n)$ be an integer-valued unbounded P-recursive function. Show that $f(n)$ is composite for infinitely many positive integers n. (This surely must be true, since otherwise there is a simple recurrence for generating arbitrarily large primes.) Perhaps the result is already known, but I have been unable to find it in the literature.

5. (a) [2+] Let $f_d(n)$ be the number of walks in the first quadrant of $\mathbb{Z}^d$ (i.e., all coordinates nonnegative) starting at the origin and with steps $\pm e_i$, where e_i is the ith unit coordinate vector. Show that for fixed d, the function $f_d(n)$ is P-recursive.

   (b) [3–] Find a simple formula for $f_2(n)$ and a three-term recurrence relation with polynomial coefficients satisfied by $f_2(n)$.

6. [2] Let $f\colon \mathbb{N} \to \mathbb{Q}$ be P-recursive, and let d be the least integer for which there is a recurrence

$$P_d(n)f(n+d) + P_{d-1}(n)f(n+d-1) + \cdots + P_0(n)f(n) = 0, \ n \geq 0,$$

with $P_i(n) \in \mathbb{C}[n]$ and $P_d(n) \neq 0$. Show that there exists such a recurrence (of the same degree d) with $P_i(n) \in \mathbb{Z}[n]$ for $0 \leq i \leq d$.

7. (a) [3–] Show that $f(n) = n^n$ is not P-recursive.

(b) [2+] Does there exist a P-recursive function $f\colon \mathbb{N} \to \mathbb{R}$ such that $f(n) \sim n^n$, i.e., $\lim_{n\to\infty} \frac{f(n)}{n^n} = 1$?

(c) [3–] Same as (b), except $f\colon \mathbb{N} \to \mathbb{Q}$.

8. [2] Let $f\colon \mathbb{N} \to \mathbb{C}$ be P-recursive. Show that $\frac{\log f(n)}{n \log n}$ is bounded as $n \to \infty$. Thus for instance 2^{n^2} is not P-recursive.

9. (a) [3–] Show that for $\alpha \in \mathbb{R}$ there exists a P-recursive function $f\colon \mathbb{N} \to \mathbb{R}$ satisfying $f(n) \sim n^\alpha$.

   (b) [2–] Let $\mathcal{A}$ be the set of all $\alpha \in \mathbb{R}$ for which there exists a P-recursive function $f\colon \mathbb{N} \to \mathbb{Z}$ satisfying $f(n) \sim n^\alpha$. Show that $\mathcal{A}$ is a submonoid of the additive reals.

   (c) [3] Show that $2^{-k} \in \mathcal{A}$ for all $k \in \mathbb{N}$ and that $\sqrt{17} \in \mathcal{A}$.

   (d) [5–] What more can be said about the monoid $\mathcal{A}$?

10. [2+] Let $f(n) = n! + \frac{1}{n!}$. Using just hand computation, find a nontrivial linear recurrence with polynomial coefficients satisfied by $f(n)$.

11. [2] Let $u \in K[[x]]$ be D-finite, say

$$p_d(x)u^{(d)} + p_{d-1}(x)u^{(d-1)} + \cdots + p_1(x)u' + p_0(x)u = 0.$$

Find a nontrivial homogeneous linear differential equation with polynomial coefficients satsified by $u+1$. (You can write the coefficients in any form that makes it clear that they are polynomials in x.)

12. (a) [2] Let $1 \le n_1 < n_2 < n_3 < \cdots$ $(n_i \in \mathbb{P})$, and let

$$F(x) = \sum_{i\ge 0} a_i x^{n_i} \in K[[x]],$$

where $\mathrm{char}(K) = 0$ and $a_i \neq 0$ for all $i \ge 0$. Show that if

$$\limsup_{i\to\infty}(n_{i+1} - n_i) = \infty,$$

then $F(x)$ is not D-finite. For instance, $\sum_{n\ge 0} x^{\binom{n}{2}}$ is not D-finite.

   (b) [2] Show that (a) need not be true if $\mathrm{char}(K) > 0$. Show in fact that (a) can fail for algebraic $F(x)$.

(c) [5–] Suppose that $\limsup_{i\to\infty} n_i^{1/i} = \infty$ Is it true that then $F(x)$ is not D-finite for *any* field K? (Perhaps this is already known.)

13. [3+] Fix a subset $\mathcal{F}$ of $\mathfrak{S}_k$, and let $\mathrm{Av}_n(\mathcal{F})$ denote the number of permutations $w \in \mathfrak{S}_n$ that avoid the patterns in $\mathcal{F}$ (in the sense the second paragraph of EC1, page 43, or `en.wikipedia.org/wiki/Permutation_pattern`). Show that $\mathrm{Av}(\mathcal{F})$ need not be D-finite.

CHAPTER 7

1. Define symmetric functions P_n by the formula

$$2P_n = \sum_{k=0}^{n} e_k h_{n-k}.$$

(a) [2–] Show that

$$1 + 2\sum_{n\geq 1} P_n(x)t^n = \prod_i \frac{1+x_it}{1-x_it}.$$

(b) [2] Recall the notation $f(1^m) = f(x_1 = \cdots = x_m = 1, x_i = 0 \text{ for } i > m)$. Show that $\frac{2}{n+1}P_n(1^{n+1})$ is equal to the Schröder number r_n.

(c) [2+] Let $A = P_1t + P_3t^3 + P_5t^5 + \cdots$. Show that

$$P_2t^2 + P_4t^4 + P_6t^6 + \cdots = \frac{-1+\sqrt{1+4A^2}}{2}.$$

2. Let $d \geq 1$, and write $\mathfrak{T}_{n,d} = \{w \in \mathfrak{S}_n \,:\, w^d = 1\}$. Let L_λ be the Lyndon symmetric function of Supplementary Problem 127.

(a) [2+] Show that

$$\sum_{\substack{m,n,e\geq 1\\ me|d}} \frac{1}{mne}\mu(e)p_{ne}^m = \sum_{j\geq 1} \frac{1}{j} p_{j/\gcd(j,d)}^{\gcd(j,d)},$$

where μ denotes the usual number-theoretic Möbius function.

(b) [3–] Deduce that

$$\sum_{\substack{\lambda\vdash n\\ \lambda_i | d}} L_\lambda = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} p_{\rho(w^d)}.$$

(c) [2+] Fix n, and let $d(T)$ denote the number of descents of the SYT T. Using the notation of Corollary 7.23.8, show that

$$\sum_{\substack{T \text{ is an SYT}\\ \text{of shape } \lambda\vdash n}} q^{d(T)} s_\lambda = \sum_{S\subseteq[n-1]} q^{\#S} s_{B_S}.$$

(d) [3–] Show that

$$\frac{1}{n!}\sum_{w\in\mathfrak{S}_n} q^{-1}A_{\kappa(w^d)}(q)(1-q)^{n-\kappa(w^d)} = \sum_{w\in\mathfrak{T}_{n,d}} q^{\operatorname{des}(w)}. \qquad (1)$$

Here $\kappa(w^d)$ denotes the number of cycles of w^d, and $A_m(q)$ is an Eulerian polynomial.

(e) [2] Deduce that if d is even, then

$$\#\{w\in\mathfrak{T}_{n,d} \,:\, \operatorname{des}(w)=i\} = \#\{w\in\mathfrak{T}_{n,d} \,:\, \operatorname{des}(w)=n-1-i\}.$$

(f) [2+] Let $E(n,d)=\sum_{w\in\mathfrak{T}_{n,d}}\operatorname{des}(w)$. Show that if d is odd then

$$E(n,d)=\frac{n-1}{2}\#\mathfrak{T}_{n,d},$$

while if n is even then

$$E(n,d)=\frac{n-1}{2}\#\mathfrak{T}_{n,d}-\frac{1}{n}\#\{w\in\mathfrak{S}_n \,:\, \kappa(w^d)=n-1\}.$$

(g) [2+] Strengthen (e) by showing that for $S\subseteq[n-1]$ and d even,

$$\#\{w\in\mathfrak{T}_{n,d} \,:\, D(w)=S\} = \#\{w\in\mathfrak{T}_{n,d} \,:\, D(w)=[n-1]-S\},$$

where $D(w)$ denotes the descent set of w.

Hints, Solutions, References, Etc.

CHAPTER 1

1. See R. Stanley, *J. Combinatorics* **3** (2012), 277–298; arXiv:1208.3540.

3. See `mathoverflow.net/questions/29490`.

7. See F. Zanello, *Electronic J. Comb.* **25** (2018), P2.17.

12. Hint: consider $F(x)^{\langle -1\rangle}$.

13. See W. Feit and N. J. Fine, *Duke Math. J.* **27** (1960), 91–94. A generalization is due to Y. Huang, *Algebraic Combinatorics* **5** (2022), 583–592; arXiv:2110.15570.

CHAPTER 3

5. (a) Use Philip Hall's theorem (Proposition 3.8.5).

7. For N prime or the square of a prime, see `mathoverflow.net/questions/40390`. This site seems to say the result is true for all N, but I don't see a proof there. There is a broken link to a proof by Borevich. At `link.springer.com/article/10.1007/BF01094365` there is a proof for topologies (=preposets) on a finite set. Presumably this proof technique works also for posets.

CHAPTER 4

3. See R. Stanley and F. Zanello, *Ann. Comb.* **20** (2016), 623–634; arXiv:1503.06367. The answer to (a) is
$$\frac{1-x+x^2}{(1-x)^2(1-x^4)}.$$

CHAPTER 5

12. This exercise first appeared as an exercise in R. Stanley, *Algebraic Combinatorics*, second ed., Springer, New York, 2018 (Exercise 10.6).

CHAPTER 6

1. See page 781 of M. Kontsevich and D. Zagier, *Periods*, in *Mathematics Unlimited—2001 and Beyond* (B. Engquist and W. Schmid, eds), Springer, Berlin, 2001. See also OEIS A211417. The degree of $F_4(x)$ is 483840.

7. (a) This result appears as Exercise 1.6.2 in M. Kauers, *D-Finite Functions*, Springer, Cham, Switzerland, 2023.
   (b) One example is $\frac{1}{\sqrt{2}}\left(\frac{e}{4}\right)^n\frac{(2n)!}{n!}$.
   (c) A negative answer was provided by Fedor Petrov, MathOverflow 474664.

13. See I. Pak and S. Gallabrant, arXiv:1505.06508.

CHAPTER 7

1. See R. Stanley, arXiv:2405.02164.

2. (b) By EC2, equation (7.174), we have

$$\sum_n \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} p_{\rho(w^d)} = \sum_{j\geq 1}\frac{1}{j}p_{j/\gcd(j,d)}^{\gcd(j,d)}.$$

On the other hand, in Supplementary Problem 128(b) apply the homomorphism taking $p_i(y)=1$ if $i|d$ and $p_i(y)=0$ if $i\nmid d$. We get

$$\sum_n \sum_{\substack{\lambda\vdash n\\ \lambda_i|d}} L_\lambda = \exp\sum_{j\geq 1}\frac{1}{j}p_{j/\gcd(j,d)}^{\gcd(j,d)}, \tag{2}$$

and the proof follows.

(d) By Exercise 7.110 and part (c) of this problem we have

$$\sum_{S\subseteq[n-1]} q^{\#S} s_{B_S} = \sum_\lambda z_\lambda^{-1}\, q^{-1}(1-q)^{n-\ell} A_\ell(q)\, p_\lambda. \tag{3}$$

Take the scalar product of equation (3) with $\frac{1}{n!}\sum_{w\in\mathfrak{S}_n} p_{\rho(w^d)}$ and apply Supplementary Exercise 127(a). The proof follows from the orthogonality relation $\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda\mu}$ (Proposition 7.9.3).

Is there a simpler proof avoiding symmetric functions?

(e) Use the fact that $q^{m+1}A_m(1/q) = A_m(q)$ and that $n - \kappa(w^d)$ is even when d is even.

(f) Differentiate equation (1) with respect to q, set $q = 1$, and simplify.

(g) A permutation $w \in \mathfrak{S}_n$ satisfies $w^d = 1$ if and only if every cycle of w has length dividing d. Set

$$g_{nd} = \sum_{\substack{\lambda\vdash n\\ \lambda_i | d}} L_\lambda.$$

Since $\omega s_{B_S} = s_{B_{[n-1]-S}}$, it follows from Supplementary Problem 127(a) that we need to show that $\omega g_{nd} = g_{nd}$. Now from (a), (b), and equation (2),

$$\sum_{n\geq 0} g_{nd} = \exp \sum_{j\geq 1} \frac{1}{j} p_{j/\gcd(j,d)}^{\gcd(j,d)}.$$

Since $p_{j/\gcd(j,d)}^{\gcd(j,d)}$ is ω-invariant when d is even, the proof follows.

Is there a combinatorial proof?