ERRATA AND ADDENDA

to

Enumerative Combinatorics, volume 1

by

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longer corrections listed separately in the second printing, hardcover edition of 1997, pp. 319–325

- p. vii, line 4–. Insert "(in two volumes)" after "include".
- p. 2, Example 1.1.3, l. 10. Change "$f(0) = f(1) =$" to "$f(0) = 1, f(1) =$".
- p. 19, second paragraph of second proof, l. 2. Change b_k to b_{n-k}.
- p. 25, l. 12–. Change "unoriented" to "unordered".
- p. 41, last paragraph. The statement that pictorial representations (actually incense diagrams) were used in *The Tale of Genji* to represent the 52 partitions of a 5-element set is misleading. These diagrams were not used by Lady Murasaki herself, but rather were introduced by the Wasanists during the Wasan period of old Japanese mathematics, from the late 1600's well into the 1700's.
- p. 42, ref. 16. The correct reference should be P. R. Stein, A brief history of enumeration, in *Science and Computers*, a volume dedicated to Nicholas Metropolis (G.-C. Rota, ed.), Academic Press, 1986, pp. 169–206.
- p. 46, l. 2–. Change this line to:

  [3–] b. Give a combinatorial proof.
- p. 46, Exercise 11(a). Add at end: (The elements of N and X are assumed to be distinguishable.)
- p. 46, Exercise 14(g). The sum shoud be:

$$\sum (2^{a_1-1} - 1) \cdots (2^{a_k-1} - 1).$$

- p. 49, Exercise 28. It is assumed that every number $1, 2, \ldots, k$ occurs at least once.
- p. 54, Exercise 8(d). Macdonald's conjecture (for general q) for the root system G_2 has been verified independently by L. Habsieger, *C.R. Acad. Sci. Paris (Série I)* **303** (1986), 211–213, and D. Zeilberger, *SIAM J. Math. Anal.* **18** (1988), 880–883, and for

the root systems B_n, C_n, and D_n by K. Kadell (preprint). For F_4 a proof has been given by F. G. Garvan and G. H. Gonnet, *Bull. Amer. Math. Soc. (new series)* **24** (1991), 343–347. For $q = 1$ a proof for all root systems was given by E.M. Opdam, *Invent. math.* **98** (1989), 1–18. The general case of Macdonald's conjecture was finally proved by I. Cherednik in 1993 and published in *Ann. Math.* **141** (1995), 191–216. Many other papers have been written on these conjectures. See the above-mentioned paper by Garvan and Gonnet for some additional references.

- p. 56, Exercise 14(f), l. 3. Change "between" to "in".

- p. 57, Exercise 16, l. 1. Change $a_1 + \cdots + a_i$ to $a_k + a_{k-1} + \cdots + a_{k+1-i}$.

- p. 57, Exercise 17. Insert at the end of this solution:

  The first of several persons to find a combinatorial proof were K. L. Collins and M. Hovey, *Combinatorica* **31** (1991), 31–32.

- p. 57, Exercise 19. A solution to (d) was found by several persons. A more general bijection will be discussed in Chapter 5.

- p. 59, Exercise 26. Insert at end: "The bijection given here also appears in A. H. M. Hoare, Amer. Math. Monthly **93** (1986), 475–476." Another proof appears in L. Solomon, Istituto Nazionale di Alta Matematica, Symposia Mathematica, vol. 13 (1974), 453–466 (lemma on p. 461).

- p. 69, two lines below equation (15). Insert "$f(i, i) = 1$ and" after "satisfying".

- p. 73, second proof, l. 4. Change "k spaces" to "$m - k$ spaces".

- p. 74, l. 11–. Should be:

$$B = \{(i, j) : 1 \leq j \leq m,\ 1 \leq i \leq b_j\}.$$

- p. 76, l. 9–. Delete "(where we take $(\mathbf{j})!$ in the variable x)".

- p. 76. Change $(\mathbf{k-1})!$ to $(1-x)(1-x^2)\cdots(1-x^{k-1})$ (twice), and change $(\mathbf{k})!$ to $(1-x)(1-x^2)\cdots(1-x^k)$ (three times).

- p. 77. Change $(\mathbf{k})!^{\mathbf{2}}$ to $(1-x)^2(1-x^2)^2\cdots(1-x^k)^2$.

- p. 84, l. 7–8. Replace "and let j be the least integer > 1 for which L_i and L_j intersect" with "and let x be the least integer such that L_i intersects some L_k with $k > i$ at a point (x, y), and then of all such k let j be the minimum".

- p. 85, Notes. A survey of the menage problem appears in J. Dutka, *Math. Intelligencer* **8** (1986), no. 3, 18–25 and 33.

- p. 96, l. 8–. Regarding the statement that "the coefficient -2 depends only on the *partial order relation* among A, B, C, D," it is perhaps unclear that the assumption $D = A \cap B = A \cap C = B \cap C = A \cap B \cap C$ remains in effect.

- p. 98, l. 4–. Change "horizontal" to "vertical".

- p. 106, paragraph 3. In the definition of "join-irreducible," we make the convention that $\hat{0}$ is *not* join-irreducible.

- p. 132, l. 1 of proof of Thm. 3.12.1. Change "from Proposition 3.5.1" to "by definition".

- p. 135. For additional information on Eulerian posets, see R. Stanley, in *Polytopes: Abstract, Convex and Computational* (T. Bisztriczky, et al., eds.), Kluwer, Dordrecht/Boston/London, 1994, pp. 301–333.

- p. 136, l. 6. Change second $(-1)^n$ to $(-1)^{n-1}$.

- p. 140, l. 3. Change this line to

$$g(Q) + yf(Q) = \begin{cases} (a_s - a_{s-1})x^{s+1} + (a_{s-1} - a_{s-2})x^{s+2} + \cdots, & r \text{ even} \\ (a_s - a_{s-1})x^{s+2} + (a_{s-1} - a_{s-2})x^{s+3} + \cdots, & r \text{ odd} \end{cases}$$

- p. 146, Example 3.15.10, l. 5–. Change $c(n)x^n$ to $c(n)x^n/n!$.

- p. 148, l. 2. Should be:

$$\sum_{n\geq 0} \mu(kn)(B(k)x)^n/B(kn) = \left[\sum_{n\geq 0}(B(k)x)^n/B(kn)\right]^{-1}.$$

- p. 149, l. 1. Change "By definition" to "It is clear that".

- p. 149, l. 10. Insert "reverse" between "choose" and "alternating".

- p. 154, Exercise 3(a). If $f(n)$ denotes the number of nonisomorphic n-element posets, then $f(8) = 16999$, $f(9) = 183231$, $f(10) = 2567284$, $f(11) = 46749427$, $f(12) = 1104891746$, $f(13) = 33823827452$. For the last of these and further references, see C. Chaunier and N. Lygerös, *Order* **9** (1992), 203–204.

- p. 154, Exercise 5(b). Assume also in the hypothesis that every element of P is contained in a chain of length ℓ. (Equivalently, either $\ell = 0$ or else P contains no isolated points.)

- p. 165, Exercise 51. The definition of $Q_n(G)$ is not correct. Define two partial G-partitions $\alpha = \{a_1, \ldots, a_r\}$ and $\beta = \{b_1, \ldots, b_s\}$ to be *equivalent* if their underlying partial partitions are the same (so $r = s$), say $\{A_1, \ldots, A_r\}$, and if for each $1 \leq j \leq r$, there is some $w \in G$ (depending on j) such that $a_j(x) = w \cdot b_j(x)$ for all $x \in A_j$. The elements of $Q_n(G)$ are equivalence classes of partial G-partitions. Representing a class by one of its elements, define $\alpha = \{a_1, \ldots, a_r\} \leq \beta = \{b_1, \ldots, b_s\}$ in $Q_n(G)$ if every block A_i of the underlying partial partition $\{A_1, \ldots, A_r\}$ of α is either (1) contained in a block B_j of the underlying partial partition σ of β, in which case there is a $w \in G$ for which $a_i(x) = w \cdot b_j(x)$ for all $x \in A_i$, or else (2) every block of σ is disjoint from A_i. (Thus $Q_n(G)$ has a top element consisting of the empty set.) The poset $Q_n(G)$ is called a *Dowling lattice*.

- p. 166, l. 11–. Change "meet-semilattice" to "lattice". (In fact, if $x \neq \hat{1}$, then the interval $[\hat{0}, x]$ of L is a geometric lattice.)

- p. 167, Exercise 56. For a recent extensive treatment of arrangements, see P. Orlik, *CBMS Lecture Notes* **72**, Amer. Math. Soc. 1989. Even more extensive is the book P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer, 1992.

- p. 167, Exercise 56(d). The arrangement is not free for $n \geq 9$. This is a result of G. M. Ziegler, *Advances in Math.* **101** (1993), 50–58.

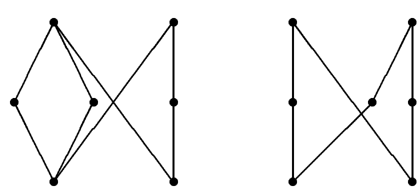
- p. 167, Exercise 56(g). This was a question of P. Orlik. A counterexample was discovered by P. Edelman and V. Reiner, *Proc. Amer. Math. Soc.* **118** (1993), 927–929.

- p. 168, Exercise 63. We must define

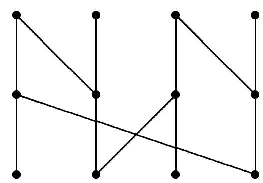
$$\binom{\lambda_i - \mu_j + n}{i - j + n} = 0$$

  if $\lambda_i - \mu_j + n < 0$.

- p. 168, Exercise 64. Change the rating of this exercise from [5–] to [3]. Of the 2045 nonisomorphic seven-element posets, it has been checked using John Stembridge's Posets package for Maple that there is a unique pair P, Q which satisfy $\beta(J(P), S) = \beta(J(Q), S)$ for all $S \subseteq [6]$. The Hasse diagrams of P and Q look as follows:



- p. 175, Exercise 6. Ziegler's example is incorrect. However, he has produced several correct examples, the smallest of which is



Ziegler also has an example of length one with 24 elements, and an example which is a graded lattice of length three with 26 elements.

- p. 175, Exercise 6, l. 2. Delete "presumably".
- p. 179, Exercise 19(d). J. Stembridge has given another proof of this result in *European J. Combinatorics* **7** (1986), 377–387 (Corollary 2.2).
- p. 179, Exercise 19(f), l. 1–. Change ", to appear" to "**7** (1986), 377–387", and insert after "results": "(see in particular Corollary 2.4)".
- p. 179, Exercise 20. A survey of meet-distributive lattices is given by P. Edelman, *Contemporary Math.* **57** (1986), 127–150.
- p. 186, l. 11. Delete "essentially".
- p. 186. l. 11–12. A further reference is J. Kung, *Math. Proc. Cambridge Philos. Soc.* **101** (1987), 221–231.
- p. 186, l. 12. Change "The version given here ..." to "The solution given here ..."
- p. 186, Exercise 39(b). One can give a simple direct proof (avoiding Möbius functions) of the following stronger result: Let L be a finite lattice with n elements, such that for all $x > \hat{0}$ in L, x is the join of atoms of the interval $[\hat{0}, x]$. Then every $x > \hat{0}$ satisfies $|V_x| \leq n/2$.
- p. 187, Exercise 42. The value $|\mu(\hat{0}, \hat{1})| = (k-1)(\ell^{k-1} - 1)$ was achieved by G. Ziegler, *J. Combinatorial Theory (A)* **56** (1991), 203–222.
- pp. 191–192, Exercise 53(c). The following result is due to J. Shareshian, in a preprint entitled "On the Möbius number of the subgroup lattice of the symmetric group," extending work begun in his 1994 Ph.D. thesis at Rutgers University and also work of H. Pahlings.

  Theorem. *Write* $\mu_n = \mu(\hat{0}, \hat{1})$.

  (i) *Let p be an odd prime. Then* $\mu_p = p!/2$.

  (ii) *Let $n = 2p$, where p is an odd prime, $p \neq 11$. Then*

$$\mu_n = \begin{cases} -n!, & \text{if } n-1 \text{ is prime and } p \equiv 3 \pmod 4 \\ -\frac{n!}{2}, & \text{otherwise.} \end{cases}$$

(iii) *Let* $n = 2^a$, *where* $a \in \mathbb{N}$. *Then* $\mu_n = -n!/2$.

Note that (ii) shows that the formula stated in (c) is false, e.g., for $n = 14$.

- p. 192, Exercise 54(b), l.2. Change ", to appear" to "**42** (1986), 215–222".

- p. 192, l. 5– to 4–. Change "is ... proof?" to "This follows from the fact that the localization of a free module is free (as observed by L. Billera). Earlier Terao had given a deeper proof. See Thm. 3.8.3 of G. M. Ziegler, Ph.D. thesis, M.I.T., 1987. This thesis provides a highly readable account of the algebraic properties of hyperplane arrangements, including the theory of the module $\Omega(X)$. For a more recent extensive treatment of arrangements, see P. Orlik, *CBMS Lecture Notes* **72**, Amer. Math. Soc. 1989, and P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer, 1972."

- p. 194, Exercise 60(a). The paper of D. Kelly appears in *Order* **3** (1986), 155–158.

- p. 195, Exercise 66, line 5. Delete $v_1, v_2, \ldots$.

- p. 197, Exercise 68. The study of non-crossing partitions goes back at least to H. W. Becker, *Bull. Amer. Math. Soc.* **58** (1952), 39 (where they are called "planar rhyme schemes"). Further results on non-crossing partitions are given by H. Prodinger, *Discrete Math.* **46** (1983), 205–206; N. Dershowitz and S. Zaks, *Discrete Math.* **62** (1986), 215–218; R. Simion and D. Ullman, *Discrete Math.* **98** (1991), 193–206; P. H. Edelman and R. Simion, *Discrete Math.* **126** (1994), 107–119; and R. Simion, *J. Combinatorial Theory (A)* **66** (1994), 270–301. For a surprising connection between non-crossing partitions and Voiculescu's theory of "free random variables," see R. Speicher, *Math. Annalen* **298** (1994), 611–628, and A. Nica and R. Speicher, A "Fourier transform" for multiplicative functions on non-crossing partitions, preprint.

- p. 199, Exercise 75(d), l. 2– to 1–. Change "will publish ... to appear" to "have published their proof in *Advances in Math.* **63** (1987), 42–99".

- p. 203, proof of Theorem 4.1.1. The proof that $\dim V_3 = d$ is incomplete, since it is unclear that the functions $n^k \gamma_i^n$ (as functions of n) are linearly independent. The argument in the book certainly yields $\dim V_3 \leq d$. Similarly it is unclear that $\dim V_4 = d$, since we must show that the power series $x^j/(1-\gamma_i x)^k$ are linearly independent. To show this last statement, assume otherwise. Then we have a relation

$$\sum_i \frac{P_i(x)}{(1-\gamma_i x)^{k_i}} = 0,$$

where $P_i(x) \in \mathbb{C}[x]$, and the sum over i is finite. We may assume each term of the sum is reduced to lowest terms, so $P_i(\gamma_i^{-1}) \neq 0$. Multiply by $\prod_i (1-\gamma_i x)^{k_i}$, yielding a *polynomial* identity. Hence we can set $x = \gamma_i^{-1}$, yielding

$$P_i(\gamma_i^{-1}) \prod_{j \neq i} (1 - \gamma_j \gamma_i^{-1})^{k_j} = 0,$$

a contradiction. Thus $\dim V_4 = d$. Since $V_4 \subseteq V_3$ and $\dim V_3 \leq d$, we have $\dim V_3 = d$, and the proof goes through as before.

- p. 205, l. 5 of proof. At end of line, change $L(x)$ to $-L(x)$.

- p. 212, Lemma 4.5.1(b). The proof is incorrect. For a correct proof, let $(B_1, \ldots, B_k)$ be as in (a). Then ρ is obtained by arranging the elements of B_1 in *decreasing* order, then the elements of B_2 in decreasing order, and so on.

- p. 213, l. 2–. Change $\tau(a_i) \leq \cdots \leq \tau(a_k) < \tau(a_{k+1}) \leq \cdots \leq \tau(a_j)$ to $\tau(a_i) \geq \cdots \geq \tau(a_k) > \tau(a_{k+1}) \geq \cdots \geq \tau(a_j)$.

- p. 220, l. 13. Change "$x = x_0 < x_1 < \cdots < x_k$ in P with bottom x" to "$x_0 < x_1 < \cdots < x_k = x$ in P with top x".

- p. 227, equations (28a) and (28b). Change D_S to D_F. The same should be done in the next paragraph (note) and in the proof of Lemma 4.6.13 (four times).

- p. 230, l. 12. Change the exponent $d - \dim \sigma$ to $d - \dim \sigma + 1$.

- p. 232, proof of Lemma 4.6.18, l. 7. Insert after "non-zero entry in" the following: "column j_1, say $\pi_{i_2 j_1}$. Since row i_2 has the same sum as column j_1, there is another nonzero entry in".

- p. 232, l. 3–. Multiply $H_n(r)$ by $(-1)^{n-1}$.

- p. 235, l. 13–14. The parenthesized statement "with respect to ..." should be replaced by "with respect to the embedding of $\mathcal{P}$ in its affine span".

- p. 245, l. 4–. Insert "we" after "If".

- p. 261, paragraph 1. Proposition 4.2.3 is a special case of §44 on p. 609 of G. Pólya, *Math. Zeit.* **29** (1928–29), 549–640. It is also a special case of the less general (than Pólya) §3 of R. M. Robinson, *Trans. Amer. Math. Soc.* **153** (1971), 451–468.

- p. 262, paragraph 6. For more information on powers of Fibonacci numbers, see L. Carlitz, *Duke Math. J.* **22** (1962), 521–537, and A. F. Horadam, *Duke Math. J.* **32** (1965), 437–446.

- p. 264, Exercise 3(a). For an interesting article on the Skolem-Mahler-Lech theorem, see G. Myerson and A. J. van der Poorten, *Amer. Math. Monthly* **102** (1995), 698–705. For a proof, see J. W. S. Cassels, *Local Fields*, Cambridge University Press, Cambridge, 1986.

- p. 280, solution to Exercise 14(a). Change the last sentence, viz., "(An erroneous …, 89–91.)", to: "(A special case was proved by D. Zeilberger, *Discrete Math.* **34** (1981), 89–91. The precise hypotheses used in this paper are not clearly stated.)"

- p. 284, end of Exercise 21(a). Insert: Further information about the rational functions $A_S(q)$ may be found in L. Butler, *J. Combinatorial Theory (A)* **50** (1989), 132–161.

- p. 284, Exercise 21(b). In R. Stanley, *J. Amer. Math. Soc.* **1** (1988), 919–961 (Theorem 3.2), it is shown algebraically that

$$\sum_{n\geq 0} \frac{A_{[n]}(q)x^n}{n!} = \exp\left(\frac{x}{1-q} + \frac{x^2}{2(1-q^2)}\right).$$

  A bijective proof was first given in B. Sagan and R. Stanley, *J. Combinatorial Theory (A)* **55** (1990), 161–193 (Corollary 4.5). It follows that

$$B_{[n]}(q) = \phi_n(q) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2k)!k!2^k(1-q)^{n-2k}(1-q^2)^k}.$$