

CHESS AND MATHEMATICS

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excerpt

6 Retrograde analysis.

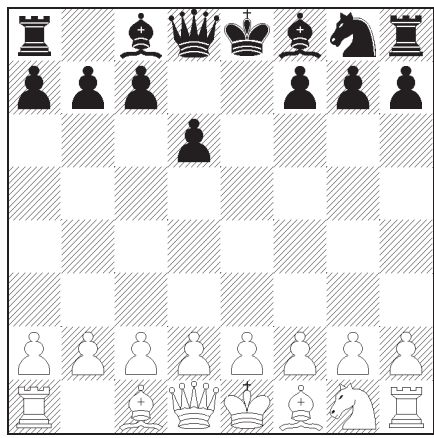
(b) *Proof games*. A fascinating genre of retro problems which has achieved maturity only relatively recently is known as “Proof Games” (PG’s). The object is to find a chess game of specified length (number of moves) that achieves the given position. In order for the problem to be sound, the game must be unique. (Occasionally there will be more than one solution, but they must be thematically related; e.g., in one solution White castles king-side and in the other queen-side.) Usually the specified length n is the *minimal* possible length of a game that can achieve the position, in which case the game is often called a “Shortest Proof Game” (SPG). For some PG’s, however (such as shown in Figures 79 and [to be inserted]), it is relatively easy to find solutions in fewer moves than the specified length; a primary point of the problem is to determine how the players can waste moves. In some cases there will be unique solutions for two specified game lengths; an example is shown in [to be inserted].

The earliest PG’s were composed by Sam Loyd in the 1890’s but had duals; the earliest dual-free PG seems to have been composed by T. R. Dawson in 1913. Although some interesting PG’s were composed in subsequent years, the vast potential of the subject was not suspected until the fantastic pioneering efforts of Michel Caillaud in the early 1980’s. A close to complete collection of all PG’s published up to 1991 (around 160 problems) appears in [2]. A further interesting book on PG’s is [1], which shows the Black king being mated on each of the 64 squares such that each square of the king’s field is guarded or blocked exactly once (such mates are called *pure mates*). What makes PG’s different from other retro problems is the specified length of the game; this is an extra restriction which increases the scope for complicated themes.

We start with three short (and therefore relatively simple) PG’s. The first (Figure 77) is a classic with a surprise for the unwary solver. Figure 78 has

G. Schweig

Tukon, 1938



Position after the 4th move of Black. How did the game go?

Figure 77:

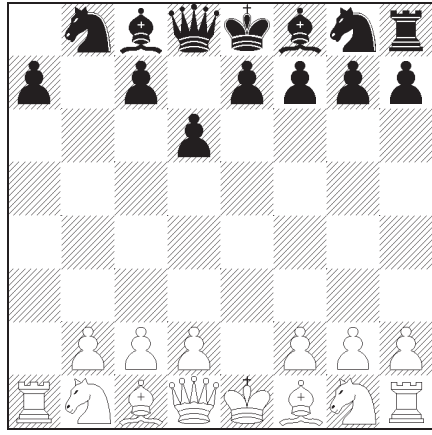
a surprise of a different (though related) nature. Figure 79 is surprisingly tricky for such a short problem. Here it is easy to reach the position in 3.5 moves, but a solution in *exactly* four moves is required.

For a somewhat more difficult problem, this time with two “surprises” (something in the position is not what it seems), see Figure 80. One of the surprises is relatively easy to spot, but the other is cleverly hidden.

Next we show two of the monumental efforts of Michel Caillaud. Figure 81 illustrates the *tempo theme*. It is based on considerations of parity (i.e., mod 2 arithmetic). One can achieve the given position with only moderate difficulty using 30 White moves and 17 Black moves, so that at 13 times Black just “passes.” Thus one must find a way for Black to “waste” 13 moves. Any simple attempt to do this turns out to have each Black move consisting of moving a piece to a square of a different color. Thus only an even number of moves can be wasted. It is necessary for Black to gain a tempo (move a piece to a square of the same color), and to do so Black embarks on an incredible rook journey of 13 moves. Figure 82 illustrates another common PG theme: multiple capture of promoted pieces (called the *Frolkin theme*).

A natural extremal problem associated with PG’s is that of finding the longest PG. This quest has an interesting history recounted in [2, Ch. VI]. The

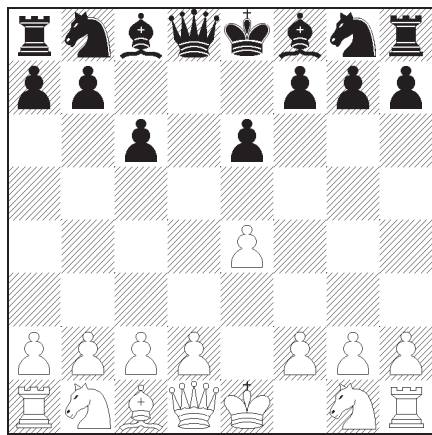
R. Müller
Rochade, 1985



Position after the 7th move of White. How did the game go?

Figure 78:

Tibor Orban
Commend. Die Schwalbe 1976

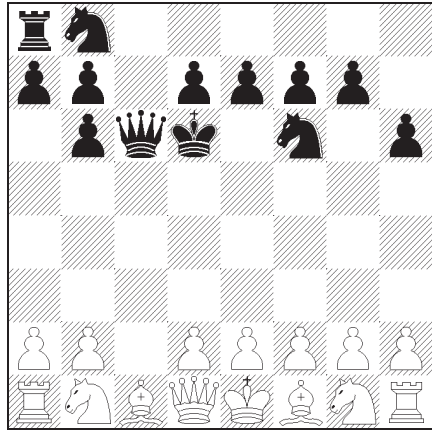


Proof game in exactly 4.0 moves

Figure 79:

G. Donati

Problem Paradise 10, Summer 1998

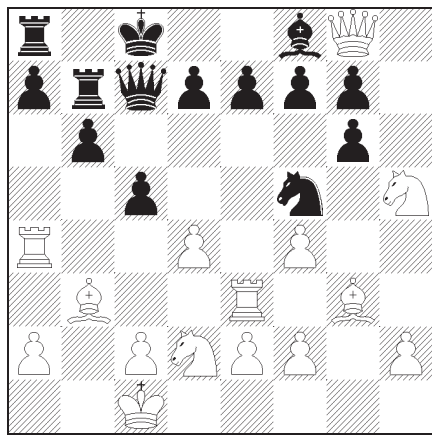


Position after the 14th move of White. How did the game go?

Figure 80:

M. Caillaud

Die Schwalbe, 1981, 1st prize

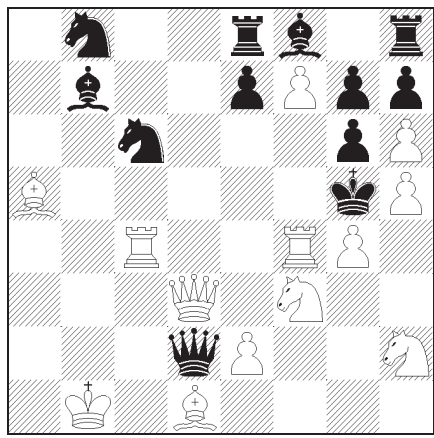


Position after the 30th move of Black. How did the game go?

Figure 81:

M. Caillaud

Europe Echecs, 1983, 1st prize



Position after the 36th move of White. How did the game go?

Figure 82:

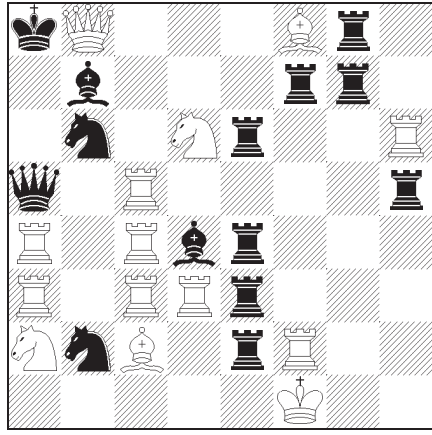
current record, based on an idea of Dmitry Pronkin, is shown in Figure 83. It is an exceptionally difficult problem full of promoted rooks. An interesting variant of PG's are problems in which one must not only reconstruct the game, but also the color of the pieces! An example of such a problem is shown in Figure 84. Two of the pieces turn out to have unexpected colors.

A recent trend in PG's are problems with more than one solution. The different solutions should be thematically related. For an example of a proof game with two solutions (both quite tricky), see Figure 85.

We mentioned in our discussion of game tasks in Section 1(b) that such tasks almost never have a unique solution. We can rectify this shortcoming by combining the genres of game tasks and proof games. Namely, we can ask for the shortest game of chess with a specified property (usually ending in a certain kind of checkmate) such that there is a *unique* game that reaches the final position in the minimum number of moves. The study of such games was initiated in May 1996 by Gianni Donati and Richard Stanley. Note that the Fool's Mate (1. f3 e6 2. g4 Qh5 mate) does not have the desired property. Although specifying the final position does resolve the question of whether White plays Pf3 or Pf4, and whether Black plays Pe6 or Pe5, it does not determine the order of White's two moves. In other words, there are two

D. Pronkin & A. Frolkin

Die Schwalbe, 1989

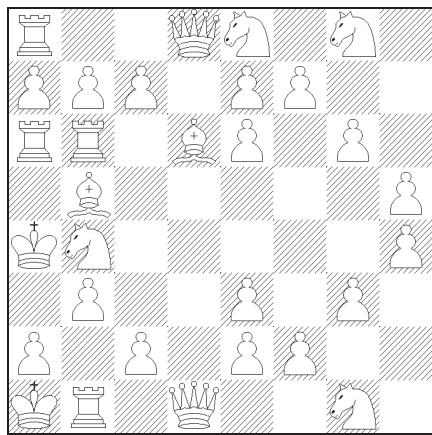


Position after the 58th move of White. How did the game go?

Figure 83:

A. Frolkin & L. Lyubashevsky

The Problemist, 1988, 1st prize

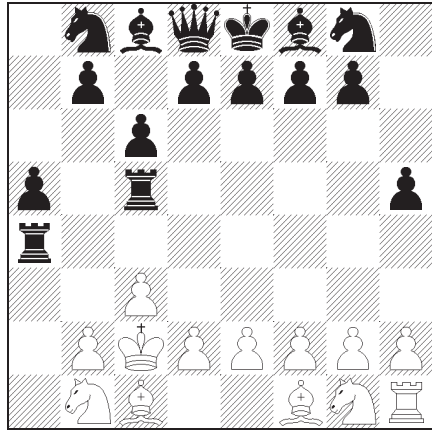


Position after the 20th move of White. Color the pieces and reconstruct the game.

Figure 84:

M. Caillaud

Phénix 28, 12/1994



Position after the 9th move of Black. How did the game go? (two solutions)

Figure 85:

games in two moves which lead to the final position. If there is a unique game leading to a given position in the minimum possible number of moves, then we call such a game a *uniquely realizable game* (URG). (We believe that our terminology is clearer than the customary “shortest proof game” in this situation.) Thus it is natural to ask what is the shortest URG ending in mate, since the Fool’s Mate is not a URG. Two answers are given by the games 1. e4 e5 2. Qh5 Ke7 3. Qxe5 mate (found independently by Joost de Heer and Richard Stanley) and 1. e4 f5 2. exf5 g5 3. Qh5 mate (found by Juha M. Saukkola). Note that the first of these games ends with a capture, while the second one doesn’t. For a URG with no captures at all ending in mate, three moves seems to be the best possible (found independently by Gianni Donati, Noam Elkies, and Göran Forslund): 1. f3 e5 2. Kf2 Qh4+ 3. Ke3 Dd4 mate. One of the most remarkable URG’s is in the category “mate by pawn promoting to knight,” found by Peter Rösler, *Problemkiste*, 8/1994 (version). The game is in fact completely determined by its last move! We state it explicitly in the following form:

(URG1) Construct a game of chess that ends with the move 6. gxf8N mate.

It is possible that this is the longest game of chess completely determined by its last move. Here is a sample of some further shortest URG’s ending

in mate. We give the composer and the number of moves (where e.g. a 5.0 move game means five White moves and five Black moves). The reader is invited to try to break these records.

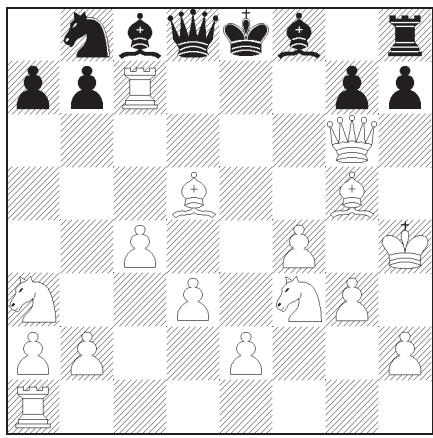
- (URG2) (N. Elkies) Find a URG in 5.0 moves in which Black mates by moving his king.
- (URG3) (O. Heimo) Find a URG in 5.0 moves in which Black mates with a rook, and in which no captures take place.
- (URG4) (N. Elkies and R. Stanley, independently) Find a URG in 4.0 moves in which Black mates with a knight.
- (URG5) (G. Wilts and N. Geissler) Find a URG in 6.0 moves in which Black mates by castling.
- (URG6) (O. Heimo) Find a URG in 7.5 moves in which White mates by castling, and in which no captures take place.
- (URG7) (G. Wilts and N. Geissler) Find a URG in 7.0 moves in which Black mates by capturing a pawn *en passant*.
- (URG8) (R. Stanley) Find a URG in 4.5 moves ending in mate in which no piece moves more than once.

As a variant of URG's we can consider *series* URG's, where White makes all the moves. White may not move into check, and may not check Black except on his last move. For instance, the shortest series URG ending in mate (first found by R. Stanley) seems to be in five moves, such as e3 Be2 Bh5 Qf3 Qxf7 mate. (We don't write the move numbers since only White moves.) Here are some series URG problems.

- (SURG1) (R. Stanley) Find a series URG in ten moves in which White mates by double check.
- (SURG2) (T. Luffingham) Find a series URG in nine moves in which White mates with a knight.
- (SURG3) (G. Donati) Find a series URG in thirteen moves in which White mates by promoting a pawn to a bishop.

N. Elkies

Retros mailing list, 1996



White makes 32 consecutive moves from the opening position.

Figure 86:

- (SURG4) (R. Stanley) Find a series URG in ten moves in which White mates by discovered check (battery mate) but not double check.
- (SURG5) (N. Elkies) Find a series URG in 32 moves such that if Black moves after the series, then Black *must* mate White in one move. The final position is shown in Figure 86.

SOLUTIONS

Figure 77. 1. Nc3 d6 2. Nd5 Nd7 3. Nxe7 Ndf6 4. Nxc8 Nxc8. The “surprise” is that the knight at g8 is not the king’s knight that began the game on that square.

Figure 78. 1. a4 d6 2. a5 Bg4 3. a6 Bxe2 4. axb7 Bxd1 5. bxa8Q Bg4 6. Qf3 Bc8 7. Qd1. Here the “surprise” is that the White queen on d1 is promoted! This problem illustrates the *Phoenix theme*, in which a piece leaves its original square to be sacrificed somewhere else, after which a pawn promotes to exactly the same piece and returns to the original square to replace the sacrificed piece. For some more elaborate examples of the Phoenix theme, see [2, Ch. III].

Figure 79. [to be inserted]

Figure 80. 1. c4 Na6 2. Qb3 Rb8 3. Qb6 cxb6 4. c5 Qc7 5. c6 Qd6 6. c7 Qc6 7. cxb8Q Kd8 8. Qa8 Kc7 9. Qc8+ Kd6 10. Qxf8 Nf6 11. Qg8 h6 12. Qh7 Qa8 13. Qc2 Nb8 14. Qd1. Again the Phoenix theme (the White queen at d1 is a promoted pawn), but also the Black rook at a8 is actually the king’s rook!

Figure 81. 1. b4 c5 2. b5 Nc6 3. bxc6 b6 4. c7 Bb7 5. c8R Bf3 6. gxf3 Rb8 7. Bh3 Rb7 8. Be6 Rc7 9. Nh3 Rc6 10. Rg1 Rd6 11. Rg4 Rd3 12. Ra4 Rd5 (the key tempo move) 13. d5 Nh6 14. Qd2 Nf5 15. Qh6 Rd6 16. Bf4 Rc6 17. Nd2 Rc7 18. 0-0-0 Rb7 19. Rg1 Rb8 20. Rg6 hxg6 21. Qh7 Ra8 22. Qg8 Rh4 23. Bg3 Re4 24. Bb3 Re6 25. Nf4 Rc6 26. Nh5 Rc7 27. f4 Rb7 28. Rc6 Qc7 29. Re6 Kd8 30. Re3 Kc8.

Figure 82. 1. h4 c5 2. h5 c4 3. Rh4 c3 4. Rc4 b5 5. g4 b4 6. Bg2 b3 7. Bc6 bxa2 8. b4 a5 9. b5 a4 10. b6 a3 11. Ba4 Nc5 12. b7 d5 13. b8Q d4 14. Qd6 d3 15. Qg6 dxc2 16. d3 fxg6 17. Bd2 c1B 18. Qb3 c2 19. Ba5 Bh6 20. Nd2 c1B 21. Nf1 Bcg5 22. f4 Kf7 23. 0-0-0 a1B 24. fxg5 a2 25. gxh6 Kf6 26. Nh2 Kg5 27. Rf1 Bf6 28. Rff4 a1B 29. Kb1 Bae5 30. d4 Bb7 31. dxe5 Qd2 32. exf6 Re8 33. f7 Nf6 34. Qd3 Nd7 35. Bd1 Nbd8 36. Ngf3+. Four Black promotions to bishop, all captured.

Figure 83. 1. a4 h5 2. a5 h4 3. a6 h3 4. axb7 hxg2 5. h4 d5 6. h5 d4 7. h6 d3 8. h7 dxc2 9. d4 a5 10. Bh6 c1R 11. e4 Rc5 12. Se2 Rh5 13. e5 c5 14. e6 Nc6

15. b8R a4 16. Rb4 a3 17. Ra4 c4 18. b4 c3 19. b5 c2 20. b6 c1R 21. b7 Rc4 22. b8R Qa5+ 23. Rbb4 Bb7 24. Nc3 0-0-0 25. exf7 e5 26. Rc1 Bc5 27. f8R a2 28. Rf3 a1R 29. Na2 g1R 30. Rfa3 Rg6 31. f4 Re6 32. f5 g5 33. f6 g4 34. f7 g3 35. f8R g2 36. Rf5 g1R 37. Bf8 Rg7 38. Ng3 e4 39. Bd3 e3 40. 0-0 e2 41. Rcc3 e1R 42. Bc2 R1e3 43. d5 Rdd7 44. d6 Rdf7 45. d7+ Kb8 46. Qd6+ Ka8 47. Qc7 Nge7 48. d8R+ Nc8 49. Rdd3 Rhg8 50. h8R Rae1 51. Rh6 R1e2 52. R1f2 Rce4 53. Kf1 Bd4 54. Rfc5 Ne5 55. Nf5 Nc4 56. Nd6 Nb2 57. Rbc4 Nb6 58. Qb8+.

Figure 84. 1. h4 h5 2. Rh3 Rh6 3. Rb3 Ra6 4. g3 g6 5. Bh3 Bh6 6. Be6 Be3 7. dxe3 dxe6 8. Bd2 Kd7 9. Bb4+ Kc6 10. Nc3 Qe8 11. Qd8 Nd7 12. 0-0-0 Ndf6 13. Kb1 Qd7 14. Ka1 Ne8 15. Rb1 Qd1 16. Nd5 Kb5 17. Bd6+ Ka4 18. Rb6 Bd7 19. Nb4 Bb5 20. b3+. The “surprise” is that the queen on d1 is Black and on d8 is White! Amazingly, there does not exist a shorter game if the queens exchange positions.

Figure 85. First solution: 1. a4 a5 2. Ra3 Ra6 3. Rh3 Rc6 4. c2 Rc5 5. Qb2 c6 6. Qg6 hg6 7. Kd1 Rh4 8. Kc2 Rxa4 9. Rh5 gh5. Second solution: 1. c3 h5 2. Qb3 Rh6 3. Qb6 ab6 4. Kd1 Rxa2 5. Kc2 Ra4 6. Ra3 Rc6 7. Rb3 Rc5 8. Rb5 c6 9. Ra5 ba5.

(URG1) 1.h4 d5 2.h5 Nd7 3.h6 Ndf6 4.hxg7 Kd7 5.Rh6 Ne8 6.gxf8N mate.

(URG2) 1.f3 e5 2.Kf2 Bb4 3.Kg3 Ke7 4.Kg4 h5+ 5.Kg5 Ke6 mate.

(URG3) 1.f4 a5 2.Kf2 Ra6 3.Kg3 Rg6+ 4.Kh4 Nf6 5.Nh3 Rg4 mate.

(URG4) 1.c4 Na6 2.c5 Nxc5 3.e3 a6 4.Ne2 Nd3 mate.

(URG5) 1.e3 c6 2.Qg4 Qa5 3.Qxd7 Nxd7 4.Ke2 Nb6 5.Kd3 Bf5+ 6.Kd4 0-0-0 mate. Curiously, the shortest known URG in which White mates by castling *kingside* takes an extra move (7.0 moves).

(URG6) 1.d4 e6 2.Bf4 Ke7 3.e3 Kf6 4.Bb5 Kf5 5.c4 Ke4 6.Qf3+ Kd3 7.Nc3 e5 8.0-0-0. Again the record for the analogous kingside task is longer, this time 10.0 moves.

(URG7) 1.f4 e5 2.Kf2 Qh4+ 3.Kf3 Qf2+ 4.Kg4 h5+ 5.Kh3 h4 6.e4 d5+ 7.g4 hxg3 e.p. mate.

(URG8) 1.e4 f6 2.Ba6 bxa6 3.d3 Bb7 4.Bh6 gxh6 5.Qh5 mate. Note that this same task can be done without captures by playing instead 2. ... b5 and 4. ... g5.

(SURG1)

(SURG2)

(SURG3)

(SURG4)

(SURG5)

References

- [1] A. A. Kislyak, *Poverzhenny monarkh* (The monarch overturned), published by the author, Tver-Kiev, 1993.
- [2] G. Wilts and A. Frolkin, *Shortest Proof Games*, privately published in Karlsruhe, 1991.