QUEUE PROBLEMS REVISITED[1]

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A *queue problem* is a chess problem in which each solution has the same set of moves, but the order of the moves can vary. The object is to count the number of solutions. The computation of the number of possible move orders should be mathematically interesting. The most interesting situation occurs when this counting problem is equivalent to a known mathematical counting problem, and we can determine the answer directly from the mathematical theory. All queue problems composed thus far have been serieshelpmates or serieshelpstalemates, in which Black makes a series of moves and White one move. The number of solutions is thus the number of Black move orders; White always has the same unique move at the end.

Queue problems were introduced by the Finnish composers Eero Bonsdorff, Arto Puusa, and Kauko Väisänen, beginning around 1983. These pioneering problems are collected in [4]. In 1993–94 the Finnish Chess Problem Society sponsored an international solving contest for mathematical chess problems, in which all the problems but one were proper queue problems. This contest featured eight new problems, composed by the above three composers together with Unto Heinonen.

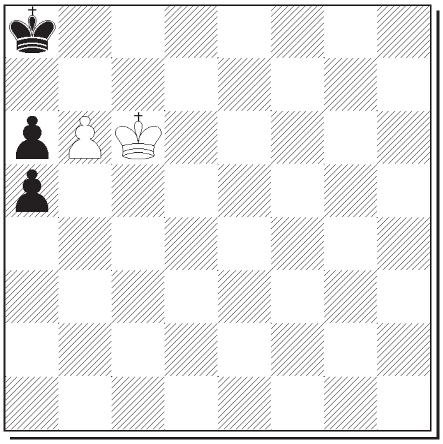
In this article we present three new queue problems illustrating three theorems from enumerative combinatorics (the mathematical subject dealing with counting the number of objects with specified properties) not involved in any previous queue problems. We also give an extension of a classic problem of Bonsdorff and Väisänen.

First we discuss a method for describing solutions to queue problems. Consider Problem A. The Black pawn at a5 must play the moves a4-a3-a2-a1B-e5-b8-a7, while the a6 pawn plays a5-a4-a3-a2-a1B-e5-b8. White then plays b7$\neq$. The a4 pawn can never pass the a5 pawn (even after promotion). We depict this situation in Figure 1, which we call the *solution poset* P. ("Poset" is an abbreviation for "partially ordered set.") The elements (vertices or points) of P correspond to the moves of Black. The pawn initially

[1]Corrected version of 12 September 2005. Problem E was unsound in the original version appearing in *Suomen Tehtäväniekat.* The present version has been computer checked by Mario Richter.

at a5 is denoted P1 and then B1 after promotion, and similarly P2 for the pawn initially at a6. A move B is written above a move A and joined to A by a sequence of descending edges if B must be played after A. Hence a solution to the problem consists of a labeling of the 14 vertices of P with the numbers $1, 2, \dots, 14$ such that if the label j can be reached from i by moving up edges, then $j > i$. The label of vertex A is the number of the move A in the solution. Such a labeling of a poset P is called a *linear extension* of P. The number of linear extensions of P is denoted $e(P)$. The number of solutions to a "pure" queue problem is thus $e(P)$, where P is the solution poset. Linear extensions of posets are a well-studied topic in combinatorics; see for instance [6, §3.5].

(A) E. Bondorff & K. Väisänen
ST solving contest, 1983



Serieshelpmate in 14: how many solutions?

The poset P of Figure 1 may be regarded as a (rotated) 2×7 rectangle. A linear extension of P is thus equivalent to filling in the squares of a 2×7 rectangle with the numbers $1, 2, \dots, 14$ so that every row and column is increasing. Figure 2 shows a linear extension of P and the equivalent 2×7 rectangle. The corresponding solution to Problem A is 1.a4 2.a3 3.a2 4.a5 5.a1B 6.a4 7.a3 8.Be5 9.a2 10.a1B 11.Bb8 12.Ba7 13.Be5 14.Bb8 b7≠. The number of $2 \times n$ rectangles with squares labelled with $1, 2, \dots, 2n$ so every row and column is increasing is well-known to be the *Catalan number* C_n,

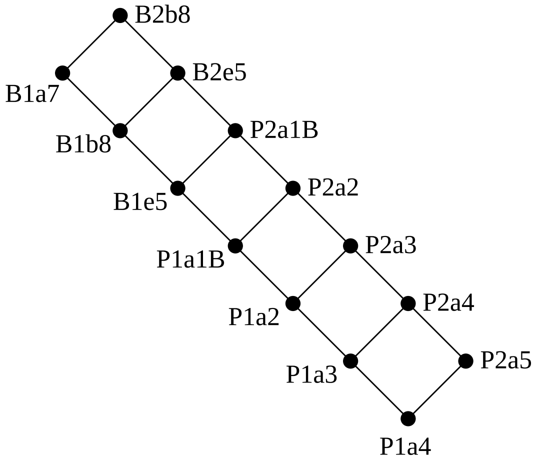


Figure 1: The solution poset for Problem A

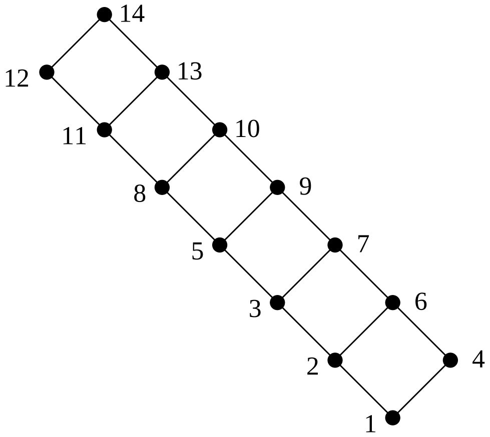
given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}.$$

Since $C_7 = 429$, the answer to Problem A is 429.

Catalan numbers are among the most ubiquitous sequences of numbers in combinatorics. For 66 combinatorial interpretations of these numbers, see [7, Exer. 5.19], available also at `http://www-math.mit.edu/~rstan/ec`. This website also has a link to a "Catalan addendum" with many additional combinatorial interpretations.

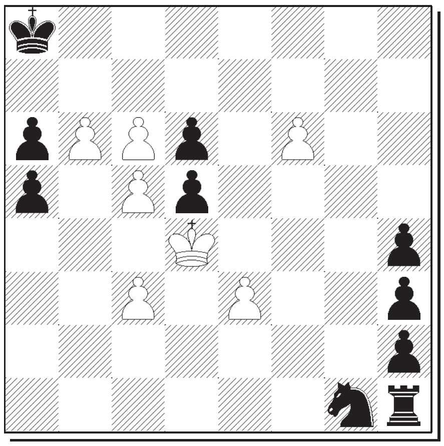
The "Catalan queue" of Problem A is so fundamental that it is interesting to ask to what extent it can be extended to a longer queue. Problem B shows our best effort in this direction: the number of solutions is $C_{17} = 129644790$. (Thanks to Noam Elkies for pointing out the necessity of the pawn at c5.)



1	2	3	5	8	11	12
4	6	7	9	10	13	14

Figure 2: A linear extension of the solution poset to Problem A

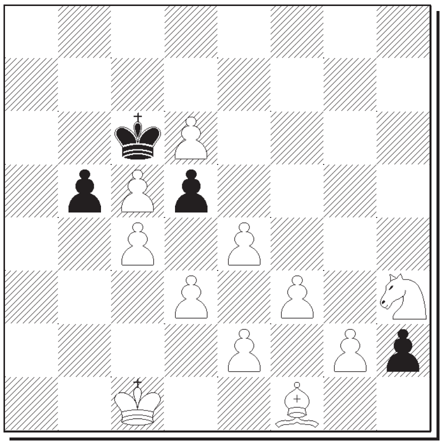
(B) R. Stanley (after E. Bonsdorff and K. Väisänen)
2003



Serieshelpmate in 34: how many solutions?

Our approach to Problem A illustrates the paradigm we will be following for the three problems below. Namely, construct the solution poset P, interpret the number $e(P)$ of linear extensions of P in terms of a known enumeration problem, and use the solution to the enumeration problem to solve the queue problem.

(C) R. Stanley, 3rd Prize
E. Bonsdorff 80th birthday tourney, 2002



Serieshelpmate in 14: how many solutions?

Consider Problem C, whose solution poset is shown in Figure 3. For $p \le q$ define a poset $P_{p,q}$ to consist of three chains $x_1 > \cdots > x_p$, $y_1 > \cdots > y_q$, and $z_1 > \cdots > z_q$, with $x_i < z_i$ and $y_i < z_i$. Kreweras [2, (85)] shows that the number of linear extensions of $P_{p,q}$ is given by

$$e(P_{p,q}) = \frac{2^{2p}(p+2q)!\,(2q-2p+2)!}{p!\,(2q+2)!\,(q-p)!\,(q-p+1)!}.$$

(A simpler proof in the case $p = q$ appears in [3].) The poset of Figure 3 is just $P_{4,4}$ with two irrelevant top elements. Hence the total number of solutions is given by

$$e(P_{4,4}) = \frac{2^8 \cdot 12! \cdot 2!}{4! \cdot 10! \cdot 0! \cdot 1!} = 2816.$$

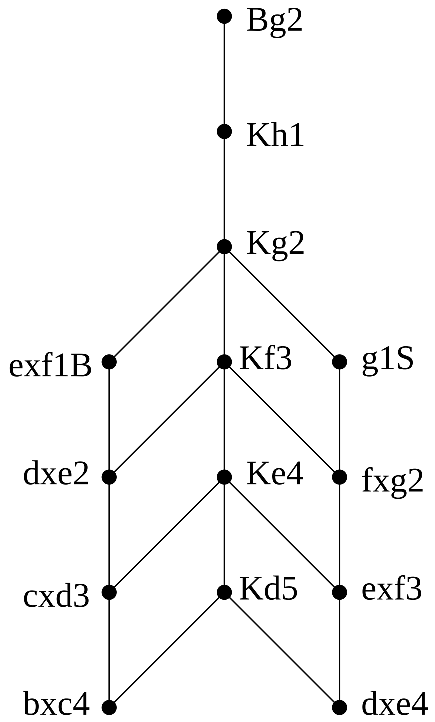
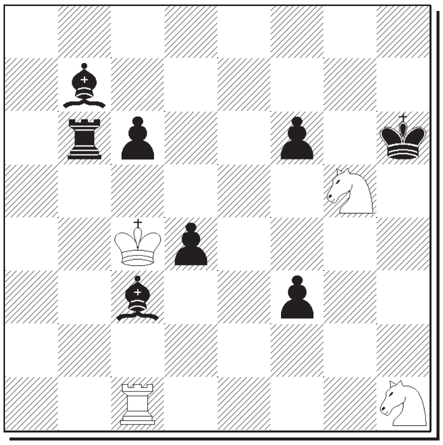


Figure 3: The solution poset for Problem C

(D) R. Stanley
2003



Serieshelpmate in 7: how many solutions?

Next we turn to Problem D. The solution poset is shown in Figure 4(a). Consider a linear extension of this poset, such as shown in Figure 4(b). If we read the labels from left-to-right along the zigzag shape of the poset, we obtain the permutation 3614275. Replace each number i in this permutation by $8-i$, obtaining $w = 5274613$. The characteristic property of w is that it first goes down, then up, then down, etc., i.e.,

$$5 > 2 < 7 > 4 < 6 > 1 < 3.$$

Such a permutation is called *alternating*. Let E_n denote the number of alternating permutations of $1, 2, \dots, n$. For instance, $E_4 = 5$, corresponding to the five alternating permutations 2143, 3241, 3142, 4231, 4132, and the number of solutions to Problem D is E_7. The numbers E_n are known as *Euler numbers* and can be computed from the recurrence

$$E_0 = 1,\ E_1 = 1,\ 2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k} \text{ if } n \geq 1.$$

It was proved by Desirée André in 1879 that

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

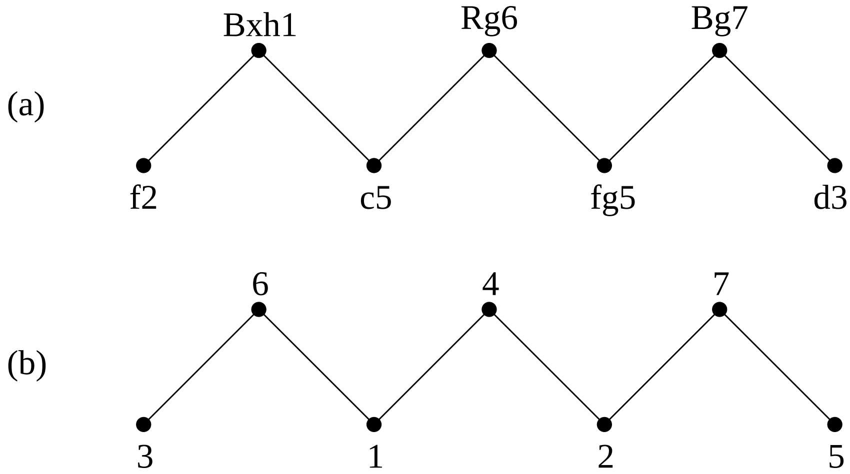


Figure 4: (a) The solution poset P for Problem D
(b) A linear extension of P

Here sec and tan denote the trigonometric functions secant and tangent. See for instance the website

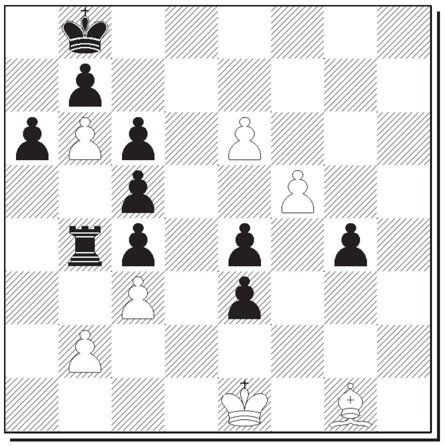
`http://mathworld.wolfram.com/AlternatingPermutation.html`.

The terms of $\sec x$ have even exponents and of $\tan x$ have odd exponents. Hence E_{2n} is sometimes called a *secant number* and E_{2n+1} a *tangent number*. In particular,

$$\tan x = x + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + 272\frac{x^7}{7!} + 7936\frac{x^9}{9!} + \cdots.$$

Hence the number of solutions to Problem D is $E_7 = 272$. Can the theme of this problem be extended to $E_8 = 1385$ or $E_9 = 7936$ solutions?

(E) R. Stanley
2004



Serieshelpmate in 14: how many solutions?

Problem E requires some knowledge of calculus to appreciate fully. The solution poset P for Problem E is shown in Figure 5(a). Figure 5(b) shows a linear extension π of P, together with a certain labeling of its elements with the labels $x, a_{12}, a_{13}, a_{23}, a_{14}, a_{24}, a_{34}$ (twice each). Suppose we list these labels in the order designated by π and adjoin an x at the beginning and end. We obtain the permutation

$$w = x\, a_{12}\, a_{13}\, a_{12}\, x\, a_{14}\, a_{24}\, a_{13}\, a_{23}\, a_{23}\, x\, a_{34}\, a_{14}\, a_{34}\, a_{24}\, x. \tag{1}$$

of four x's and two each of a_{ij} for $1 \leq i < j \leq 4$. The characteristic property of w is that each a_{ij} occurs between the ith x and the jth x. Thus if $J(n, k)$ denotes the number of permutations of n x's and $2k$ each of a_{ij} for $1 \leq i < j \leq n$, then the number of solutions to Problem E is given by $J(4, 1)$.

Now define the integral

$$I(n, k) = \int_0^1 \cdots \int_0^1 \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2k} dx_1 \cdots dx_n.$$

This integral is a special case of a famous integral due to Selberg and known as *Selberg's integral* (e.g., [1, Chap. 8]). Its value is given by

$$I(n, k) = \frac{1}{k!^n} \prod_{j=1}^{n} \frac{((j-1)k)!^2 (jk)!}{((n-j-2)k+1)!}. \tag{2}$$

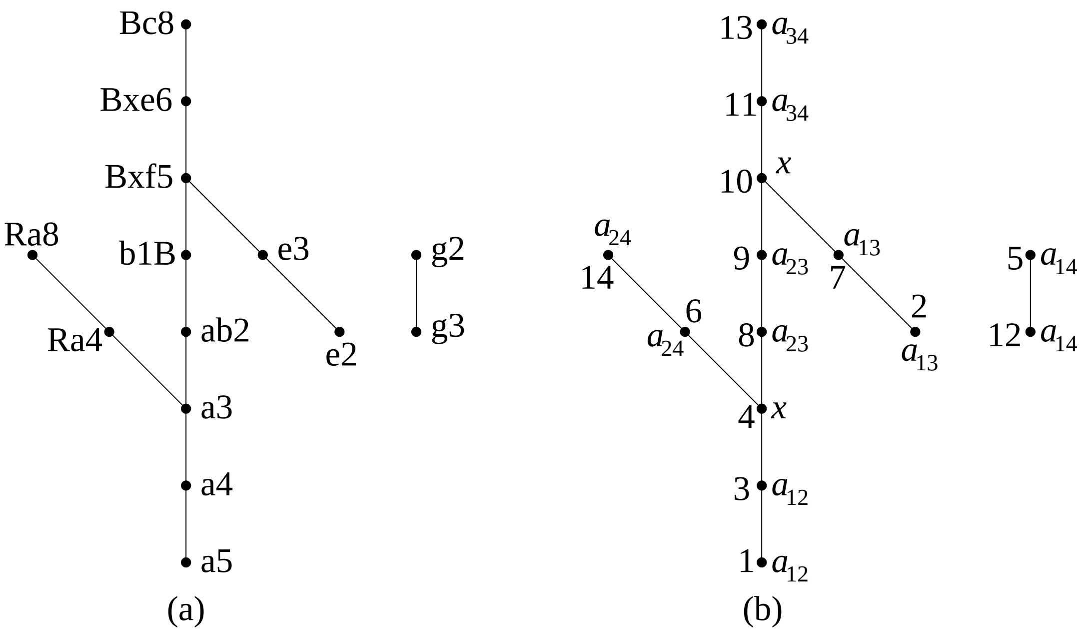


Figure 5: (a) The solution poset P for Problem E
(b) A linear extension and "Selberg labeling" of P

In particular, we obtain easily from (2) that $I(4,1) = 1/252000$.

There is also a standard way to interpret $I(n,k)$ as a probability, namely, suppose we randomly arrange n x's and $2k$ each of the symbols a_{ij} for $1 \le i < j \le n$ in a line. Then $I(n,k)$ is the probability that all the a_{ij}'s occur between the ith and jth x. (This probabilistic interpretation is much easier to derive than the explicit value (2).) Since there are a total of

$$\frac{(n+kn(n-1))!}{n!\,(2k)!^{\binom{n}{2}}}$$

ways to arrange n x's and $2k$ each of the a_{ij}'s in a line, we have

$$J(n,k) = \frac{(n+kn(n-1))!}{n!\,(2k)!^{\binom{n}{2}}} I(n,k).$$

It follows that the number of solutions to Problem E is given by

$$\begin{aligned} J(4,1) &= \frac{16!}{24 \cdot 2^6} I(4,1) \\ &= 13621608000 \cdot \frac{1}{252000} \\ &= 54054. \end{aligned}$$

References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Mathematics and Its Applications **71**, Cambridge University Press, Cambridge/New York, 1999.

[2] G. Kreweras, Sur une classe de problèmes de dénombrement liés au treillis des partitions des entiers, *Cahiers du Bureau Universitaire de Recherche Opérationnelle*, Institut de Statistique de l'Université de Paris, Paris, 1965.

[3] G. Kreweras and H. Niederhausen, Solution of an enumerative problem connected with lattice paths, *European J. Combinatorics* **2** (1981), 55–60.

[4] A. Puusa, *Queue Problems*, Finnish Chess Problem Society (Suomen Tehtäväniekat), 1992.

[5] Results of the international solving contest for mathematical chess problems (1994), Finnish Chess Problem Society (Suomen Tehtäväniekat).

[6] R. Stanley, *Enumerative Combinatorics*, vol. 1, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986; second printing, Cambridge University Press, Cambridge, 1996.

[7] R. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, New York/Cambridge, 1999.