ERRATA
for Catalan Numbers
version of 27 November 2022

• p. 1, line 8. Change $d - 1$ to $n - 1$.

• p. 40, item 132. The five examples should be

\begin{align*}
12132434 & \quad 12134234 \\
12314234 & \quad 12312434 \\
12341234 & \quad \end{align*}

• p. 51, item 200. The condition on $A$ and $B$ should be that for all $i$, the $i$th largest element of $A$ is smaller than the $i$th largest element of $B$.

• p. 59, item 17, line 6. Change $y$ to $F(x,t)$ (twice).


• p. 126, line 3 of second triangle. This should be

\begin{align*}
1 \quad 1 \quad 3 \quad 7 \quad 18 \quad \end{align*}

• p. 134, Problem A59. It should be assumed in both parts that $f(x)$ has compact support; otherwise the solution is not unique.

• p. 169, A66. Another solution follows from the identity

\[ \sum_{i=0}^{n} \frac{C_i}{4^i} = 2 - 2^{-2n-1} \binom{2n+2}{n+1}, \]

which has a straightforward generating function proof.

• p. 180, line 3. Change $C_2C_{n-1}$ to $C_2C_{n-2}$.

• p. 181, line 17. Change this displayed equation to

\[ A_n = (n + 2)(C_1C_{n-1} + C_2C_{n-2} + \cdots + C_{n-1}C_1). \]

• p. 184, line 10– (omitting footnote). Change Rendu to Rendus.

• p. 213, column 1, line 4. Change Martin to Michael.
ADDENDA

version of 6 August 2020

B1. (a) [2+] Define integers $c_n$ by

$$C(-x) = \prod_{n \geq 1} (1 - x^n)^{c_n}.$$ 

Show that

$$c_n = \frac{1}{2n} \sum_{d|n} (-1)^d \mu(n/d) \binom{2d}{d}.$$ 

(b) [2+] Show that $c_n$ is divisible by $n$.

(c) [3–] Show that $6c_n$ is divisible by $n^2$.

B2. [3] Fix $n \geq 2$. Let $X$ be a $(2n - 1)$-element set. Let $V$ be the real (any field of characteristic 0 will do) vector space with a basis consisting of all symbols

$$[a_1, \ldots, a_i, [b_1, \ldots, b_n], a_{i+1}, \ldots, a_{n-1}],$$

where $\{a_1, \ldots, a_{n-1}, b_1, \ldots, b_n\} = X$. Let $W$ be the subspace of $V$ generated by the following elements:

- $[c_1, \ldots, c_i, c_{i+1}, \ldots, c_{2n-1}] + [c_1, \ldots, c_{i+1}, c_i, \ldots, c_{2n-1}]$. In other words, the $(2n - 1)$-component “bracket” $[c_1, \ldots, c_{2n-1}]$ (where each $c_i$ is an element of $X$ with one exception which is a bracket $[b_1, \ldots, b_n]$ of elements of $X$) is antisymmetric in its entries.

- For all $a_1 < \cdots < a_{n-1}$ and $b_1 < \cdots < b_n$ such that $\{a_1, \ldots, a_{n-1}, b_1, \ldots, b_n\} = X$, the element

$$[a_1, \ldots, a_{n-1}; [b_1, \ldots, b_n]] - \sum_{i=1}^{n} [b_1, \ldots, b_{i-1}, [a_1, \ldots, a_{n-1}, b_i], b_{i+1}, \ldots, b_n].$$

Show that $\dim V/W = C_n$.

B3. [3–] Let $n$ denote the $n$-element chain $1 < 2 < \cdots < n$. Show that for $n \geq 3$, $C_n$ is the number of $n$-element subsets $S$ of the poset $n \times n$ with the following properties: (a) $S$ intersects every maximal chain of $n \times n$ and is minimal with respect to this property, (b) $S$ lies below the equator, i.e., if $(i, j) \in S$ then $i + j \leq n + 1$, and (c) $(n, 1) \in S$. 
B4. Show that $C_n$ is equal to the number of 321-avoiding alternating connected permutations $w \in S_{2n+2}$. A permutation $v = v_1 \cdots v_m \in S_m$ is connected (or indecomposable or fully supported) if $\{v_1, \ldots, v_k\} \neq [k]$ for $1 \leq k < m$. (See EC1, second ed., Exercise 1.128(a).)

31527486 31627485 41527386 41627385 51627384
Solutions


B2. This result was conjectured by Tamar Friedmann (in a more general context) and proved by Phil Hanlon in 2015. Friedmann made the stronger conjecture that the natural $S_n$-action on $V/W$ is the irreducible representation indexed by the partition $(2, 2, \ldots, 2, 1)$ of $2n-1$. Hanlon in fact proved this stronger conjecture.


B4. B. Tenner, arXiv:2008.053471, defines a map $\phi$ from 321-avoiding alternating permutations $w$ in $\mathfrak{S}_{2m-2}$ to $\mathfrak{S}_{2m}$ as follows:

$$
(\phi(w))(i) = \begin{cases} 
  w(i) + 1, & \text{if } i < 2m - 1 \text{ is odd} \\
  w(i - 2) + 1, & \text{if } i > 2 \text{ is even} \\
  1, & \text{if } i = 2 \\
  2m, & \text{if } i = 2m - 1.
\end{cases}
$$

She shows that $\phi$ is a bijection onto 321-avoiding alternating connected permutations in $\mathfrak{S}_{2m}$. The proof now follows from #146.