ERRATA

for *Catalan Numbers*

version of 26 February 2023

- p. 1, line 8. Change $d-1$ to $n-1$.
- page 5, line 1. Change x^n to $(-4x)^n$.
- p. 40, item 132. The five examples should be

  12132434 12134234 12314234 12312434 12341234

- p. 51, item 200. The condition on A and B should be that for all i, the ith largest element of A is smaller than the ith largest element of B.
- p. 59, item 17, line 6. Change y to $F(x,t)$ (twice).
- p. 99, item 210. An earlier reference is J. O. Shallit, Elementary Problem E 2972, *Amer. Math. Monthly* **89** (1982), 698; solution by D. M. Wells, **93** (1986), 217–218.
- p. 126, line 3 of second triangle. This should be

  1 1 3 7 18

- p. 134, Problem A59. It should be assumed in both parts that $f(x)$ has compact support; otherwise the solution is not unique.
- p. 169, A66. Another solution follows from the identity

$$\sum_{i=0}^{n} \frac{C_i}{4^i} = 2 - 2^{-2n-1}\binom{2n+2}{n+1},$$

  which has a straightforward generating function proof.

- p. 180, line 3. Change C_2C_{n-1} to C_2C_{n-2}.
- p. 181, line 17. Change this displayed equation to

$$A_n = (n+2)(C_1C_{n-1} + C_2C_{n-2} + \cdots + C_{n-1}C_1).$$

- p. 184, line 10– (omitting footnote). Change *Rendu* to *Rendus*.
- p. 213, column 1, line 4. Change Martin to Michael.

ADDENDA

version of 6 August 2020

B1. (a) [2+] Define integers c_n by

$$C(-x) = \prod_{n \geq 1} (1 - x^n)^{c_n}.$$

Show that

$$c_n = \frac{1}{2n} \sum_{d|n} (-1)^{d-1} \mu(n/d) \binom{2d}{d}.$$

(b) [2+] Show that c_n is divisible by n.

(c) [3–] Show that $6c_n$ is divisible by n^2.

B2. [3] Fix $n \geq 2$. Let X be a $(2n-1)$-element set. Let V be the real (any field of characteristic 0 will do) vector space with a basis consisting of all symbols

$$[a_1, \ldots, a_i, [b_1, \ldots, b_n], a_{i+1}, \ldots, a_{n-1}],$$

where $\{a_1, \ldots, a_{n-1}, b_1, \ldots, b_n\} = X$. Let W be the subspace of V generated by the following elements:

- $[c_1, \ldots, c_i, c_{i+1}, \ldots, c_{2n-1}] + [c_1, \ldots, c_{i+1}, c_i, \ldots, c_{2n-1}]$. In other words, the $(2n-1)$-component "bracket" $[c_1, \ldots, c_{2n-1}]$ (where each c_i is an element of X with one exception which is a bracket $[b_1, \ldots, b_n]$ of elements of X) is antisymmetric in its entries.
- For all $a_1 < \cdots < a_{n-1}$ and $b_1 < \cdots < b_n$ such that $\{a_1, \ldots, a_{n-1}, b_1, \ldots, b_n\} = X$, the element

$$[a_1, \ldots, a_{n-1}, [b_1, \ldots, b_n]] - \sum_{i=1}^{n} [b_1, \ldots, b_{i-1}, [a_1, \ldots, a_{n-1}, b_i], b_{i+1}, \ldots, b_n].$$

Show that $\dim V/W = C_n$.

B3. [3–] Let $\boldsymbol{n}$ denote the n-element chain $1 < 2 < \cdots < n$. Show that for $n \geq 3$, C_n is the number of n-element subsets S of the poset $\boldsymbol{n} \times \boldsymbol{n}$ with the following properties: (a) S intersects every maximal chain of $\boldsymbol{n} \times \boldsymbol{n}$ and is minimal with respect to this property, (b) S lies below the *equator*, i.e., if $(i, j) \in S$ then $i + j \leq n + 1$, and (c) $(n, 1) \in S$.

B4. Show that C_n is equal to the number of 321-avoiding alternating connected permutations $w \in \mathfrak{S}_{2n+2}$. A permutation $v = v_1 \cdots v_m \in \mathfrak{S}_m$ is *connected* (or *indecomposable* or *fully supported*) if $\{v_1, \ldots, v_k\} \neq [k]$ for $1 \leq k < m$. (See EC1, second ed., Exercise 1.128(a).)

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Solutions

B1. (c) See `http://mathoverflow.net/questions/195339`. Vasu Tewari contributed the additional references `arXiv:2004.12093` (Section 4), which gives a combinatorial interpretation of C_n/n, and

`http://people.mpim-bonn.mpg.de/stavros/publications/LMOV.asymptotics.pdf`

(Table 4).

B2. This result was conjectured by Tamar Friedmann (in a more general context) and proved by Phil Hanlon in 2015. Friedmann made the stronger conjecture that the natural $\mathfrak{S}_n$-action on V/W is the irreducible representation indexed by the partition $(2, 2, \dots, 2, 1)$ of $2n-1$. Hanlon in fact proved this stronger conjecture.

B3. See S. Ahmad and V. Welker, *Order* **33** (2016), 347–358 (Theorem 2.1).

B4. B. Tenner, `arXiv:2008.053471`, defines a map ϕ from 321-avoiding alternating permutations w in $\mathfrak{S}_{2m-2}$ to $\mathfrak{S}_{2m}$ as follows:

$$(\phi(w))(i) = \begin{cases} w(i)+1, & \text{if } i < 2m-1 \text{ is odd} \\ w(i-2)+1, & \text{if } i > 2 \text{ is even} \\ 1, & \text{if } i = 2 \\ 2m, & \text{if } i = 2m-1. \end{cases}$$

She shows that ϕ is a bijection onto 321-avoiding alternating connected permutations in $\mathfrak{S}_{2m}$. The proof now follows from #146.