ADDITIONAL POSET PROBLEMS
version of 25 October 2017

1. [2] Find a finite poset $P$ with the following property, or show that no such $P$ exists. The longest chain in $P$ has $m$ elements (for some $m \geq 1$). $P$ can be written as a union of two chains $C_1$ and $C_2$, but cannot be written in this way where $\#C_1 = m$.

2. (a) [2] How many nonisomorphic $n$-element posets contain an $(n-1)$-element antichain?
   (b) [2+] How many nonisomorphic $n$-element posets contain an $(n-1)$-element chain?
   (c) [2–] How many nonisomorphic $n$-element posets contain both an $(n-1)$-element antichain and an $(n-1)$-element chain?

3. (a) [3–] Find a finite poset $P$ with the following property. The automorphism group $\text{Aut}(P)$ of $P$ acts transitively on the set $M$ of minimal elements of $P$. Moreover, the restriction of $\text{Aut}(P)$ to $M$ does not contain a full cycle of the elements of $M$.
   (b) [5–] Does such a poset exist if all maximal chains have two elements?

4. [2+] Let $w = t_1, \ldots, t_p$ be a permutation of the elements of a finite poset $P$. Call a permutation $w'$ a permissible swap of $w$ if it is obtained from $w$ by interchanging some $t_i$ and $t_{i+1}$ where $t_i < t_{i+1}$. Clearly a sequence of permissible swaps must eventually terminate in a permutation $v$ that has no permissible swaps. Show that $v$ is independent of the sequence of permissible swaps.

5. [2+] For each permutation $w \in \mathfrak{S}_n$, let $\sigma_w$ be the simplex in $\mathbb{R}^n$ defined by
   \[
   \sigma_w = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_{w(1)} \leq x_{w(2)} \leq \cdots \leq x_{w(n)} \leq 1\}.
   \]
   For any nonempty subset $S \subseteq \mathfrak{S}_n$, define
   \[
   X_S = \bigcup_{w \in S} \sigma_w \subset \mathbb{R}^n.
   \]
Show that $X_S$ is convex if and only if $S$ is the set of linear extensions of some partial ordering of $[n]$.

6. [2+] Let $0 \leq p \leq 1$, and let $P$ be a finite $n$-element poset with $\hat{0}$ and $\hat{1}$. Let $\sigma: P \to [n]$ be a linear extension of $P$. Define a random digraph $D$ on the vertex set $[n]$ as follows. For each $s < t$ in $P$, choose the edge $s \to t$ of $D$ with probability $p$.

Now start at the vertex $\hat{0}$ of $D$. If there is an arrow from $\hat{0}$, then move to the vertex $t$ for which $\hat{0} \to t$ is an edge of $D$ and $\sigma(t)$ is as small as possible; otherwise stop. Continue this procedure (always moving from a vertex $u$ to a vertex $v$ for which $u \to v$ is an edge of $D$ and $\sigma(v)$ is as small as possible) until unable to continue. What is the probability that we end at vertex $\hat{1}$? Try to give an elegant proof avoiding recurrence relations, linear algebra, etc.

7. (a) [2+] Let $f(n)$ be the average value of $\mu_P(\hat{0}, \hat{1})$, where $P$ ranges over all (induced) subposets of the boolean algebra $B_n$ containing $\hat{0}$ and $\hat{1}$. (The number of such $P$ is $2^{2^{n-2}}$.) Define the Genocchi number $G_n$ by

$$
\sum_{n \geq 0} G_n \frac{x^n}{n!} = \frac{2x}{1 + e^x},
$$

as in Exercise 5.8(d). Show that $f(n) = 2G_{n+1}/(n+1)$.

(b) [2] It follows from (a) that $f(n) = 0$ when $n$ is even. Give a noncomputational proof.

(c) [5–] What more can be done with this model of a random poset, i.e., each element of $B_n - \{\hat{0}, \hat{1}\}$ is chosen independently with probability $1/2$ (or we could generalize to any probability $0 \leq p \leq 1$) to belong to $P$? For instance, what is the probability that $P$ contains a maximal chain of $B_n$? (This looks quite difficult to me.) What is the expected value of the rank of the top homology $H_{n-2}(\Delta(P'); \mathbb{Z})$ of the order complex $\Delta(P')$ of $P' = P - \{\hat{0}, \hat{1}\}$?

8. (a) [2] Let $U_n$ be the set of all lattice paths $\lambda$ of length $n - 1$ (i.e., with $n - 1$ steps), starting at $(0,0)$, with steps $(1,1)$ and $(1,-1)$. Thus $\# U_n = 2^{n-1}$. Regard the $n$ integer points on the path $\lambda$ as the elements of a poset $P_\lambda$, such that $\lambda$ is the Hasse diagram of $P_\lambda$. Find $\sum_{\lambda \in U_n} e(P_\lambda)$.
(b) [2+] Give $P_\lambda$ the labeling $\omega_\lambda$ by writing the numbers $1, 2, \ldots, n$ along the path. For example, when $n = 8$ one possible pair $(P_\lambda, \omega_\lambda)$ is given by

\[
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

Find $\sum_{\lambda \in U_n} \Omega_{P_\lambda, \omega_\lambda}(m)$ and $\sum_{\lambda \in U_n} W_{P_\lambda, \omega_\lambda}(q)$.

(c) [3–] Let $V_n$ consist of those $\lambda \in U_n$ which never fall below the $x$-axis. It is well-known that $V_n = (\lfloor \frac{n-1}{2} \rfloor)$. Show that $\sum_{\lambda \in V_n} e(P_\lambda)$ is equal to the number of permutations $w \in S_n$ of odd order. A formula for this number is given in EC2, Exercise 5.10(c) (the case $k = 2$).

(d) [5–] Is there a nice bijective proof or “conceptual proof” of (c)?

(e) [5–] Are there nice expressions for $\sum_{\lambda \in V_n} \Omega_{P_\lambda, \omega_\lambda}(m)$ and/or $\sum_{\lambda \in V_n} W_{P_\lambda, \omega_\lambda}(q)$?

(f) [3–] Now let $W_n$ consist of all $\lambda \in V_{2n+1}$ that end at the $x$-axis. It is well-known that $\#W_n$ is the Catalan number $C_{n-1} = \frac{1}{n} \binom{2n}{n-1}$. Show that $\sum_{\lambda \in W_n} e(P_\lambda)$ is equal to the Eulerian-Catalan number $EC_n = A(2n+1, n)/(n+1)$ of EC1, Exercise 1.53.

9. [2+] Let $P$ be a finite poset with $\hat{0}$ and $\hat{1}$. For each $t \in P$ define a polynomial $f_t(x)$ with coefficients in $\mathbb{Z}[y]$ as follows:

\[
\begin{align*}
\hat{0}(x) & = y \\
f_t(x+y) & = \sum_{s \leq t} f_s(x).
\end{align*}
\]

Express $f_1(x)$ in terms of the zeta polynomial $Z_P(n)$.

10. [2+] Let $n \geq 1$ and $d \geq 2$. Let $P_{nd}$ be the poset with elements $x_{ij}$, $1 \leq i \leq n$ and $1 \leq j \leq d$, and with cover relations $x_{ij} \leq x_{i+1,k}$ for all $1 \leq i \leq n-1$, $1 \leq j \leq d$ and $1 \leq k \leq d$ except $j = k$. Thus $P_{nd}$ is graded of rank $n-1$, and there are exactly $d(d-1)$ cover relations between consecutive ranks. Find $e(P_{nd})$. For instance, $e(P_{2,3}) = 48$ and $e(P_{3,3}) = 384$. 

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11. [3–] Let $U_k$ denote an ordinal sum of $k$ 2-element antichains, so $\#U_k = 2k$. Show that when $k = 2j$, the real parts of the zeros of the order polynomial $\Omega_{U_k}(m)$ are $0, -1, -2, \ldots, -(k - 1)$ (each occurring once), and $-(j - \frac{1}{2})$ (occurring $k$ times). Similarly when $k = 2j + 1$ the zeros of $\Omega_{U_k}(m)$ have real parts $0, -1, \ldots, -(k - 1)$, each occurring once, except that $-j$ occurs $k + 1$ times.

12. (a) [3–] Let $f(n)$ be the number of nonisomorphic $n$-element posets for which $\beta_{J(P)}(S) \leq 1$ for all $S \subseteq [n-1]$. Find a simple formula for the generating function $\sum_{n \geq 0} f(n) x^n$.

(b) [3–] Among the $f(n)$ posets $P$ of (a), find the maximum value of $e(P)$. How many posets (up to isomorphism) achieve this maximum?

13. (a) [2+] Define two labelings $\omega, \omega' : P \to [p]$ of the $p$-element poset $P$ to be equivalent if $A_{P,\omega} = A_{P,\omega'}$. For instance, one equivalence class consists of the natural labelings. Let $[\omega]$ denote the equivalence class containing $\omega$. Define a partial ordering $L_P$ on the equivalence classes by $[\omega] \leq [\omega']$ if $A_{P,\omega} \subseteq A_{P,\omega'}$. Show that $L$ is a self-dual graded poset with $\hat{0}$ and $\hat{1}$. What is the rank (length of the longest chain) of $L_P$? (For the number of elements of $L_P$, see Exercise 3.160.)

(b) [3] Find the Möbius function of $L_P$. In particular, for any $s \leq t$ we have $\mu(s, t) = 0, \pm 1$.

(c) [5–] What else can be said about the poset $L_P$?

14. [2+] Let $(P, \omega)$ be a labelled $p$-element poset. Show that there is some $S \subseteq [p-1]$ for which exactly one permutation $w \in \mathcal{L}(P, \omega)$ has descent set $S$.

Further problems may be forthcoming.