ADDITIONAL POSET PROBLEMS

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1. [2] Find a finite poset P with the following property, or show that no such P exists. The longest chain in P has m elements (for some $m \geq 1$). P can be written as a union of two chains C_1 and C_2, but cannot be written in this way where $\#C_1 = m$.

2. (a) [2] How many nonisomorphic n-element posets contain an $(n-1)$-element antichain?

   (b) [2+] How many nonisomorphic n-element posets contain an $(n-1)$-element chain?

   (c) [2–] How many nonisomorphic n-element posets contain both an $(n-1)$-element antichain and an $(n-1)$-element chain?

3. (a) [3–] Find a finite poset P with the following property. The automorphism group $\mathrm{Aut}(P)$ of P acts transitively on the set M of minimal elements of P. Moreover, the restriction of $\mathrm{Aut}(P)$ to M does not contain a full cycle of the elements of M.

   (b) [5–] Does such a poset exist if all maximal chains have two elements?

4. [2+] Let $w = t_1, \dots, t_p$ be a permutation of the elements of a finite poset P. Call a permutation w' a *permissible swap* of w if it is obtained from w by interchanging some t_i and t_{i+1} where $t_i < t_{i+1}$. Clearly a sequence of permissible swaps must eventually terminate in a permutation v that has no permissible swaps. Show that v is independent of the sequence of permissible swaps.

5. [2+] For each permutation $w \in \mathfrak{S}_n$, let σ_w be the simplex in $\mathbb{R}^n$ defined by

$$\sigma_w = \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_{w(1)} \leq x_{w(2)} \leq \cdots \leq x_{w(n)} \leq 1\}.$$

   For any nonempty subset $S \subseteq \mathfrak{S}_n$, define

$$X_S = \bigcup_{w \in S} \sigma_w \subset \mathbb{R}^n.$$

Show that X_S is convex if and only if S is the set of linear extensions of some partial ordering of $[n]$.

6. [2+] Let $0 \leq p \leq 1$, and let P be a finite n-element poset with $\hat{0}$ and $\hat{1}$. Let $\sigma\colon P \to [n]$ be a linear extension of P. Define a random digraph D on the vertex set $[n]$ as follows. For each $s < t$ in P, choose the edge $s \to t$ of D with probability p.

   Now start at the vertex $\hat{0}$ of D. If there is an arrow from $\hat{0}$, then move to the vertex t for which $\hat{0} \to t$ is an edge of D and $\sigma(t)$ is as small as possible; otherwise stop. Continue this procedure (always moving from a vertex u to a vertex v for which $u \to v$ is an edge of D and $\sigma(v)$ is as small as possible) until unable to continue. What is the probability that we end at vertex $\hat{1}$? Try to give an elegant proof avoiding recurrence relations, linear algebra, etc.

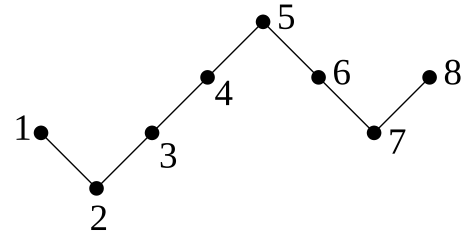
7. (a) [2+] Let $f(n)$ be the average value of $\mu_P(\hat{0}, \hat{1})$, where P ranges over all (induced) subposets of the boolean algebra B_n containing $\hat{0}$ and $\hat{1}$. (The number of such P is 2^{2^n-2}.) Define the *Genocchi number* G_n by
$$\sum_{n\geq 0} G_n \frac{x^n}{n!} = \frac{2x}{1+e^x},$$
as in Exercise 5.8(d). Show that $f(n) = 2G_{n+1}/(n+1)$.

   (b) [2] It follows from (a) that $f(n) = 0$ when n is even. Give a noncomputational proof.

   (c) [5–] What more can be done with this model of a random poset, i.e., each element of $B_n - \{\hat{0}, \hat{1}\}$ is chosen independently with probability $1/2$ (or we could generalize to any probability $0 \leq p \leq 1$) to belong to P? For instance, what is the probability that P contains a maximal chain of B_n? (This looks quite difficult to me.) What is the expected value of the rank of the top homology $H_{n-2}(\Delta(P'); \mathbb{Z})$ of the order complex $\Delta(P')$ of $P' = P - \{\hat{0}, \hat{1}\}$?

8. (a) [2] Let U_n be the set of all lattice paths λ of length $n-1$ (i.e., with $n-1$ steps), starting at $(0,0)$, with steps $(1,1)$ and $(1,-1)$. Thus $\#U_n = 2^{n-1}$. Regard the n integer points on the path λ as the elements of a poset P_λ, such that λ is the Hasse diagram of P_λ. Find $\sum_{\lambda\in U_n} e(P_\lambda)$.

(b) [2+] Give P_λ the labeling ω_λ by writing the numbers $1, 2, \dots, n$ along the path. For example, when $n = 8$ one possible pair $(P_\lambda, \omega_\lambda)$ is given by



Find $\sum_{\lambda \in U_n} \Omega_{P_\lambda, \omega_\lambda}(m)$ and $\sum_{\lambda \in U_n} W_{P_\lambda, \omega_\lambda}(q)$.

(c) [5–] Let V_n consist of those $\lambda \in U_n$ which never fall below the x-axis. It is well-known that $V_n = \binom{n-1}{\lfloor (n-1)/2 \rfloor}$. Show that $\sum_{\lambda \in V_n} e(P_\lambda)$ is equal to the number of permutations $w \in \mathfrak{S}_n$ of odd order. A formula for this number is given in EC2, Exercise 5.10(c) (the case $k = 2$).

(d) [5–] Is there a nice bijective proof or "conceptual proof" of (c)?

(e) [5–] Are there nice expressions for $\sum_{\lambda \in V_n} \Omega_{P_\lambda, \omega_\lambda}(m)$ and/or $\sum_{\lambda \in V_n} W_{P_\lambda, \omega_\lambda}(q)$?

(f) [3–] Now let W_n consist of all $\lambda \in V_{2n+1}$ that end at the x-axis. It is well-known that $\#W_n$ is the Catalan number $C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}$. Show that $\sum_{\lambda \in W_n} e(P_\lambda)$ is equal to the Eulerian-Catalan number $\mathrm{EC}_n = A(2n+1, n)/(n+1)$ of EC1, Exercise 1.53.

9. [2+] Let P be a finite poset with $\hat{0}$ and $\hat{1}$. For each $t \in P$ define a polynomial $f_t(x)$ with coefficients in $\mathbb{Z}(y)$ (the ring of rational functions in y with integer coefficients) as follows:

$$\begin{aligned} f_{\hat{0}}(x) &= y \\ f_t(0) &= 0, \quad t > \hat{0} \\ f_t(x+y) &= \sum_{s \le t} f_s(x). \end{aligned}$$

Express $f_{\hat{1}}(x)$ in terms of the zeta polynomial $Z_P(n)$.

10. [2+] Let $n \ge 1$ and $d \ge 2$. Let P_{nd} be the poset with elements x_{ij}, $1 \le i \le n$ and $1 \le j \le d$, and with cover relations $x_{ij} \lessdot x_{i+1,k}$ for all

$1 \le i \le n-1$, $1 \le j \le d$ and $1 \le k \le d$ *except* $j = k$. Thus P_{nd} is graded of rank $n-1$, and there are exactly $d(d-1)$ cover relations between consecutive ranks. Find $e(P_{nd})$. For instance, $e(P_{2,3}) = 48$ and $e(P_{3,3}) = 384$.

11. [3–] Let U_k denote an ordinal sum of k 2-element antichains, so $\#U_k = 2k$. Show that when $k = 2j$, the real parts of the zeros of the order polynomial $\Omega_{U_k}(m)$ are $0, -1, -2, \dots, -(k-1)$ (each occurring once), and $-(j - \frac{1}{2})$ (occuring k times). Similarly when $k = 2j+1$ the zeros of $\Omega_{U_k}(m)$ have real parts $0, -1, \dots, -(k-1)$, each occurring once, except that $-j$ occurs $k+1$ times.

12. (a) [3–] Let $f(n)$ be the number of nonisomorphic n-element posets for which $\beta_{J(P)}(S) \le 1$ for all $S \subseteq [n-1]$. Find a simple formula for the generating function $\sum_{n \ge 0} f(n) x^n$.

    (b) [3–] Among the $f(n)$ posets P of (a), find the maximum value of $e(P)$. How many posets (up to isomorphism) achieve this maximum?

13. (a) [2+] Define two labelings $\omega, \omega' \colon P \to [p]$ of the p-element poset P to be *equivalent* if $\mathcal{A}_{P,\omega} = \mathcal{A}_{P,\omega'}$. For instance, one equivalence class consists of the natural labelings. Let $[\omega]$ denote the equivalence class containing ω. Define a partial ordering L_P on the equivalence classes by $[\omega] \le [\omega']$ if $\mathcal{A}_{P,\omega} \subseteq \mathcal{A}_{P,\omega'}$. Show that L_P is a self-dual graded poset with $\hat{0}$ and $\hat{1}$. What is the rank (length of the longest chain) of L_P? (For the number of elements of L_P, see Exercise 3.160.)

    (b) [3] Find the Möbius function of L_P. In particular, for any $s \le t$ we have $\mu(s,t) = 0, \pm 1$.

    (c) [5–] What else can be said about the poset L_P?

14. [2+] Let (P, ω) be a labelled p-element poset. Show that there is some $S \subseteq [p-1]$ for which exactly one permutation $w \in \mathcal{L}(P, \omega)$ has descent set S.

15. (a) [2] Let P be an Eulerian poset with more than one element. Show that $\#P$ is even.

(b) [2+] Let P be an Eulerian poset of rank n, and let $0 \leq i \leq n-1$. Define

$$h_i = \sum_{\substack{S \subseteq [n-1] \\ \#S = i}} \beta_P(S).$$

Show that $h_i \equiv \binom{n-1}{i} \pmod 2$.

16. (a) [2+] Give an example of a graded poset P, say with $\hat{0}$ and $\hat{1}$, such that the flag h-vector β_P is nonnegative, but some interval of P fails to have this property.

    (b) [5–] Is there a finite graded poset P with $\hat{0}, \hat{1}$ such that the flag h-vector is nonnegative for every interval of P but P is not R-labellable? It seems likely that such a P exists, but it is difficult to show that some poset is not R-labellable.

17. (a) [2] Show that for any $M > 0$ there exists a finite graded poset P, say of rank n, such that $\beta_P(S) < 0$ for at least M subsets $S \subseteq [0, n]$.

    (b) [5–] What can one say about the sign distribution of flag h-vectors of graded posets of rank n? In other words, define $\rho_P \colon P \to \{+, -, 0\}$ by

$$\rho_P(S) = \begin{cases} +, & \text{if } \beta_P(S) > 0 \\ -, & \text{if } \beta_P(S) < 0 \\ 0, & \text{if } \beta_P(S) = 0. \end{cases}$$

What can be said about the possible functions ρ_P? E.g., what is the maximum number of S for which $\beta_P(S) < 0$ (as a function of n)?

18. [2+] Let P be a finite graded poset of rank n with $\hat{0}, \hat{1}$. Fix $j \in \mathbb{P}$. Suppose that for every interval $[s, t]$, the number of elements of rank j (where rank is computed with respect to the interval $[s, t]$) is equal to the number of elements of corank j. Show that

$$\sum_{\substack{t \in P \\ \mathrm{rank}(t) = j}} Z_{[t, \hat{1}]}(m) = \sum_{\substack{t \in P \\ \mathrm{rank}(t) = n-j}} Z_{[\hat{0}, t]}(m).$$

Here $Z_{[u,v]}(m)$ denotes the zeta polynomial of the interval $[u, v]$. Try to find an elegant proof.

19. [2+] True or false? A supersolvable Eulerian lattice is a boolean algebra.

20. [2+] Show that every interval of a supersolvable lattice is supersolvable.

21. [2] Let $n \geq 1$. How many nonisomorphic Eulerian posets are there of rank three with n atoms? How many are lattices? State your answers in terms of the number of partitions $p(m)$ of m for certain nonnegative integers m. *Optional:* how many are Cohen-Macaulay? If you have some familiarity with Cohen-Macaulay posets then this is very easy.

22. (a) [2+] Find the smallest positive integer n for which there exists an Eulerian poset P of rank n whose rank-generating function $F(P, x)$ is not unimodal. NOTE. A real polynomial $\sum_{i=0}^{n} a_i x^i$ is *unimodal* if for some j we have

$$a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq a_{j+2} \geq \cdots \geq a_n.$$

    (b) [3] Does there exist an Eulerian lattice whose rank-generating function is not unimodal?

23. [2++] Give an example of an Eulerian P poset with the following properties: (i) for some $d \geq 2$, P has d atoms and rank $d+1$, (ii) $P - \{\hat{1}\}$ is simplicial, and (iii) the rank-generating function of P is *not* equal to $(1+x)^d + x^d + x^{d+1}$.

24. [2] Let $P_0 \cup P_1 \cup \cdots \cup P_8$ be an Eulerian poset of rank eight with the following properties: (i) $P - \{\hat{1}\}$ is simplicial, and (ii) $P_0 \cup P_1 \cup P_2 \cup P_3$ agrees with the the boolean algebra B_{11} truncated above rank 3. Find the rank-generating function of P. *Bonus* (not needed to receive full credit). Does P actually exist? Could it be a lattice?

25. [2+] Let P be a locally finite graded poset with $\hat{0}$ and containing an infinite chain, satisfying the two conditions:

    - All intervals $[\hat{0}, t]$ of rank n have the same number $D(n)$ of maximal chains.
    - All intervals $[s, t]$ of rank n with $s > \hat{0}$ have the same number $B(n)$ of maximal chains.

An example is an obvious limit as $m \to \infty$ of the face lattice of an m-cube. Here $D(n) = 2^{n-1}(n-1)!$ $(n > 0)$ and $B(n) = n!$. Find a generalization of Theorem 3.18.4 for these posets. It should involve generating functions

$$\begin{aligned} F(x) &= \sum_{n \geq 0} f(n) \frac{x^n}{B(n)} \\ G(x) &= \sum_{n \geq 1} g(n) \frac{x^n}{D(n)}. \end{aligned}$$

26. (a) [2] Let $1 = A(1) \leq A(2) \leq A(3)$. Show that there exists a binomial poset P (except for the axiom of containing an infinite chain) of rank three with these "atom numbers," i.e., $B(m) = A(1)A(2)\cdots A(m)$ for $1 \leq m \leq 3$.

    (b) [5–] Let $n \geq 1$ and $1 = A(1) \leq A(2) \leq \cdots \leq A(n)$. Define $B(m) = A(1)A(2)\cdots A(m)$ for $1 \leq m \leq n$. Set $B(0) = 1$. Suppose that $\frac{B(m)}{B(k)B(m-k)} \in \mathbb{Z}$ for all $0 \leq k \leq m \leq n$. Does there exist a binomial poset P (except for the axiom of containing an infinite chain) of rank n with factorial function B? (This seems unlikely to me. As mentioned in class and in the text, an open case is $A(m) = F_{m+1}$, a Fibonacci number.)

    (c) [5] For which positive integers q does there exist a binomial *lattice* P (except for the axiom of containing an infinite chain) of rank three with $A(1) = 1$, $A(2) = q + 1$, $A(3) = q^2 + q + 1$?

    (d) (*Bonus*: does not count for pset credit.) For which positive integers q does there exist a binomial *lattice* P (except for the axiom of containing an infinite chain) of rank four with $A(1) = 1$, $A(2) = q + 1$, $A(3) = q^2 + q + 1$, $A(4) = q^3 + q^2 + q + 1$? (Rating [3] or even [3+] from scratch, but if you know a certain fact it is much easier.)

27. [2+] Give an example of a differential poset whose automorphism group is trivial.