ADDITIONAL POSET PROBLEMS

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1. [2] Find a finite poset \( P \) with the following property, or show that no such \( P \) exists. The longest chain in \( P \) has \( m \) elements (for some \( m \geq 1 \)). \( P \) can be written as a union of two chains \( C_1 \) and \( C_2 \), but cannot be written in this way where \( \#C_1 = m \).

2. (a) [2] How many nonisomorphic \( n \)-element posets contain an \( (n-1) \)-element antichain?
   
   (b) [2+] How many nonisomorphic \( n \)-element posets contain an \( (n - 1) \)-element chain?
   
   (c) [2–] How many nonisomorphic \( n \)-element posets contain both an \( (n-1) \)-element antichain and an \( (n-1) \)-element chain?

3. (a) [3–] Find a finite poset \( P \) with the following property. The automorphism group \( \text{Aut}(P) \) of \( P \) acts transitively on the set \( M \) of minimal elements of \( P \). Moreover, the restriction of \( \text{Aut}(P) \) to \( M \) does not contain a full cycle of the elements of \( M \).
   
   (b) [5–] Does such a poset exist if all maximal chains have two elements?

4. [2+] Let \( w = t_1, \ldots, t_p \) be a permutation of the elements of a finite poset \( P \). Call a permutation \( w' \) a permissible swap of \( w \) if it is obtained from \( w \) by interchanging some \( t_i \) and \( t_{i+1} \) where \( t_i < t_{i+1} \). Clearly a sequence of permissible swaps must eventually terminate in a permutation \( v \) that has no permissible swaps. Show that \( v \) is independent of the sequence of permissible swaps.

5. [2+] For each permutation \( w \in \mathfrak{S}_n \), let \( \sigma_w \) be the simplex in \( \mathbb{R}^n \) defined by
   
   \[ \sigma_w = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_{w(1)} \leq x_{w(2)} \leq \cdots \leq x_{w(n)} \leq 1\}. \]
   
   For any nonempty subset \( S \subseteq \mathfrak{S}_n \), define
   
   \[ X_S = \bigcup_{w \in S} \sigma_w \subset \mathbb{R}^n. \]
Show that $X_S$ is convex if and only if $S$ is the set of linear extensions of some partial ordering of $[n]$.

6. [2+] Let $0 \leq p \leq 1$, and let $P$ be a finite $n$-element poset with $\hat{0}$ and $\hat{1}$. Let $\sigma : P \rightarrow [n]$ be a linear extension of $P$. Define a random digraph $D$ on the vertex set $[n]$ as follows. For each $s < t$ in $P$, choose the edge $s \rightarrow t$ of $D$ with probability $p$.

Now start at the vertex $\hat{0}$ of $D$. If there is an arrow from $\hat{0}$, then move to the vertex $t$ for which $\hat{0} \rightarrow t$ is an edge of $D$ and $\sigma(t)$ is as small as possible; otherwise stop. Continue this procedure (always moving from a vertex $u$ to a vertex $v$ for which $u \rightarrow v$ is an edge of $D$ and $\sigma(v)$ is as small as possible) until unable to continue. What is the probability that we end at vertex $\hat{1}$? Try to give an elegant proof avoiding recurrence relations, linear algebra, etc.

7. (a) [2+] Let $f(n)$ be the average value of $\mu_P(\hat{0}, \hat{1})$, where $P$ ranges over all (induced) subposets of the boolean algebra $B_n$ containing $\hat{0}$ and $\hat{1}$. (The number of such $P$ is $2^{2^{n-2}}$.) Define the Genocchi number $G_n$ by

$$
\sum_{n \geq 0} G_n \frac{x^n}{n!} = \frac{2x}{1 + e^x},
$$

as in Exercise 5.8(d). Show that $f(n) = 2G_{n+1}/(n + 1)$.

(b) [2] It follows from (a) that $f(n) = 0$ when $n$ is even. Give a noncomputational proof.

(c) [5–] What more can be done with this model of a random poset, i.e., each element of $B_n - \{\hat{0}, \hat{1}\}$ is chosen independently with probability $1/2$ (or we could generalize to any probability $0 \leq p \leq 1$) to belong to $P$? For instance, what is the probability that $P$ contains a maximal chain of $B_n$? (This looks quite difficult to me.) What is the expected value of the rank of the top homology $H_{n-2}(\Delta(P'); \mathbb{Z})$ of the order complex $\Delta(P')$ of $P' = P - \{\hat{0}, \hat{1}\}$?

8. (a) [2] Let $U_n$ be the set of all lattice paths $\lambda$ of length $n - 1$ (i.e., with $n - 1$ steps), starting at $(0, 0)$, with steps $(1, 1)$ and $(1, -1)$. Thus $\#U_n = 2^{n-1}$. Regard the $n$ integer points on the path $\lambda$ as the elements of a poset $P_\lambda$, such that $\lambda$ is the Hasse diagram of $P_\lambda$. Find $\sum_{\lambda \in U_n} e(P_\lambda)$. 

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(b) [2+] Give $P_{\lambda}$ the labeling $\omega_{\lambda}$ by writing the numbers $1, 2, \ldots, n$ along the path. For example, when $n = 8$ one possible pair $(P_{\lambda}, \omega_{\lambda})$ is given by

![Diagram showing labeling](image)

Find $\sum_{\lambda \in U_n} \Omega_{P_{\lambda}, \omega_{\lambda}}(m)$ and $\sum_{\lambda \in U_n} W_{P_{\lambda}, \omega_{\lambda}}(q)$.

(c) [3–] Let $V_n$ consist of those $\lambda \in U_n$ which never fall below the $x$-axis. It is well-known that $V_n = \left(\left\lfloor \frac{n-1}{2} \right\rfloor \right)$. Show that $\sum_{\lambda \in V_n} e(P_{\lambda})$ is equal to the number of permutations $w \in S_n$ of odd order. A formula for this number is given in EC2, Exercise 5.10(c) (the case $k = 2$).

(d) [5–] Is there a nice bijective proof or “conceptual proof” of (c)?

(e) [5–] Are there nice expressions for $\sum_{\lambda \in V_n} \Omega_{P_{\lambda}, \omega_{\lambda}}(m)$ and/or $\sum_{\lambda \in V_n} W_{P_{\lambda}, \omega_{\lambda}}(q)$?

(f) [3–] Now let $W_n$ consist of all $\lambda \in V_{2n+1}$ that end at the $x$-axis. It is well-known that $\#W_n$ is the Catalan number $C_{n-1} = \frac{1}{n} \binom{2(n-1)}{n-1}$. Show that $\sum_{\lambda \in W_n} e(P_{\lambda})$ is equal to the Eulerian-Catalan number $EC_n = A(2n+1, n+1)/(n+1)$ of EC1, Exercise 1.53.

9. [2+] Let $P$ be a finite poset with $\hat{0}$ and $\hat{1}$. For each $t \in P$ define a polynomial $f_t(x)$ with coefficients in $\mathbb{Z}[y]$ as follows:

$$
\begin{align*}
    f_{\hat{0}}(x) &= y \\
    f_t(x+y) &= \sum_{s \leq t} f_s(x).
\end{align*}
$$

Express $f_1(x)$ in terms of the zeta polynomial $Z_P(n)$.

10. [2+] Let $n \geq 1$ and $d \geq 2$. Let $P_{nd}$ be the poset with elements $x_{ij}$, $1 \leq i \leq n$ and $1 \leq j \leq d$, and with cover relations $x_{ij} \preceq x_{i+1,k}$ for all $1 \leq i \leq n-1$, $1 \leq j \leq d$ and $1 \leq k \leq d$ except $j = k$. Thus $P_{nd}$ is graded of rank $n-1$, and there are exactly $d(d-1)$ cover relations between consecutive ranks. Find $e(P_{nd})$. For instance, $e(P_{2,3}) = 48$ and $e(P_{3,3}) = 384$. 
11. Let $U_k$ denote an ordinal sum of $k$ 2-element antichains, so $\#U_k = 2k$. Show that when $k = 2j$, the real parts of the zeros of the order polynomial $\Omega_{U_k}(m)$ are $0, -1, -2, \ldots, -(k - 1)$ (each occurring once), and $-(j - \frac{1}{2})$ (occurring $k$ times). Similarly when $k = 2j + 1$ the zeros of $\Omega_{U_k}(m)$ have real parts $0, -1, \ldots, -(k - 1)$, each occurring once, except that $-j$ occurs $k + 1$ times.

Further problems may be forthcoming.