ADDITIONAL POSET PROBLEMS

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1. [2] Find a finite poset \( P \) with the following property, or show that no such \( P \) exists. The longest chain in \( P \) has \( m \) elements (for some \( m \geq 1 \)). \( P \) can be written as a union of two chains \( C_1 \) and \( C_2 \), but cannot be written in this way where \( \#C_1 = m \).

2. (a) [2] How many nonisomorphic \( n \)-element posets contain an \( (n - 1) \)-element antichain?

(b) [2+] How many nonisomorphic \( n \)-element posets contain an \( (n - 1) \)-element chain?

(c) [2–] How many nonisomorphic \( n \)-element posets contain both an \( (n - 1) \)-element antichain and an \( (n - 1) \)-element chain?

3. (a) [3–] Find a finite poset \( P \) with the following property. The automorphism group \( \text{Aut}(P) \) of \( P \) acts transitively on the set \( M \) of minimal elements of \( P \). Moreover, the restriction of \( \text{Aut}(P) \) to \( M \) does not contain a full cycle of the elements of \( M \).

(b) [5–] Does such a poset exist if all maximal chains have two elements?

4. [2+] Let \( w = t_1, \ldots, t_p \) be a permutation of the elements of a finite poset \( P \). Call a permutation \( w' \) a permissible swap of \( w \) if it is obtained from \( w \) by interchanging some \( t_i \) and \( t_{i+1} \) where \( t_i < t_{i+1} \). Clearly a sequence of permissible swaps must eventually terminate in a permutation \( v \) that has no permissible swaps. Show that \( v \) is independent of the sequence of permissible swaps.

5. [2+] For each permutation \( w \in \mathfrak{S}_n \), let \( \sigma_w \) be the simplex in \( \mathbb{R}^n \) defined by

\[ \sigma_w = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_{w(1)} \leq x_{w(2)} \leq \cdots \leq x_{w(n)} \leq 1 \}. \]

For any nonempty subset \( S \subseteq \mathfrak{S}_n \), define

\[ X_S = \bigcup_{w \in S} \sigma_w \subset \mathbb{R}^n. \]
Show that $X_S$ is convex if and only if $S$ is the set of linear extensions of some partial ordering of $[n]$.

6. [2+] Let $0 \leq p \leq 1$, and let $P$ be a finite $n$-element poset with $\hat{0}$ and $\hat{1}$. Let $\sigma: P \to [n]$ be a linear extension of $P$. Define a random digraph $D$ on the vertex set $[n]$ as follows. For each $s < t$ in $P$, choose the edge $s \to t$ of $D$ with probability $p$.

Now start at the vertex $\hat{0}$ of $D$. If there is an arrow from $\hat{0}$, then move to the vertex $t$ for which $\hat{0} \to t$ is an edge of $D$ and $\sigma(t)$ is as small as possible; otherwise stop. Continue this procedure (always moving from a vertex $u$ to a vertex $v$ for which $u \to v$ is an edge of $D$ and $\sigma(v)$ is as small as possible) until unable to continue. What is the probability that we end at vertex $\hat{1}$? Try to give an elegant proof avoiding recurrence relations, linear algebra, etc.

7. (a) [2+] Let $f(n)$ be the average value of $\mu_P(\hat{0}, \hat{1})$, where $P$ ranges over all (induced) subposets of the boolean algebra $B_n$ containing $\hat{0}$ and $\hat{1}$. (The number of such $P$ is $2^{2^n-2}$.) Define the Genocchi number $G_n$ by

$$
\sum_{n \geq 0} G_n \frac{x^n}{n!} = \frac{2x}{1+e^x},
$$

as in Exercise 5.8(d). Show that $f(n) = 2G_{n+1}/(n+1)$.

(b) [2] It follows from (a) that $f(n) = 0$ when $n$ is even. Give a noncomputational proof.

(c) [5–] What more can be done with this model of a random poset, i.e., each element of $B_n - \{\hat{0}, \hat{1}\}$ is chosen independently with probability 1/2 (or we could generalize to any probability $0 \leq p \leq 1$) to belong to $P$? For instance, what is the probability that $P$ contains a maximal chain of $B_n$? (This looks quite difficult to me.) What is the expected value of the rank of the top homology $H_{n-2}(\Delta(P'); \mathbb{Z})$ of the order complex $\Delta(P')$ of $P' = P - \{\hat{0}, \hat{1}\}$?

8. (a) [2] Let $U_n$ be the set of all lattice paths $\lambda$ of length $n - 1$ (i.e., with $n - 1$ steps), starting at $(0, 0)$, with steps $(1, 1)$ and $(1, -1)$. Thus $\# U_n = 2^{n-1}$. Regard the $n$ integer points on the path $\lambda$ as the elements of a poset $P_\lambda$, such that $\lambda$ is the Hasse diagram of $P_\lambda$. Find $\sum_{\lambda \in U_n} e(P_\lambda)$. 

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(b) \([2+]\) Give \(P\) the labeling \(\omega\) by writing the numbers 1, 2, \ldots, \(n\) along the path. For example, when \(n = 8\) one possible pair \((P, \omega)\) is given by

\[1, 2, 3, 4, 5, 6, 7, 8\]

Find \(\sum_{\lambda \in U_n} \Omega_{P, \omega}(m)\) and \(\sum_{\lambda \in U_n} W_{P, \omega}(q)\).

(c) \([3-]\) Let \(V_n\) consist of those \(\lambda \in U_n\) which never fall below the \(x\)-axis. It is well-known that \(V_n = \binom{n-1}{\lfloor (n-1)/2 \rfloor}\). Show that \(\sum_{\lambda \in V_n} e(P, \omega)\) is equal to the number of permutations \(w \in \mathfrak{S}_n\) of odd order. A formula for this number is given in EC2, Exercise 5.10(c) (the case \(k = 2\)).

(d) \([5-]\) Is there a nice bijective proof or “conceptual proof” of (c)?

(e) \([5-]\) Are there nice expressions for \(\sum_{\lambda \in V_n} \Omega_{P, \omega}(m)\) and/or \(\sum_{\lambda \in V_n} W_{P, \omega}(q)\)?

(f) \([3-]\) Now let \(W_n\) consist of all \(\lambda \in V_{2n+1}\) that end at the \(x\)-axis. It is well-known that \(#W_n\) is the Catalan number \(C_{n-1} = \frac{1}{n} \binom{2(n-1)}{n-1}\). Show that \(\sum_{\lambda \in W_n} e(P, \omega)\) is equal to the Eulerian-Catalan number \(EC_n = A(2n+1, n+1)/(n+1)\) of EC1, Exercise 1.53.

9. \([2+]\) Let \(P\) be a finite poset with \(\hat{0}\) and \(\hat{1}\). For each \(t \in P\) define a polynomial \(f_t(x)\) with coefficients in \(\mathbb{Z}[y]\) as follows:

\[f_0(x) = y,\]
\[f_t(x+y) = \sum_{s \leq t} f_s(x)\]
11. [3–] Let $U_k$ denote an ordinal sum of $k$ 2-element antichains, so $\#U_k = 2k$. Show that when $k = 2j$, the real parts of the zeros of the order polynomial $\Omega_{U_k}(m)$ are $0, -1, -2, \ldots, -(k - 1)$ (each occurring once), and $-(j - \frac{1}{2})$ (occurring $k$ times). Similarly when $k = 2j + 1$ the zeros of $\Omega_{U_k}(m)$ have real parts $0, -1, \ldots, -(k - 1)$, each occurring once, except that $-j$ occurs $k + 1$ times.

12. (a) [3–] Let $f(n)$ be the number of nonisomorphic $n$-element posets for which $\beta_{J(P)}(S) \leq 1$ for all $S \subseteq [n - 1]$. Find a simple formula for the generating function $\sum_{n \geq 0} f(n)x^n$.

(b) [3–] Among the $f(n)$ posets $P$ of (a), find the maximum value of $e(P)$. How many posets (up to isomorphism) achieve this maximum?

13. (a) [2+] Define two labelings $\omega, \omega': P \to [p]$ of the $p$-element poset $P$ to be equivalent if $A_{P,\omega} = A_{P,\omega'}$. For instance, one equivalence class consists of the natural labelings. Let $[\omega]$ denote the equivalence class containing $\omega$. Define a partial ordering $L_P$ on the equivalence classes by $[\omega] \leq [\omega']$ if $A_{P,\omega} \subseteq A_{P,\omega'}$. Show that $L$ is a self-dual graded poset with $\hat{0}$ and $\hat{1}$. What is the rank (length of the longest chain) of $L_P$? (For the number of elements of $L_P$, see Exercise 3.160.)

(b) [3] Find the Möbius function of $L_P$. In particular, for any $s \leq t$ we have $\mu(s, t) = 0, \pm 1$.

(c) [5–] What else can be said about the poset $L_P$?

14. [2+] Let $(P, \omega)$ be a labelled $p$-element poset. Show that there is some $S \subseteq [p - 1]$ for which exactly one permutation $w \in L(P, \omega)$ has descent set $S$.

Further problems may be forthcoming.