THE MATHEMATICAL LEGACY OF RODICA SIMION

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- real zeros and the Poset Conjecture
- pattern-avoiding permutations
- noncrossing partitions
- shuffle posets

REAL ZEROS AND THE POSET CONJECTURE

A multiindexed Sturm sequence of polynomails and unimodality of certain combinatorial sequences, *JCT(A)* **36** (1984), 15–22. (From U. Penn. Ph.D. thesis.)

$\mathfrak{S}_n$: symmetric group on $1, 2, \dots, n$

For $w = a_1 \cdots a_n \in \mathfrak{S}_n$, define

$$\mathbf{des}(w) = \#\{i \,:\, a_i > a_{i+1}\}.$$

E.g., des(**4**1**75**236) = 3.

Eulerian polynomial:

$$A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{1+\mathrm{des}(w)}.$$

Theorem (Harper). *All zeros of $A_n(x)$ are real.*

Generalize to **multisets**:

$$\begin{aligned} \boldsymbol{M} &= \{1^{m_1}, 2^{m_2}, \ldots\}, \ \sum m_i = n \\ \boldsymbol{\mathfrak{S}_M} &= \{\text{permutations of } M\} \\ \#\mathfrak{S}_M &= \binom{n}{m_1, m_2, \ldots}. \end{aligned}$$

For $w = a_1 \cdots a_n \in \mathfrak{S}_M$ define as before

$$\begin{aligned} \mathbf{des}(\boldsymbol{w}) &= \#\{i \,:\, a_i > a_{i+1}\} \\ \boldsymbol{A_M(x)} &= \sum_{w \in \mathfrak{S}_M} x^{1+\mathrm{des}(w)}. \end{aligned}$$

Theorem (Simion). *All zeros of $A_M(x)$ lie in $[-1, 0]$. Moreover, $A_M(x)$ and $A_{M \cup \{j\}}(x)$ have interlaced zeros.*

The Poset Conjecture (Neggers, RPS, c. 1970). Let P be a partial ordering of $1,\ldots,n$. Let

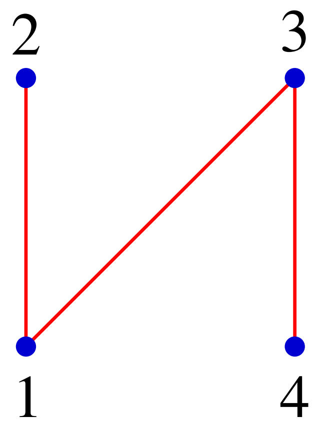
$$\mathcal{L}_P = \{w = w_1\cdots w_n \in \mathfrak{S}_n : \; i \overset{P}{<} j \Rightarrow w^{-1}(i) < w^{-1}(j) \; \text{(i.e., } i \text{ precedes } j \text{ in } w)\}.$$

$$W_P(x) = \sum_{w\in\mathcal{L}_P} x^{\mathrm{des}(w)}.$$

Note. Let $\boldsymbol{a}$ = a-element chain,

$$P = \boldsymbol{a_1} + \boldsymbol{a_2} + \cdots.$$

Then $W_P(x) = A_M(x)/x$, where $M = \{a_1, a_2, \ldots\}$.



w	$\text{des}(w)$
1**4**23	1
4123	1
1**43**2	2
41**3**2	2
12**4**3	1

$$W_P(x) = 3x + 2x^2 : \quad \text{all zeros real!}$$

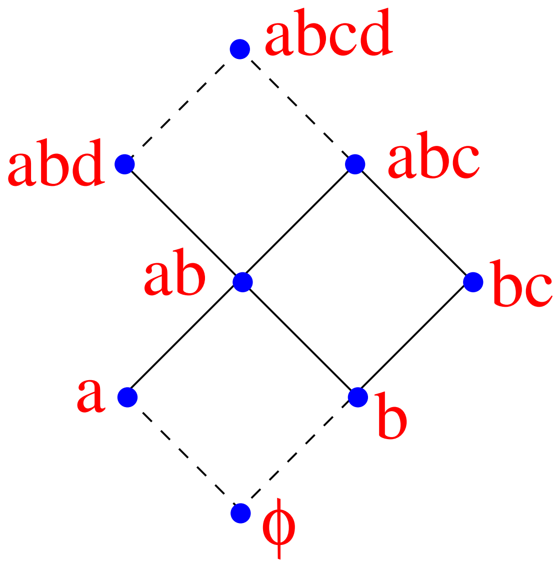
Poset Conjecture. For any poset P on $1, \ldots, n$, all zeros of $W_P(x)$ are real. (True for $|P| \leq 8$. There are 431,723,379 8-element labelled posets.)

Let Q be a finite poset.

chain polynomial: $\boldsymbol{C_Q(x)} = \sum_{\sigma} x^{\#\sigma}$,

where σ ranges over all chains of Q.

Special case (open). Let L be a finite distributive lattice (a finite collection of sets closed under $\cup$ and $\cap$, ordered by inclusion). Then all zeros of $C_L(x)$ are real.



$$C_L(x) = (1 + 6x + 10x^2 + 5x^3)(1 + x)^2$$

Also open: All zeros of $C_L(x)$ are real if L is a finite **modular** lattice.

PATTERN-AVOIDING PERMUTATIONS

Restricted permutations, *Europ. J. Combinatorics* **6** (1985), 383–406 (with F. Schmidt).

Partially ordered sets associated with permutations, *Europ. J. Combinatorics* **10** (1989), 375-391.

Let $\boldsymbol{u} = a_1 a_2 \cdots a_k \in \mathfrak{S}_k$.

Definition. $w = b_1 b_2 \cdots b_n \in \mathfrak{S}_n$ ***avoids*** u if no subsequence $b_{i_1} \cdots b_{i_k}$ is in the same relative order as u, i.e.,

$$a_r < a_s \Leftrightarrow b_{i_r} < b_{i_s}.$$

$12 \cdots k$-avoiding: no increasing subsequence of length k

$$\begin{aligned} \boldsymbol{S_n(u)} &= \{u\text{-avoiding } w \in \mathfrak{S}_n\} \\ \boldsymbol{A_n(u)} &= \# S_n(u) \\ \boldsymbol{S_n(u,v)} &= S_n(u) \cap S_n(v) \\ \boldsymbol{A_n(u,v)} &= \#(S_n(u) \cap S_n(v)) \end{aligned}$$

Hammersley (1972), Knuth (1973), Rotem (1975), D. G. Rogers (1978):

$$A_n(123) = \boldsymbol{C_n} = \frac{1}{n+1}\binom{2n}{n}$$

Knuth (1968): $A_n(213) = C_n$

Rotem (1981): $A_n(231, 312) = 2^{n-1}$

Simion-Schmidt (1985): first ***systematic*** consideration of pattern-avoidance (for patterns in $\mathfrak{S}_3$)

- $A_n(\mathcal{X})$ for any subset $\mathcal{X} \subseteq \mathfrak{S}_3$ (including several nontrivial bijections)

  E.g, $A_n(123, 132, 213) = F_{n+1}$ (Fibonacci number).

- $E_n(u) - O_n(u)$ for all $u \in \mathfrak{S}_3$, where

  $$\boldsymbol{E_n(u)} = \#\{w \in S_n(u) : w \text{ is even}\}$$
  $$\boldsymbol{O_n(u)} = \#\{w \in S_n(u) : w \text{ is odd}\}.$$

  E.g., $E_n(132) - O_n(132) = C_{(n-1)/2}$.

- $I_n(u)$ for all $u \in \mathfrak{S}_3$, where

  $$\boldsymbol{I_n(u)} = \#\{w \in S_n(u) : w^2 = \mathrm{id}\}.$$

  E.g., $I_n(123) = I_n(132) = \binom{n}{\lfloor n/2 \rfloor}$,

  $$I_n(231) = 2^{n-1}.$$

- $EI_n(231) - OI_n(231) =$

$$\frac{7-2\sqrt{-7}}{28}\alpha^n + \frac{7+3\sqrt{-7}}{28}\beta^n,$$

where

$$\alpha = \frac{1+\sqrt{-7}}{2}, \quad \beta = \frac{1-\sqrt{-7}}{2}.$$

- $$\#\{w \in \mathfrak{S}_n \,:\, w \text{ avoids } \boldsymbol{no}\ u \in \mathfrak{S}_3\} =$$

$$n! - 6C_n + 5\cdot 2^n + 4\binom{n}{2} - 2F_{n+1} - 14n + 20,$$

$$n \geq 5.$$

NONCROSSING PARTITIONS

Chains in the lattice of noncrossing partitions, *Discrete Math.* **126** (1994), 107–119 (with P. Edelman).

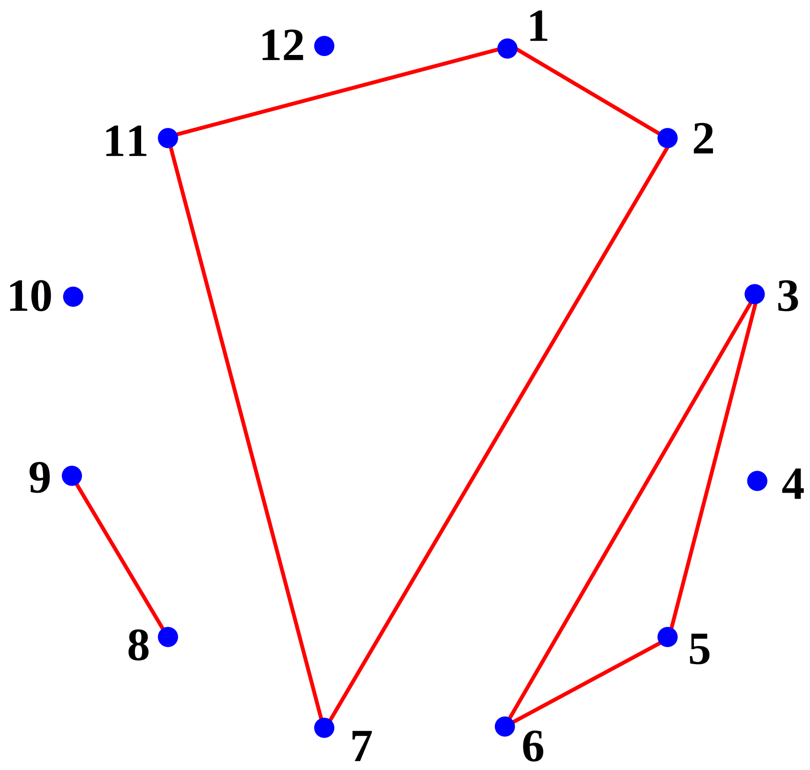
Combinatorial statistics on non-crossing partitions, *J. Combinatorial Theory (A)* **66** (1994), 270–301.

Noncrossing partitions, *Discrete Math.* **217** (2000), 367–409.

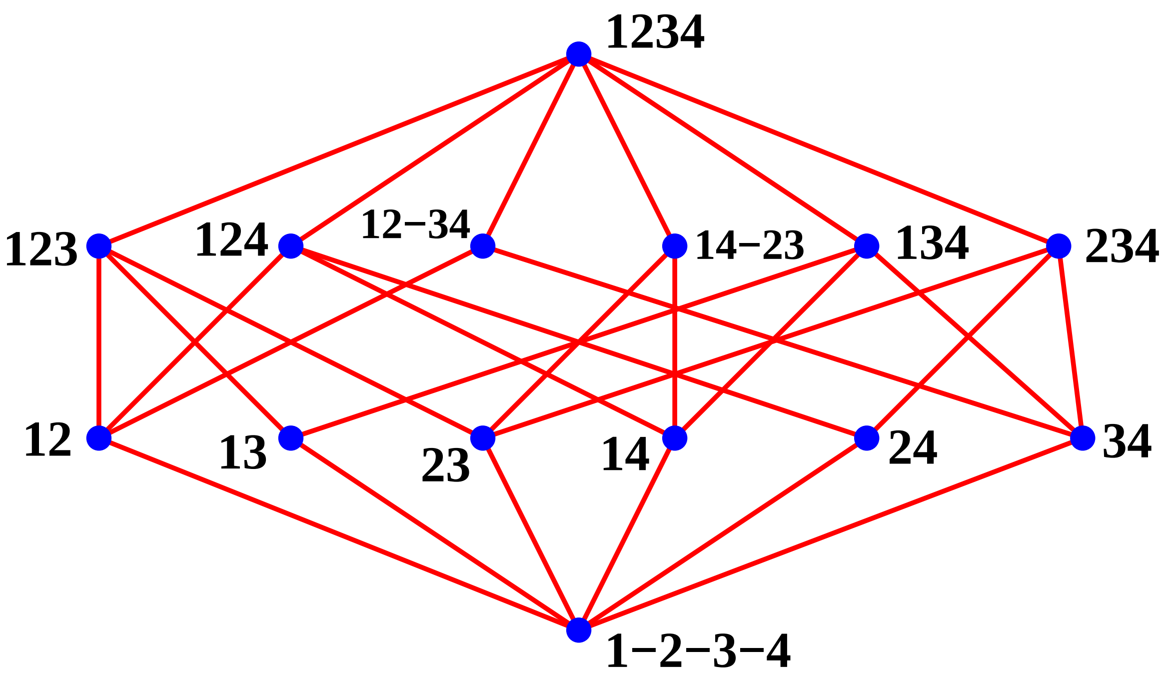
On the structure of the lattice of noncrossing partitions, *Discrete Math.* **98** (1991), 193–206 (with D. Ullman).

Definition. **Noncrossing partition** of $[n] = \{1, 2, \ldots, n\}$: a partition $\pi \in \Pi_n$ such that for $a < b < c < d$,

$$a \sim c,\ b \sim d \Rightarrow a \sim b \sim c \sim d.$$



$\mathbf{NC}_n$: set of noncrossing partitions of $[n]$, ordered by refinement (a graded lattice of rank $n-1$)

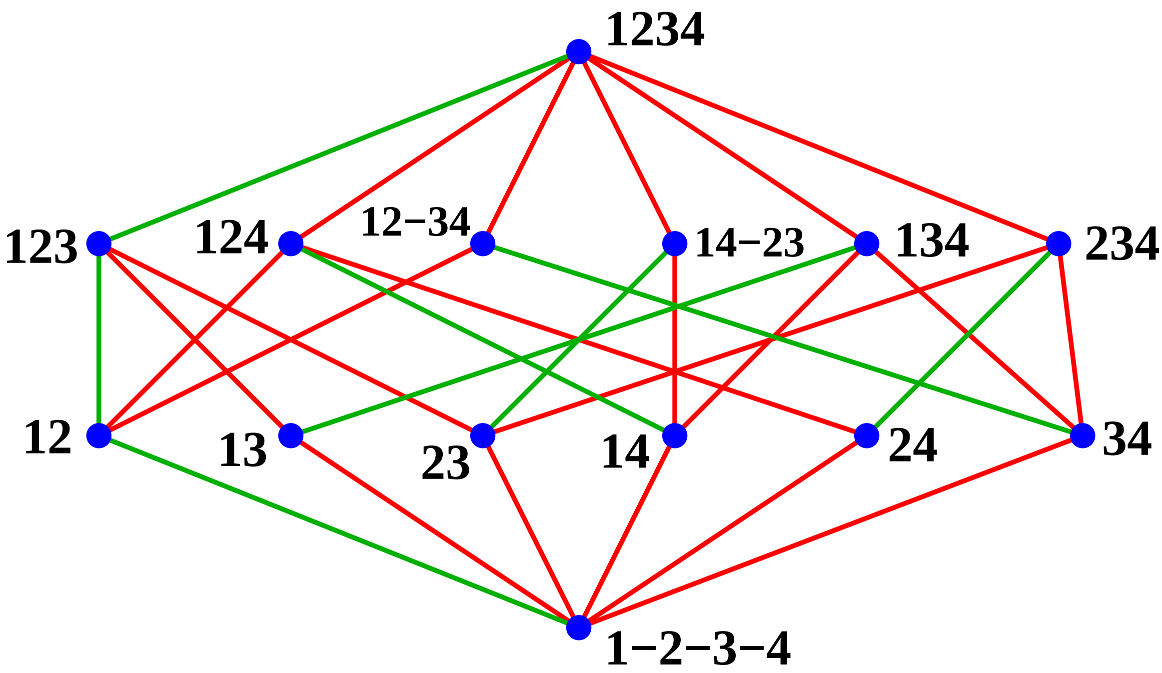


Sample properties:

- $\#\mathrm{NC}_n = C_n$
- $\#\{\pi \in \mathrm{NC}_n \, : \, \mathrm{rank}(\pi) = n-k\}$
$$= \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$
(Narayana number)
- $\#\{\hat{0} = \pi_0 \le \pi_1 \le \cdots \le \pi_k = \hat{1}\}$
$$= \frac{(kn)_{n-1}}{n!}$$ **(zeta polynomial)**
- $$\#\{\hat{0} < \pi_1 < \cdots < \pi_k < \hat{1} \, : \, \mathrm{rk}(\pi_i) = m_i\}$$
$$= \frac{1}{n}\binom{n}{m_1}\binom{n}{m_2 - m_1}\cdots\binom{n}{n-1-m_k}$$
(flag f-vector)

- $$\#\{\pi \in \mathrm{NC}_n : \mathrm{type}(\pi) = (1^{m_1}2^{m_2}\cdots)\}$$
  $$= \frac{n(n-1)\cdots(n-\sum m_i + 2)}{m_1!\,m_2!\cdots}$$
- $\mu(\hat{0},\hat{1}) = (-1)^{n-1}C_{n-1}$
- NC_n is ***locally self-dual***, i.e., every interval is self-dual. (Order-reversing **involution** due to Simion-Ullman.)

Definition. A **symmetric chain decomposition** of a graded poset P of rank n is a partitioning of P into chains $x_i < x_{i+1} < \cdots < x_{n-i}$, where $\mathrm{rk}(x_j) = j$.



Note. If P has a symmetric chain decomposition, then P is rank-unimodal and **Sperner**, i.e.,

$$\max \#(\text{antichain}) = \max \#P_i = \#P_{\lfloor n/2 \rfloor}.$$

Theorem (Edelman-Simion). NC_n *admits a symmetric chain decomposition for all* $n \geq 1$.

Definition. Let P be graded of rank n with $\hat{0}$ and $\hat{1}$. Let

$$\mathcal{E} = \{(s,t) \in P \times P \,:\, s \lessdot t\}.$$

An ***R*-labeling** of P is a map $\lambda : \mathcal{E} \to \mathbb{Z}$ such that for all $s < t$ there is a unique saturated chain

$$s = s_0 \lessdot s_1 \lessdot \cdots \lessdot s_k = t$$

satisfying

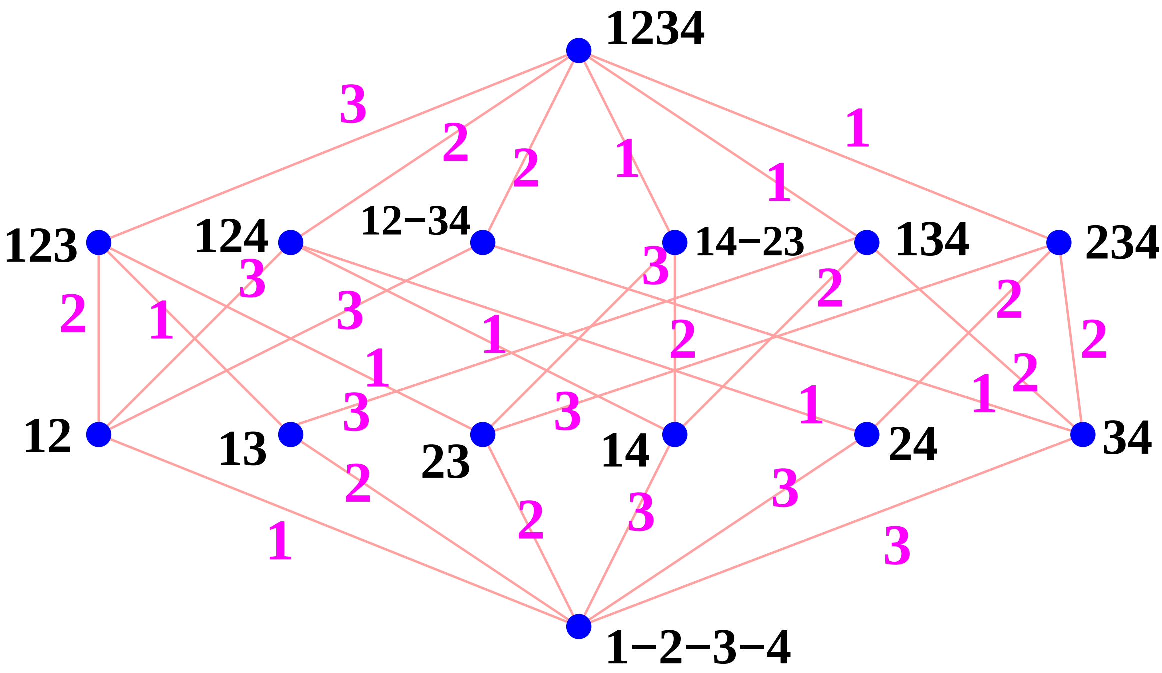
$$\lambda(s_0, s_1) < \lambda(s_1, s_2) < \cdots < \lambda(s_{k-1}, s_k).$$

Then, e.g., $(-1)^n \mu_P(\hat{0}, \hat{1})$ is the number of maximal chains

$$\hat{0} = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_n = \hat{1}$$

satisfying

$$\lambda(t_0, t_1) > \lambda(t_1, t_2) > \cdots > \lambda(t_{n-1}, t_n).$$



Björner: NC_n has an R-labeling. Namely, let σ be obtained from π by merging B and B'. Let

$$\lambda(\pi, \sigma) = \max\{\min B, \min B'\} - 1.$$

Every maximal chain label is a permutation of $1, 2, \ldots, n$.

Theorem (Edelman-Simion). *The number of maximal chains of* NC_n *labelled by* $\sigma \in \mathfrak{S}_{n-1}$ *is the number of noncrossing partitions of* $1, \dots, n-1$*, each of whose blocks is a decreasing subsequence of* σ*.*

$\boldsymbol{\sigma = 3421}$: 1-2-3-4, 21-3-4, 1-32-4, 31-2-4, 1-42-3, 41-2-3, 41-32, 421-3 (**not** 31-42)

Sample corollary. *In* NC_n *there are* $\binom{b-a+2}{2} - 1$ *maximal chains labelled by the transposition* (a, b)*.*

SHUFFLE POSETS

Flag-symmetry of the poset of shuffles and a local action of the symmetric group, *Discrete Math.* **204** (1999), 369–396 (with RPS).

$$\boldsymbol{A} = \{a_1 < \cdots < a_m\}$$
$$\boldsymbol{B} = \{b_1 < \cdots < b_n\}$$

(ordered alphabets)

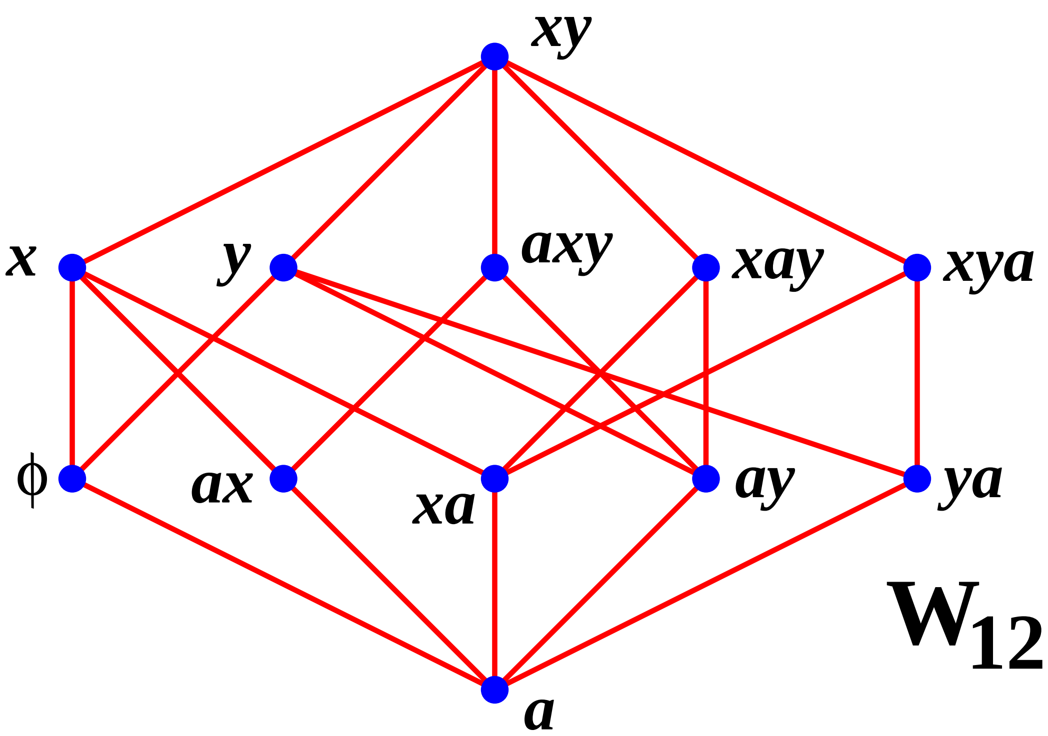
Definition (C. Greene).

$\boldsymbol{W_{mn}} = \{$shuffles of subwords of A and $B\}$

E.g., $b_2a_4a_5b_3a_8b_5b_6a_9$ and $\emptyset$.

Let $s \lessdot t$ in W_{mn} if t can be obtained from s by removing an A-letter or adding a B-letter.

$W_{m0} \cong B_m$, $W_{0n} \cong B_n$ (boolean algebras)



W_{mn} is graded of rank $m + n$, with $\hat{0} = A$ and $\hat{1} = B$.

Let P be a graded poset of rank n with $\hat{0}$ and $\hat{1}$.

Definition (R. Ehrenborg). $\boldsymbol{F_P} =$

$$\sum_{\hat{0}=t_0\leq t_1\leq\cdots\leq t_{k-1}<t_k=\hat{1}} x_1^{\mathrm{rk}(t_0,t_1)} x_2^{\mathrm{rk}(t_1,t_2)} \cdots.$$

In general, F_P is a "quasisymmetric function." Define P to be **locally rank-symmetric** if every interval $[s,t]$ is rank-symmetric, i.e.,

$$\#\{u \in [s,t] \,:\, \mathrm{rk}(s,u) = i\}$$
$$= \#\{v \in [s,t] \,:\, \mathrm{rk}(v,t) = i\}.$$

Proposition. *P locally rank-symmetric $\Rightarrow F_P$ is a symmetric function. In this case*

$$F_P = \sum_{\lambda \vdash n} \alpha_P(\lambda) m_\lambda,$$

where for $\lambda = (\lambda_1, \ldots, \lambda_\ell)$,

$$\alpha_P(\lambda) = \#\{\hat{0} < t_1 < \cdots < t_{\ell-1} < \hat{1} :$$
$$\mathrm{rk}(t_i) = \lambda_1 + \cdots + \lambda_i.$$

Observation (R. Simion). W_{mn} is locally rank-symmetric (but not in general locally self-dual).

Theorem.

$$F_{W_{mn}} = \sum_{k\geq 0} \binom{m}{k}\binom{n}{k} e_2^k\, e_1^{m+n-2k}.$$

Corollary (C. Greene) (a)

$$\#W_{mn} = \sum_{k\geq 0} \binom{m}{k}\binom{n}{k} 2^{m+n-2k}$$

(b) #max. chains in W_{mn}

$$= \sum_{k\geq 0} \binom{m}{k}\binom{n}{k} \frac{(m+n)!}{2^k}$$

(c) $\mu_{W_{mn}}(\hat{0},\hat{1}) = (-1)^{m+n}\binom{m+n}{m}$

Two further results:

- A CL-labeling of W_{mn} which induces a "local" action of $\mathfrak{S}_{m+n}$ on the maximal chains of W_{mn} with (Frobenius) characteristic $\omega F_{W_{mn}}$.
- An algebra of multiplicative functions on W_{mn} (as $m, n \to \infty$) isomorphic to $\{F \in \mathbb{C}[[x, y]] : f(0,0) = 1\}$, under the operation $F * G$ defined by
$$\frac{1}{F*G} = \frac{1}{\tilde{F}G_0} + \frac{1}{F_0\tilde{G}} - \frac{1}{F_0G_0},$$
where
$$\boldsymbol{F_0} = F(x,0), \quad \boldsymbol{G_0} = G(0,y)$$
$$\tilde{\boldsymbol{F}}(x,y) = F(x, yG_0)$$
$$\tilde{\boldsymbol{G}}(x,y) = G(xF_0, y).$$