

LECTURE NOTES FOR *REPRÉSENTATIONS DE $GL_2(\mathbb{Q}_p)$ ET (ϕ, Γ) -MODULES*

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1. ORDERED NOTES

- L (0.1, pg. 2) - a finite extension of \mathbb{Q}_p .
- G (2.1, pg. 11) - $\mathbf{GL}_2(F)$.
- Z (2.1, pg. 11) - The center of $\mathbf{GL}_2(F)$, ie $\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in F^* \right\}$.
- B (2.1, pg. 11) - The standard Borel of $\mathbf{GL}_2(F)$, ie $\begin{pmatrix} F^* & F \\ 0 & F^* \end{pmatrix}$.
- U^+ (2.1, pg. 11) - The subgroup of G consisting of upper triangular unipotent matrices, ie $\begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix}$.
- A (2.1, pg. 11) - The subgroup of diagonal matrices in G .
- Δ (2.1, pg. 11) - The dihedral group generated by A and w .
- K (2.1, pg. 11) - $\mathbf{GL}_2(\mathcal{O}_F)$.
- K_n (2.1, pg. 11) - The subgroup of K consisting of matrices congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ modulo π^n , where $n \in \mathbb{Z}$.
- I_n (2.1, pg. 11) - The subgroup of K consisting of lower triangular matrices modulo π^n , where $n \geq 1$.
- $D(a, n)$ (2.2, pg. 12) - An elementary open subset of $\mathbf{P}^1(F)$, given by $a + \pi^n \mathcal{O}_F$ for $a \in F$ and $n \in \mathbb{Z}$.
- $D(\infty, n)$ (2.2, pg. 12) - An elementary open subset of $\mathbf{P}^1(F)$, defined as the complement of $D(0, 1 - n)$. The image of $D(0, n)$ under w .
- elementary open in $\mathbf{P}^1(F)$ (2.2, pg. 12) - $D(a, n)$ or its complement for some $a \in F$ and $n \in \mathbb{Z}$.
- \mathcal{S} (2.2, pg. 12) - homothety classes of lattices in $Fe_1 \oplus Fe_2$, where a homothety is the action of a scalar matrix. Isomorphic to G/KZ .
- $d(s, s')$ (2.2, pg. 12) - If $s, s' \in \mathcal{S}$ and Λ is a representative for s , there is a unique Λ' representing s' with $\Lambda' \subset \Lambda$ and Λ/Λ' a cyclic \mathcal{O}_F -module, ie isomorphic to $\mathcal{O}_F/\pi^n \mathcal{O}_F$ for some n . Define $d(s, s') = n$.
- σ_n (2.2, pg. 12) - the homothety class of the lattice $(\pi^n e_1, e_2)$.
- \mathcal{T} (2.2, pg. 12) - The tree (building) of $\mathbf{PGL}_2(F)$. The vertices of \mathcal{T} are the homothety classes of lattices in $Fe_1 \oplus Fe_2$. The oriented edges are pairs $[s, s']$ with $d(s, s') = 1$.
- $\ell(I)$ (2.2, pg. 12) - For an oriented segment $I = [s, s']$, define the length of I to be $\ell(I) = d(s, s')$. The action of G on the oriented segments preserves the length, is transitive on segments of a given length, and the stabilizer of $[(e_1, e_2), (\pi^n e_1, e_2)]$ is I_n , and the G -set of oriented segments of length n is isomorphic to $G/I_n Z$.

- s_x (2.2, pg. 13) - Given $s \in \mathcal{T}$, there is a unique lattice Λ_s in the class of s such that the projection of Λ_s onto Fe_2 parallel to Fe_1 is \mathcal{O}_{Fe_2} . $\Lambda_s \cap Fe_1$ will be of the form $\pi^n \mathcal{O}_{Fe_1}$ and there will be a $b \in F$, uniquely defined up to $\pi^n \mathcal{O}_F$, such that Λ_s has \mathcal{O}_F -basis $\{\pi^n e_1, e_2 + be_1\}$. Fix a choice of b . For $x \in k_F$, define s_x to be the class of the lattice $(\pi^{n+1}e_1, e_2 + (b + \pi^n \hat{x})e_1)$ where $\hat{x} \in \mathcal{O}_F$ lifts x . Define s_∞ to be the class of the lattice $(\pi^{n-1}e_1, e_2 + be_1)$. The edges emanating from s are the set $\{[s, s_x] : x \in \mathbf{P}^1(k_F)\}$.
- $D_{[s, s']}$ (2.2, pg. 13) - See s_x for the definition of b . If $s' = s_\infty$ then define $D_{[s, s']}$ to be the elementary open of $\mathbf{P}^1(F)$ given by the complement of $b + \pi^n \mathcal{O}_F$. If $s' = s_x$ for $x \in k_F$ then set $D_{[s, s']} = b + \pi^n \hat{x} + \pi^{n+1} \mathcal{O}_F$. Note that there is a typo (p in place of b in Colmez's definition).
- $\mathcal{T}_{[s_0, s_1]}$ (2.2, pg.13) - The subtree issuing from $[s_0, s_1]$, the vertices of which are the vertices $s \in \mathcal{T}$ with $s_1 \in [s_0, s]$. Note that $s_0 \notin \mathcal{T}_{[s_0, s_1]}$ but $s_1 \in \mathcal{T}_{[s_0, s_1]}$.
- extremity (2.2, pg. 13) - If \mathcal{T}' is a subtree of \mathcal{T} , we say an edge $[s_0, s_1]$ of \mathcal{T}' is an extremity if $\mathcal{T}' - \{s_0\} \subset \mathcal{T}_{[s_0, s_1]}$.
- \mathcal{T}_U (2.2, pg. 13) - If U is an elementary open of $\mathbf{P}^1(F)$ then it corresponds to an edge $[s_0, s_1]$. Set $\mathcal{T}_U = \mathcal{T}_{[s_0, s_1]}$.
- ray from s (2.2, pg. 13) - If $s \in \mathcal{T}$, define a ray from s to be a nested union of oriented segments J_n with $\ell(J_n) \rightarrow \infty$ as $n \rightarrow \infty$.
- central character (2.3, pg. 13) - If Π is a Λ -representation then we say that $\omega: Z \rightarrow \Lambda^*$ is a central character of Π if every $g \in Z$ acts by multiplication by $\omega(g)$.
- locally constant (representation) (2.3, pg. 13) - A representation Π of G is locally constant (or lisse) if the stabilizer of each element $v \in \Pi$ is open in G .
- admissible (representation) (2.3, pg. 13) - A Λ -representation Π of G is admissible if Π^{K_n} is of finite type over Λ for each $n \in \mathbb{N}$.
- $\text{Rep}_{\mathcal{O}_L} G$ (2.3, pg. 13) - The category of locally constant, admissible, finite length \mathcal{O}_L -representations of G admitting a central character.
- Π (2.3, pg. 13) - For a ring Λ , a Λ -representation Π of G is a Λ -module equipped with a left, Λ -linear action of G . We often implicitly set $\Lambda = \mathcal{O}_L$.
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Lemma (2.4). *If $\Pi \in \text{Rep}_{\mathcal{O}_L} G$ then there exists $W \subset \Pi$ of finite type over \mathcal{O}_L that is stable under KZ and generates Π as a G -module.*

- $\mathcal{W}(\Pi)$ (2.4, pg. 14) - For $\Pi \in \text{Rep}_{\mathcal{O}_L} G$, denote by $\mathcal{W}(\Pi)$ the set of finite type sub- \mathcal{O}_L -modules of Π that are stable under KZ and generate Π as a G -module.
- $I(W)$ (2.4, pg. 14) - For $W \in \mathcal{W}(\Pi)$, set $I(W) = \text{Ind}_{KZ}^G W$.
- $R(W, \Pi)$ (2.4, pg. 14) - The kernel of the morphism of G -modules from $I(W)$ to W defined by $\phi \mapsto \sum_{g \in G/KZ} g \cdot \phi(g^{-1})$.
- $[g, v]$ (2.4, pg. 14) - If $W \in \mathcal{W}(\Pi)$, $v \in W$ and $g \in G$ let $[g, v]$ be the element of $I(W)$ defined by

$$[g, v](h) = \begin{cases} hg \cdot v & \text{if } hg \in KZ, \\ 0 & \text{if } hg \notin KZ. \end{cases}$$

- $[g, W]$ (2.4, pg. 14) - If $W \in \mathcal{W}(\Pi)$ and $g \in G$, set $[g, W] = \{[g, v] : v \in W\}$. This is a submodule of $I(W)$ depending only on the class of g in $G/KZ \cong \mathcal{T}$. It's image under the map to Π is the translate $g \cdot W$.

- $[s, W]$ (2.4, pg. 14) - Since $[g, W]$ depends only on the class of g in $G/KZ \cong \mathcal{T}$ we can define $[s, W]$ in the natural way.
- supported on \mathcal{T}' (2.4, pg. 14) - Since $I(W) = \bigoplus_{s \in \mathcal{T}} [s, W]$, so we can write $x \in I(W)$ as $x = \sum_{s \in \mathcal{T}} x_s$ with $x_s \in [s, W]$. If \mathcal{T}' is a subtree of \mathcal{T} , then we say that x is supported on \mathcal{T}' if $x_s = 0$ for $s \notin \mathcal{T}'$.
- support of x (2.4, pg. 15) - The smallest subtree \mathcal{T}' of \mathcal{T} such that x is supported on \mathcal{T}' .
- $W^{[n]}$ (2.4, pg. 15) - If $W \subset \Pi$ is stable under K and $n \in \mathbb{N}$, set $W^{[n]}$ to be the image in Π of the submodule $\sum_{d(s, \sigma_0) \leq n} [s, W] \subset I(W)$.
- standard presentation (2.4, pg. 15) - We say that $I(W)/R(W, \Pi)$ is a standard presentation of Π if $R(W, \Pi)$ is generated as an $\mathcal{O}_L[G]$ -module by $W \cap \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \cdot W$. Equivalently, $R(W, \Pi)$ is generated by $R^{(0)}(W, \Pi)$.
- $R^{(0)}(W, \Pi)$ (2.4, pg. 15) - For $W \in \mathcal{W}(\Pi)$, define

$$R^{(0)}(W, \Pi) = \{ [\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x] - [\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}, y] : y \in W \cap \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot W, x = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \cdot y \}$$
- $R^{(n)}(W, \Pi)$ (2.4, pg. 15) - The kernel of the natural map $\bigoplus_{d(s, \sigma_0) \leq n} [s, W] \rightarrow W^{[n]}$.
- complexity $\leq n$ (2.4, pg. 15) - If $n \in \mathbb{N}$ we say that Π is of complexity $\leq n$ if there is a $W \in \mathcal{W}(\Pi)$ such that $R^{(n)}(W, \Pi)$ generates the $\mathcal{O}_L[G]$ -module $R(W, \Pi)$.
- $\mathcal{W}^{(n)}(\Pi)$ (2.4, pg. 15) - For $n \in \mathbb{N}$, denote by $\mathcal{W}^{(n)}(\Pi)$ the set of $W \in \mathcal{W}(\Pi)$ such that $R^{(n)}(W, \Pi)$ generates the $\mathcal{O}_L[G]$ -module $R(W, \Pi)$.
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Proposition (2.7). *Given the following data:*

- A finite type \mathcal{O}_L -module W with action of KZ ,
- A sub- \mathcal{O}_L -module W' of W stable under $I^+(1)$ and $\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$,
- An isomorphism $\iota: W' \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot W'$ such that $\iota(g \cdot x) = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \iota(x)$ for all $x \in W'$ and $g \in I^+(1)$, and such that $\iota(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \iota(v)) = \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix} \cdot v$ for all $v \in W'$;

and making the following definitions:

- $R(W, W', \iota)$ as the sub- $\mathcal{O}_L[G]$ -module of $I(W)$ generated by the $[\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}, v] - [\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \iota(v)]$ for $v \in W'$,
- $\Pi = I(W)/R(W, W', \iota)$,
- \overline{W} and \overline{W}' the images of W and W' in Π ,

then $I(\overline{W})/R(\overline{W}, \Pi)$ is a standard presentation of Π and $\overline{W}' = \overline{W} \cap \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \overline{W}$.

- ι (2.5, pg. 15) - Define $\iota: W \rightarrow \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \cdot W$ by $\iota(v) = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \cdot v$.
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Lemma (2.6). *Let $\Pi \in \text{Rep}_{\mathcal{O}_L} G$, suppose $W \in \mathcal{W}(\Pi)$, and set $W' = W \cap \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot W$. Then*

- (i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot W' = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \cdot W'$; in particular, $\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \cdot W'$ is contained in W .
- (ii) W' is stabilized by $I^+(1)$ and $\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$.
- H (2.5, pg. 16) - The subgroup of G generated by Z , $I^-(1)$ and the matrix $\begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix}$.
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Lemma (2.8). *If a system of representatives for G/H is fixed, then every element R of $R(W, W', \iota)$ can be expressed uniquely as*

$$R = \sum_{g \in G/H} g \cdot \left(\left[\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}, v_g \right], \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \iota(v_g) \right] \right).$$

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Lemma (2.11). *If $W \in \mathcal{W}^{(0)}(\Pi)$, then $W^{[1]} \in \mathcal{W}^{(0)}(\Pi)$.*

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Proposition (2.13). *Let $0 \rightarrow \Pi_1 \rightarrow \Pi \rightarrow \Pi_2 \rightarrow 0$ be an exact sequence of objects in $\text{Rep}_{\mathcal{O}_L} G$.*

(i) *If Π admits a standard presentation, then so do Π_1 and Π_2 .*

(ii) *If Π_1 and Π_2 admit standard presentations, then so does Π .*

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Lemma (2.16). *If $W \in \mathcal{W}^{(1)}(\Pi)$ then $W^{[1]} \in \mathcal{W}^{(1)}(\Pi)$.*

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Proposition (2.18). *Let $0 \rightarrow \Pi_1 \rightarrow \Pi \rightarrow \Pi_2 \rightarrow 0$ be an exact sequence of objects in $\text{Rep}_{\mathcal{O}_L} G$. If Π_1 and Π_2 are of complexity ≤ 1 , then so is Π . More precisely, if $W_2 \in \mathcal{W}^{(1)}(\Pi_2)$, and if W_1 is a finite type sub- \mathcal{O}_L -module of Π_1 then there exists $W \in \mathcal{W}^{(1)}(\Pi)$, containing W_1 , with image W_2 in Π_2 .*

- Π^\vee (2.7, pg. 21) - If Π is an \mathcal{O}_L -representation of G , define the dual of Π by $\Pi^\vee = \text{Hom}(\Pi, L/\mathcal{O}_L)$. It is given the structure of a G module by $(g \cdot \mu)(v) = \mu(g^{-1} \cdot v)$. Π^\vee is given the weak convergence topology, making it a compact \mathcal{O}_L -module.
- $\langle \mu, v \rangle$ (2.7, pg. 21) - If $\mu \in \Pi^\vee$ and $v \in \Pi$ then $\langle \mu, v \rangle$ is the result of applying μ to v .
- $\Gamma(\mathcal{S}', \mathcal{F}(W, \Pi))$ (2.7, pg. 21) - If Π is of complexity ≤ 1 and $W \in \mathcal{W}^{(1)}(\Pi)$ then define $\Gamma(\mathcal{S}', \mathcal{F}(W, \Pi))$ to be the set of $\mu \in \prod_{s \in \mathcal{S}'} [s, W]^\vee$ such that $\langle \mu, x \rangle = 0$ for all $x \in G \cdot R^{(1)}(W, \Pi)$ supported on \mathcal{S}' .
- zero on U (2.7, pg. 21) - We say that $\mu \in \Gamma(\mathcal{S}, \mathcal{F}(W, \Pi))$ is zero on U if the restriction to $[s, W]$ is identically zero for all $s \in \mathcal{S}_U$. Equivalently, if $g_U \in G$ sends \mathcal{O}_F to U then we require $\langle \mu, g_U h \cdot v \rangle = 0$.
- $B(s, N)$ (2.7, pg. 22) - For $s \in \mathcal{S}$ and $N \in \mathbb{N}$ define $B(s, N)$ to be the set of vertices of \mathcal{S} of distance at most N from s . Colmez doesn't explicitly define this notation on page 22.
- compact support (2.7, pg. 22) - We say that $\mu \in \Gamma(\mathcal{S}, \mathcal{F}(W, \Pi))$ is compactly supported in F if there exists $a \in \mathbb{N}$ such that μ is zero on $D(\infty, a)$, and compactly supported on F^* if it's zero on $D(0, a)$ and $D(\infty, a)$. It turns out that this notion is independent of W : $\mu \in \Pi^\vee$ is compactly supported in F if there is a $W \in \mathcal{W}^{(1)}(\Pi)$ such that μ , as an element of $\Gamma(\mathcal{S}, \mathcal{F}(W, \Pi))$, is compactly supported in F , and similarly for F^* .
- Π_c^\vee (2.7, pg. 22) - The set of elements of Π^\vee compactly supported in F^* . See compact support.
- χ (3.1, pg. 25) - a character $\mathbb{Q}_p^* \rightarrow k_L^*$.
- $W_{r, \chi}$ (3.1, pg. 25) - the KZ -module $(\text{Sym}^r k_L^2) \otimes \chi \circ \det$, where the action of K factors through $\mathbf{GL}_2(\mathbb{F}_p)$.

- P (3.1, pg. 25) - used to represent an element of $\text{Sym}^r k_L^2$ thought of as a polynomial in X of degree $\leq r$.
- $P(\infty)$ (3.1, pg. 26) - if $P \in W_{r,\chi}$ then $P(\infty)$ is the coefficient of X^r .
- T_p (3.1, pg. 26) - If $0 \leq r \leq p-1$, Barthel and Livné construct $T_p: I(W_{r,\chi}) \rightarrow I(W_{r,\chi})$, commuting with the action of G and such that, if $g \in G$ and $P \in W_{r,\chi}$,

$$T_p([g, P]) = \sum_{i=0}^{p-1} P(-i)[g\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right), 1] + P(\infty)[g\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), X^r].$$

- $\Pi(r, \lambda, \chi)$ (3.1, pg. 26) - For $\lambda \in k_L$, $0 \leq r \leq p-1$ and $\chi: \mathbb{Q}_p^* \rightarrow k_L^*$ define $\Pi(r, \lambda, \chi) = I(W_{r,\chi}) / (T_p - \lambda) \cdot (I(W_{r,\chi}))$.
- $\text{LC}_c(\mathbb{Q}_p, k_L)$ (3.2, pg. 26) - The k_L vectors space of locally constant functions with compact support in \mathbb{Q}_p and values in k_L .
- $\text{LC}_c(\delta_1 \otimes \delta_2)$ (3.2, pg. 26) - $\text{LC}_c(\mathbb{Q}_p, k_L)$ equipped with left and (and corresponding right) actions of B ,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot_{\delta_1 \otimes \delta_2} \phi(x) = \delta_1(a)\delta_2(d)\phi\left(\frac{dx-b}{a}\right),$$

$$\phi \star_{\delta_1 \otimes \delta_2} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(x) = \delta_1^{-1}(a)\delta_2^{-1}(d)\phi\left(\frac{ax+b}{d}\right).$$

- ϕ_i (3.2, pg. 27) - For $i \in \mathbb{Z}_p/p\mathbb{Z}_p$, set $\phi_i = \mathbf{1}_{i+p\mathbb{Z}_p} \in \text{LC}_c(\mathbb{Q}_p, k_L)$.
- $Y(\delta_1, \delta_2)$ (3.2, pg. 27) - The k_L vector space $\bigoplus_{i \in \mathbb{Z}_p/p\mathbb{Z}_p} k_L \cdot \phi_i$ with the action of $ZB(\mathbb{Z}_p)$ obtained by restriction from $\text{LC}_c(\delta_1 \otimes \delta_2)$.
- $R_{\delta_1, \delta_2, 0}$ (3.2, pg. 27 and 3.3, pg. 28) - Define

$$R_{\delta_1, \delta_2, 0} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \phi_0\right] - \sum_{i \in \mathbb{Z}_p/p\mathbb{Z}_p} \left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \delta_1(p)^{-1}\phi_i\right].$$

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Proposition (3.5). *The $k_L[B]$ -module $\text{LC}_c(\delta_1 \otimes \delta_2)$ is the quotient of $\text{Ind}_{ZB(\mathbb{Z}_p)}^B Y(\delta_1, \delta_2)$ by the sub- $k_L[B]$ -module generated by $R_{\delta_1, \delta_2, 0}$.*

Proof. Let J be a system of representatives for $\mathbb{Q}_p/\mathbb{Z}_p$. Then

- The matrices $\begin{pmatrix} p^n & p^n c \\ 0 & 1 \end{pmatrix}$ for $n \in \mathbb{Z}$ and $c \in J$ form a family of representatives for G/KZ .
- $\begin{pmatrix} p^n & p^n c \\ 0 & 1 \end{pmatrix} \cdot \phi_i = \delta_1(p)^n \mathbf{1}_{p^n(i+c)+p^{n+1}\mathbb{Z}_p}$.
- The $p^n(i+c)$, where $c \in J$ and $i \in \{0, 1, \dots, p-1\}$ form a system of representatives for $\mathbb{Q}_p/p^{n+1}\mathbb{Z}_p$.
- Considered as a k_L -vector space, we have

$$\text{LC}_c(\mathbb{Q}_p, k_L) = \left(\bigoplus_{n \in \mathbb{Z}} \bigoplus_{b \in \mathbb{Q}_p/p^{n+1}\mathbb{Z}_p} k_L \cdot \mathbf{1}_{b+p^{n+1}\mathbb{Z}_p} \right) / \left(\bigoplus_{n \in \mathbb{Z}} \bigoplus_{b \in \mathbb{Q}_p/p^n\mathbb{Z}_p} k_L \cdot (\mathbf{1}_{b+p^n\mathbb{Z}_p} - \sum_{i=0}^{p-1} \mathbf{1}_{b+p^n i+p^{n+1}\mathbb{Z}_p}) \right).$$

$$- \mathbf{1}_{b+p^n\mathbb{Z}_p} - \sum_{i=0}^{p-1} \mathbf{1}_{b+p^n i+p^{n+1}\mathbb{Z}_p} = \delta_1(p)^{1-n} \begin{pmatrix} p^{n-1} & b \\ 0 & 1 \end{pmatrix} \cdot R_{\delta_1, \delta_2, 0}.$$

□

- ω (3.3, pg. 27) - Define $\omega: \mathbb{Q}_p^* \rightarrow \mathbb{F}_p^*$ to be the reduction modulo p of the character $x \mapsto |x|$.

- $B(\delta_1, \delta_2)$ (3.3, pg. 27) - For $\delta_1, \delta_2 \in \widehat{\mathcal{F}}(k_L)$, define $B(\delta_1, \delta_2)$ to be the vector space of functions with values in k_L , locally constant on \mathbb{Q}_p , such that $x \mapsto (\omega^{-1}\delta_1\delta_2^{-1})(x) \cdot \phi(1/x)$ extends to 0 and defines a locally constant function on \mathbb{Q}_p . Equal to $\text{LC}_c(\mathbb{Q}_p, k_L) \oplus k_L \cdot \phi_\infty$. Define a right action of G by

$$(\phi \star_{\delta_1, \delta_2} \begin{pmatrix} a & b \\ c & d \end{pmatrix})(x) = (\omega\delta_1^{-1})(ad - bc)(\omega^{-1}\delta_1\delta_2^{-1})(cx + d)\phi\left(\frac{ax + b}{cx + d}\right),$$

and a right action by $g \cdot_{\delta_1, \delta_2} \phi = \phi \star_{\delta_1, \delta_2} g^{-1}$.

- ϕ_∞ (3.3, pg. 27) - The function on \mathbb{Q}_p defined by:

$$\phi_\infty(x) = \begin{cases} (\omega^{-1}\delta_1\delta_2^{-1})(x) & \text{if } x \notin \mathbb{Z}_p, \\ 0 & \text{if } x \in \mathbb{Z}_p. \end{cases}$$

- ϕ_v (3.3, pg. 27) - For $v \in \text{Ind}_B^G \delta_1 \otimes \delta_2$, define $\phi_v: \mathbb{Q}_p \rightarrow k_L$ by $\phi_v(x) = v\left(\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}\right)$.
- $W(\delta_1, \delta_2)$ (3.3, pg. 28) - The k_L -subspace of $B(\delta_1, \delta_2)$ generated by ϕ_∞ and the ϕ_i for $i \in \mathbb{Z}_p/p\mathbb{Z}_p$.
- $R_{\delta_1, \delta_2, \infty}$ (3.3, pg. 28) Define

$$R_{\delta_1, \delta_2, \infty} = \left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \delta_1(p)^{-1}\phi_\infty \right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \phi_\infty \right] - \sum_{i \in (\mathbb{Z}_p/p\mathbb{Z}_p)^*} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (\omega^{-1}\delta_1\delta_2^{-1})(i)\phi_i \right].$$

- $\text{LC}_c(\mathbb{Q}_p^*, k_L)$ (3.3, pg. 29) - the vector space of locally constant functions on \mathbb{Q}_p^* with values in k_L and compactly supported in \mathbb{Q}_p^* .
- $\text{LC}(\mathbf{P}^1(\mathbb{Q}_p), k_L)$ (3.4, pg. 29) - $B(\omega, 1)$, the vector space of locally constant functions on $\mathbf{P}^1(\mathbb{Q}_p)$ with a left action of G defined by $g \cdot \phi = \phi \star g^{-1}$ and $\phi \star \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \phi\left(\frac{ax+b}{cx+d}\right)$.
- St (3.4, pg. 29) - The quotient of $\text{LC}(\mathbf{P}^1(\mathbb{Q}_p), k_L)$ by the subspace of constant functions.
- $W_0(\omega, 1)$ (3.4, pg. 29) - Defined by $W(\omega, 1)/k_L \cdot \mathbf{1}_{\mathbf{P}^1(\mathbb{Q}_p)}$.
- R_0 (3.5, pg. 30) - Define R_0 as a submodule of $I(W_{r, \chi}) \oplus I(W_{p-1-r, \chi\omega^r})$ by
$$R_0 = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (0, Y^{p-1-r}) \right] - \left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, (1, 0) \right].$$
- R_1 (3.5, pg. 30) - Define R_1 as a submodule of $I(W_{r, \chi}) \oplus I(W_{p-1-r, \chi\omega^r})$ by
$$R_0 = \left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, (0, 1) \right] - (-1)^r \chi(p)^2 \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (X^r, 0) \right].$$
- P_r (3.5, pg. 30) - The polynomial of degree r given by $\frac{(-X+1)\cdots(-X+r)}{r!}$.
- $f(r, \chi)$ (3.5, pg. 30) - The vector $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot 1 \in \Pi(r, 0, \chi)$.
- W (3.5, pg. 30) - $W_{r, \chi} \oplus W_{p-1-r, \chi\omega^r}$. Represent an element as (P, Q) where P is a polynomial in X of degree $\leq r$ and Q a polynomial in Y of degree $\leq p-1-r$.