LECTURE NOTES FOR *REPRÉSENTATIONS DE* $GL_2(\mathbb{Q}_P)$ *ET* (ϕ, Γ)– *MODULES*

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1. Outline

In this section I outline Colmez's arguments without any of the proofs.

2. Representations of $\mathbf{GL}_2(F)$. In this section Colmez defines a bunch of subgroups of G. See the Definitions section.

2.1. $\mathbf{GL}_2(F)$and its subgroups.

Proposition (2.1). (i) *The subgroup of* G *generated by* $\left(\begin{smallmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 1 & 0 \\ \mathscr{O}_F & 1 \end{smallmatrix}\right)$ *is* $\mathbf{SL}_2\mathscr{O}_F$.
(ii) *The subgroup of* G *generated by* $\left(\begin{smallmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 1 & 0 \\ \pi^{-1}\mathscr{O}_F & 1 \end{smallmatrix}\right)$ *is* $\mathbf{SL}_2(F)$.

2.2. The tree of $\mathbf{PGL}_2(F)$. Colmez defines the "tree" of $\mathbf{PGL}_2(F)$, whose vertices are homothety classes of lattices in $Fe_1 \oplus Fe_2$, and with oriented edges between lattices of index index p in some scaling of the other. See $\mathscr{T}, D(a,n), \mathscr{I}, d(s,s'), \sigma_n, \ell(I), \mathscr{T}_{]s_0,s_1)}$, and s_x in the Definitions section for more details.

2.3. Representations of G.

Lemma (2.2). *If* M *is an* $\mathscr{O}_L$*-module of finite length with a continuous action of* U^+ *then* U^+ *acts trivially on* M.

Lemma (2.3). *Let* $\Pi \in \mathrm{Rep}_{\mathscr{O}_L} G$. *If* $M \subset \Pi$ *is a sub-*$\mathscr{O}_L$*-module of finite length stable under* Δ *then* M *is stable under* G *and fixed by* $\mathbf{SL}_2(F)$.

2.4. The presentation of a representation of G.

Lemma (2.4). *If* $\Pi \in \mathrm{Rep}_{\mathscr{O}_L} G$ *then there exists* $W \subset \Pi$ *of finite type over* $\mathscr{O}_L$*that is stable under* KZ *and generates* Π *as a* G*-module.*

2.5. Representations admitting a standard presentation.

Lemma (2.6). *Let* $\Pi \in \mathrm{Rep}_{\mathscr{O}_L} G$, *suppose* $W \in \mathscr{W}(\Pi)$, *and set* $W' = W \cap \left(\begin{smallmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot W$. *Then*
(i) $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot W' = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot W'$; *in particular,* $\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot W'$ *is contained in* W.
(ii) W' *is stablized by* $I^+(1)$ *and* $\left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)$.

Proposition (2.7). *Given the following data:*

- *A finite type* $\mathscr{O}_L$*-module* W *with action of* KZ,
- *A sub-*$\mathscr{O}_L$*-module* W' *of* W *stable under* $I^+(1)$ *and* $\left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)$,
- *An isomorphism* $\iota \colon W' \to \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot W'$ *such that* $\iota(g \cdot x) = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right) g \left(\begin{smallmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \iota(x)$ *for all* $x \in W'$ *and* $g \in I^+(1)$, *and such that* $\iota(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot \iota(v)) = \left(\begin{smallmatrix} 0 & \pi \\ \pi & 0 \end{smallmatrix}\right) \cdot v$ *for all* $v \in W'$;

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and making the following definitions:

- $R(W,W',\iota)$ *as the sub-*$\mathscr{O}_L[G]$*-module of* $I(W)$ *generated by the* $[\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right), v] - [\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \iota(v)]$ *for* $v \in W'$,
- $\Pi = I(W)/R(W,W',\iota)$,
- $\overline{W}$ *and* $\overline{W}'$ *the images of* W *and* W' *in* Π,

then $I(\overline{W})/R(\overline{W},\Pi)$ *is a standard presentation of* Π *and* $\overline{W}' = \overline{W} \cap \left(\begin{smallmatrix}\pi^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)\cdot \overline{W}$.

Lemma (2.8)**.** *If a system of representatives for* G/H *is fixed, then every element* R *of* $R(W,W',\iota)$ *can be expressed uniquely as*

$$R = \sum_{g\in G/H} g \cdot \left([\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right), v_g], [\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \iota(v_g)]\right).$$

Lemma (2.10)**.** *If* $W \in \mathscr{W}(\Pi)$*, the following conditions are equivalent:*

(i) $W \in \mathscr{W}^{(0)}(\Pi)$;
(ii) *Given any subtree* $\mathscr{T}'$ *of* $\mathscr{T}$*, extremity* $[s_0,s_1]$ *of* $\mathscr{T}'$*, and* $R = \sum_{s\in\mathscr{T}'}[s,x_s] \in R(W,\Pi)$ *with support included in* $\mathscr{T}'$*, we have* $s_1 \cdot x_{s_1} \in s_0 \cdot W$.

Lemma (2.11)**.** *If* $W \in \mathscr{W}^{(0)}(\Pi)$*, then* $W^{[1]} \in \mathscr{W}^{(0)}(\Pi)$.

Corollary (2.12)**.** *If* Π *admits a standard presentation, then for all* $W' \in \mathscr{W}(\Pi)$*, there exists* $W'' \in \mathscr{W}^{(0)}(\Pi)$ *containing* W'.

Proposition (2.13)**.** *Let* $0 \to \Pi_1 \to \Pi \to \Pi_2 \to 0$ *be an exact sequence of objects in* $\mathrm{Rep}_{\mathscr{O}_L} G$.

(i) *If* Π *admits a standard presentation, then so do* Π_1 *and* Π_2.
(ii) *If* Π_1 *and* Π_2 *admit standard presentations, then so does* Π.

2.6. Representations of complexity ≤ 1.

Lemma (2.14)**.** *If* $W \in \mathscr{W}(\Pi)$*, the following conditions are equivalent.*

(i) $W \in \mathscr{W}^{(1)}(\Pi)$;
(ii) *Given any subtree* $\mathscr{T}'$ *of* $\mathscr{T}$*, extremity* $[s_0,s_1]$ *of* $\mathscr{T}'$*, and* $R \in R(W,\Pi)$ *with support included in* $\mathscr{T}'$*, there exists* $R_0 \in R^{(1)}(W,\Pi)$ *such that* $R - s_1 \cdot R_0$ *has support in* $\mathscr{T}' - \{s_0\}$.

Corollary (2.15)**.** *If* $W \in \mathscr{W}^{(1)}(\Pi)$ *and if* $R \in R(W,\Pi)$ *has support* $\mathscr{T}'$*, then there is a finite family of pairs* $\{(g_i,R_i) : i \in I\}$ *with* $g_i \in G$ *and* $R_i \in R^{(1)}(W,\Pi)$ *such that*

- *for all* $i \in I$*,* $g_i \cdot R_i$ *is supported in* $\mathscr{T}'$*, and*
- $R = \sum_{i\in I} g_i \cdot R_i$.

Lemma (2.16)**.** *If* $W \in \mathscr{W}^{(1)}(\Pi)$ *then* $W^{[1]} \in \mathscr{W}^{(1)}(\Pi)$.

Corollary (2.17)**.** *If* Π *is of complexity* ≤ 1*, then for all* $W' \in \mathscr{W}(\Pi)$*, there exists* $W'' \in \mathscr{W}^{(1)}(\Pi)$ *containing* W'.

Proposition (2.18)**.** *Let* $0 \to \Pi_1 \to \Pi \to \Pi_2 \to 0$ *be an exact sequence of objects in* $\mathrm{Rep}_{\mathscr{O}_L} G$*. If* Π_1 *and* Π_2 *are of complexity* ≤ 1*, then so is* Π*. More precisely, if* $W_2 \in \mathscr{W}^{(1)}(\Pi_2)$*, and if* W_1 *is a finite type sub-*$\mathscr{O}_L$*-module of* Π_1 *then there exists* $W \in \mathscr{W}^{(1)}(\Pi)$*, containing* W_1*, with image* W_2 *in* Π_2.

2.7. Duals.

Lemma (2.19). *If $\mathscr{T}'$ is a subtree of $\mathscr{T}$, then*

$$\left(\sum_{s\in\mathscr{T}'} s\cdot W\right)^\vee = \Gamma((,\mathscr{F}(W,\Pi))\mathscr{T}').$$

Lemma (2.20). *If $g\in G$ and if $\mu\in\Pi^\vee$ then μ is zero on $\mathscr{T}_U$ if and only if $g\cdot\mu$ is zero on $\mathscr{T}_{g\cdot U}$.*

Lemma (2.21). *Let $\mathscr{T}'$ be a subtree of $\mathscr{T}$, $\{[s_{i,0},s_{i,1}] : i\in I\}$ the extremities of $\mathscr{T}'$ and suppose the restriction of $\mu\in\left(\sum_{s\in\mathscr{T}'} s\cdot W\right)^\vee$ to $s_{i,0}\cdot W + s_{i,1}\cdot W$ is zero for all $i\in I$. Then there exists $\tilde{\mu}\in\Pi^\vee$ such that $\tilde{\mu}$ restricted to $\sum_{s\in\mathscr{T}'} s\cdot W$ is μ and $\tilde{\mu}$ is identically zero on $\mathscr{T}_{]s_{i,1},s_{i,0})}$.*

Lemma (2.22). *If $W,W'\in\mathscr{W}^{(1)}(\Pi)$ and $\mu\in\Pi^\vee$ then the following conditions are equivalent.*

(i) *There exists $a\in\mathbb{N}$ such that μ, considered as an element of $\Gamma(\mathscr{T},\mathscr{F}(W,\Pi))$, is zero on $D(0,a)$.*

(ii) *There exists $a'\in\mathbb{N}$ such that μ, considered as an element of $\Gamma(\mathscr{T},\mathscr{F}(W',\Pi))$, is zero on $D(0,a')$.*

Corollary (2.23). *Let $W,W'\in\mathscr{W}^{(1)}(\Pi)$ and $\mu\in\Pi^\vee$. Then μ is compactly supported in F (resp. F^*) as an element of $\Gamma(\mathscr{T},\mathscr{F}(W,\Pi))$ if and only if it is compactly supported as an element of $\Gamma(\mathscr{T},\mathscr{F}(W',\Pi))$.*

Proposition (2.24). *If $\Pi\in\mathrm{Rep}_{\mathscr{O}_L}G$ is of complexity ≤ 1, the following conditions are equivalent.*

(i) $\Pi^{\mathbf{SL}_2(F)}=0$,

(ii) $\Pi_c^\vee$ *is dense in* $\Pi^\vee$.

Lemma (2.25). *If $\Pi^{\mathbf{SL}_2(F)}=0$ then for all finite type sub-$\mathscr{O}_L$-modules M,M' of Π, there exists $n\in\mathbb{N}$ with*

$$M'\cap\left(\sum_{m\geq n}\left(\begin{smallmatrix}\pi^m & 0\\ 0 & 1\end{smallmatrix}\right)\cdot M\right)=0 \quad \textit{and} \quad M'\cap\left(\sum_{m\geq n}\left(\begin{smallmatrix}\pi^{-m} & 0\\ 0 & 1\end{smallmatrix}\right)\cdot M\right)=0$$

2.8. The Jacquet functor $\Pi\mapsto J(\Pi)$.

3. Representations of $\mathbf{GL}_2(\mathbb{Q}_p)$.

Theorem (3.1). *Every object of $\mathrm{Rep}_{\mathscr{O}_L}G$ admits a standard presentation.*

The irreducible objects of $\mathrm{Rep}_{\mathscr{O}_L}G$.

Theorem (3.2). (i) *The representation $\Pi(r,\lambda,\chi)$ is irreducible unless $r=0$ and $\lambda=\pm1$, in which case $\Pi(r,\lambda,\chi)$ is an extension of the infinite-dimensional irreducible representation $\mathrm{St}\otimes\chi\mu_\lambda\circ\det$ by the character $\chi\mu_\lambda\circ\det$; or $r=p-1$ and $\lambda=\pm1$, in which case $\Pi(r,\lambda,\chi)$ is an extension of $\chi\mu_\lambda\circ\det$ by $\mathrm{St}\otimes\chi\mu_\lambda\circ\det$.*

(ii) *Every irreducible object in $\mathrm{Rep}_{k_L}G$ is isomorphic to a Jordan-Hölder factor of some $\Pi(r,\lambda,\chi)$.*

Proposition (3.4). (i) *The only isomorphisms between supersingulars are*

$$\Pi(r,0,\chi)\cong\Pi(r,0,\chi\mu_{-1})\cong\Pi(p-1-r,0,\chi\omega^r)\cong\Pi(p-1-r,0,\chi\omega^r\mu_{-1})$$

(ii) *There are no isomorphisms between supersingulars and subobjects of principal series, or between Jordan-Hölder factors of $\mathrm{Ind}_B^G\delta_1\otimes\delta_2$ and $\mathrm{Ind}_B^G\delta_1'\otimes\delta_2'$ for $(\delta_1,\delta_2)\neq(\delta_1',\delta_2')$.*

3.2. Representations of the Borel.

Proposition (3.5). *The $k_L[B]$-module $\mathrm{LC}_c(\delta_1 \otimes \delta_2)$is the quotient of $\mathrm{Ind}_{ZB(\mathbb{Z}_p)}^{B} Y(\delta_1, \delta_2)$ by the sub-$k_L[B]$-module generated by $R_{\delta_1,\delta_2,0}$.*

3.3. The principal series in characteristic p.

Proposition (3.6).

(i) *$B(\delta_1, \delta_2)$is an object of $\mathrm{Rep}_{\mathscr{O}_L} G$ with central character $\omega^{-1}\delta_1\delta_2$.*
(ii) *$\mathrm{LC}_c(\mathbb{Q}_p, k_L)$is stable under the action of B, and there is an exact sequence of $k_L[B]$-modules*

$$0 \to \mathbf{LC}_c(\delta_1\omega^{-1} \otimes \delta_2) \to B(\delta_1, \delta_2) \to \delta_2 \otimes \delta_1\omega^{-1} \to 0$$

Proposition (3.7). *$W(\delta_1, \delta_2) \in \mathscr{W}(B(\delta_1, \delta_2))$ and $R(W(\delta_1, \delta_2), B(\delta_1, \delta_2))$ is generated as an $\mathscr{O}_L[G]$-module by $R_{\delta_1,\delta_2,0}$ and $R_{\delta_1,\delta_2,\infty}$.*

Corollary (3.8). *$I(W(\delta_1, \delta_2))/R(W(\delta_1, \delta_2), B(\delta_1, \delta_2))$ is a standard presentation of $B(\delta_1, \delta_2)$.*

3.4. The Steinberg.

Proposition (3.10). *The representation St admits a standard presentation and, more precisely, $W_0(\omega, 1) \in \mathscr{W}^{(0)}(\mathrm{St})$ and $R(W_0(\omega, 1), \mathrm{St})$ is generated by $[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \phi_0] - \sum_{i=0}^{p-1} [\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), \phi_i]$.*

3.5. The Supersingulars.

Proposition (3.11). *If $0 \le r \le p-1$, and if $\chi\colon \mathbb{Q}_p^* \to k_L^*$, then we have the isomorphisms of representations of G,*

$$\Pi(r, 0, \chi) \cong \frac{I(W_{r,\chi}) \oplus I(W_{p-1-r,\chi\omega^r})}{(R_0, R_1)} \cong \Pi(p-1-r, 0, \chi\omega^r).$$

2. Proofs

Proof of Proposition 3.5. Let J be a system of representatives for $\mathbb{Q}_p/\mathbb{Z}_p$. Then

- The matrices $\left(\begin{smallmatrix} p^n & p^n c \\ 0 & 1 \end{smallmatrix}\right)$ for $n \in \mathbb{Z}$ and $c \in J$ for a family of representatives for G/KZ.
- $\left(\begin{smallmatrix} p^n & p^n c \\ 0 & 1 \end{smallmatrix}\right) \cdot \phi_i = \delta_1(p)^n \mathbf{1}_{p^n(i+c)+p^{n+1}\mathbb{Z}_p}$.
- The $p^n(i+c)$, where $c \in J$ and $i \in \{0, 1, \ldots, p-1\}$ form a system of representatives for $\mathbb{Q}_p/p^{n+1}\mathbb{Z}_p$.
- Considered as a k_L-vector space, we have

$$\mathrm{LC}_c(\mathbb{Q}_p, k_L) = \left(\bigoplus_{n\in\mathbb{Z}} \bigoplus_{b\in\mathbb{Q}_p/p^{n+1}\mathbb{Z}_p} k_L \cdot \mathbf{1}_{b+p^{n+1}\mathbb{Z}_p} \right) / \left(\bigoplus_{n\in\mathbb{Z}} \bigoplus_{b\in\mathbb{Q}_p/p^n\mathbb{Z}_p} k_L \cdot (\mathbf{1}_{b+p^n\mathbb{Z}_p} - \sum_{i=0}^{p-1} \mathbf{1}_{b+p^n i+p^{n+1}\mathbb{Z}_p}) \right).$$

- $\mathbf{1}_{b+p^n\mathbb{Z}_p} - \sum_{i=0}^{p-1} \mathbf{1}_{b+p^n i+p^{n+1}\mathbb{Z}_p} = \delta_1(p)^{1-n} \left(\begin{smallmatrix} p^{n-1} & b \\ 0 & 1 \end{smallmatrix}\right) \cdot R_{\delta_1,\delta_2,0}$.

□

Proof of Proposition 3.6. The map $v \mapsto \phi_v$ defines a G-equivariant isomorphism of $\mathrm{Ind}_B^G \delta_1 \otimes \delta_2$ to $B(\delta_2\omega, \delta_1)$, and therefore $B(\delta_1, \delta_2) \cong \mathrm{Ind}_B^G \delta_2 \otimes \delta_1\omega^{-1}$. Moreover, evaluation on $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ defines a B-equivariant, surjective map $\mathrm{Ind}_B^G \delta_2 \otimes \delta_1\omega^{-1} \to \delta_2 \otimes \delta_1\omega^{-1}$. After applying the aforementioned isomorphism, the kernel of the map $B(\delta_1, \delta_2) \to \delta_2 \otimes \delta_1\omega^{-1}$ is $\mathrm{LC}_c(\delta_2 \otimes \delta_1\omega^{-1})$. □

Proof of Proposition 3.7. To simplify notation, let $\Pi = B(\delta_1,\delta_2)$, $W = W(\delta_1,\delta_2)$ and $\delta = \omega^{-1}\delta_1\delta_2^{-1}$ for the duration of this proof. Quick calculations show that

$$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\cdot\phi_i = \phi_{i+1} \text{ if } i \in \mathbb{Z}_p/p\mathbb{Z}_p,$$

$$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\cdot\phi_{infty}(x) = \begin{cases}\delta(x-1) = \delta(x)\delta(1-x^{-1}) & \text{if } x \notin \mathbb{Z}_p,\\ 0 & \text{if } x \in \mathbb{Z}_p,\end{cases}$$

$$\text{if } a \in (\mathbb{Z}_p/p\mathbb{Z}_p)^*, \text{ then } \left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\cdot\phi_i = \begin{cases}\omega^{-1}(a)\delta_1(a)\phi_{ai} & \text{if } i \in \mathbb{Z}_p/p\mathbb{Z}_p\\ \delta_2(a)\phi_\infty & \text{if } i = \infty\end{cases}$$

$$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\cdot\phi_i = \delta_1(-1)\delta(i)^{-1}\phi_{i^{-1}} \text{ if } i \in (\mathbb{Z}_p/p\mathbb{Z}_p)^*, \quad \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\cdot\phi_0 = \phi_\infty, \quad \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\cdot\phi_\infty = \phi_0$$

Therefore W is stable under $ZG(\mathbb{Z}_p)$ and thus $W \in \mathscr{W}(\Pi)$.

We now have $\delta_1(p)^{-1}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\cdot\phi_i = \mathbf{1}_{pi+p^2\mathbb{Z}_p}$, and

$$\delta_1(p)^{-1}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\cdot\phi_\infty(x) = \phi_\infty(x/p) = \begin{cases}\delta(x) & \text{if } x \notin p\mathbb{Z}_p,\\ 0 & \text{if } x \in \mathbb{Z}_p,\end{cases} = \phi_\infty(x) + \sum_{i\in(\mathbb{Z}_p/p\mathbb{Z}_p)^*}\delta(i)\phi_i(x).$$

One deduces that $R_0 = R_{\delta_1,\delta_2,0}$ and $R_\infty = R_{\delta_1,\delta_2,\infty}$ are contained in $R(W,\Pi)$.

If one considers Π modulo $\mathrm{LC}_c(\delta_2\otimes\delta_1\omega^{-1})$, then it is generated as a B-module by $\overline{\phi_\infty}$, and R_∞ is generated by $[\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right), \delta_1(p)^{-1}\overline{\phi_\infty}] - [\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \overline{\phi_\infty}]$. Let $R'(W,\Pi)$ be the sub-$k_L[B]$-module of $R(W,\Pi)$ generated by R_0 and R_∞. The quotient of $I(W)$ by $R'(W,\Pi)$, by Proposition 3.5, fits into the exact sequence of $k_L[B]$-modules

$$0 \to \mathrm{LC}_c(\delta_2\otimes\delta_1\omega^{-1}) \to I(W)/R'(W,\Pi) \to k_L\cdot\overline{\phi_\infty} \to 0.$$

Since $\Pi = I(W)/R(W,\Pi)$ is a quotient of $I(W)/R'(W,\Pi)$ which, by virtue of Proposition 3.6, embeds in an exact sequence of $k_L[B]$-modules, we deduce that the natural map of $I(W)/R'(W,\Pi)$ to Π is an isomorphism, and thus $R(W,\Pi) = R'(W,\Pi)$. □

Proof of Proposition 3.10. We have that $W_0(\omega,1) = \oplus_{i\in\mathbb{Z}_p/p\mathbb{Z}_p} k_L\cdot\phi_o$, where the action of Z is trivial and

$$\begin{aligned}
&\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\cdot\phi_i = \phi_{i+1} \text{ if } i \in \mathbb{Z}_p/p\mathbb{Z}_p,\\
&\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\cdot\phi_i = \phi_{ai} \text{ if } a \in \mathbb{Z}_p^* \text{ and } i \in \mathbb{Z}_p/p\mathbb{Z}_p,\\
&\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\cdot\phi_i = \phi_{i^{-1}} \text{ if } i \in (\mathbb{Z}_p/p\mathbb{Z}_p)^*,\\
&\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\cdot\phi_0 = -\sum_{i\in\mathbb{Z}_p/p\mathbb{Z}_p}\phi_i.
\end{aligned}$$

This gives the desired result. □

Proof of Proposition 3.11.

Lemma (3.12)**.** *In* $\mathbb{F}_p$*we have* $P_r(\infty) = \frac{(-1)^r}{r!}$ *and*

$$P_r(-i) = \begin{cases}(-1)^i\binom{p-1-r}{i} & \text{if } 0 \le i \le p-1-r,\\ 0 & \text{if } p-r \le i \le p-1.\end{cases}$$

Proof. Both sides are clearly zero if $p-r \le i \le p-1$. Moreover, if $0 \le i \le p-1-r$ then modulo p one has

$$\binom{p-1-r}{i} = \frac{(p-1-r)\cdots(p-i-r)}{i!} = (-1)^i\frac{(r+1)\cdots(r+i)}{i!} = (-1)^i\frac{(r+i)!}{r!\cdot i!} = (-1)^i P_r(-i).$$

□

Lemma (3.13). *The sub-KZ-module of $\Pi(r,0,\chi)$ generated by $f(r,\chi)$ is isomorphic to $W_{p-1-r,\chi\omega^r}$.*

Proof. Since $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\cdot 1$ is invariant under $\left(\begin{smallmatrix}1+p\mathbb{Z}_p&\mathbb{Z}_p\\p\mathbb{Z}_p&1+p\mathbb{Z}_p\end{smallmatrix}\right)$, the vector $f(r,\chi)$ is invariant under $\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1+p\mathbb{Z}_p&\mathbb{Z}_p\\p\mathbb{Z}_p&1+p\mathbb{Z}_p\end{smallmatrix}\right)\left(\begin{smallmatrix}p^{-1}&0\\0&1\end{smallmatrix}\right)$ $\left(\begin{smallmatrix}1+p\mathbb{Z}_p&p\mathbb{Z}_p\\p\mathbb{Z}_p&1+p\mathbb{Z}_p\end{smallmatrix}\right)$. Moreover, we have

$$\begin{aligned}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\cdot f(r,\chi) &= \left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\cdot 1 = \chi(-1)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\cdot X^r,\\ \left(\begin{smallmatrix}a&0\\0&a\end{smallmatrix}\right)\cdot f(r,\chi) &= \left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\cdot 1 = \chi(a)a^r f(r,\chi) = (\chi\omega^r(a))f(r,\chi),\\ (\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)-1)^{p-1-r}\cdot f(r,\chi) &= \sum_{i=1}^{p-1}(-1)^{p-1-r-i}\binom{p-1-r}{i}\left(\begin{smallmatrix}p&i\\0&1\end{smallmatrix}\right)\cdot 1\\ &= (-1)^{p-1-r}\sum_{i=0}^{p-1}P_r(-i)\left(\begin{smallmatrix}p&i\\0&1\end{smallmatrix}\right)\cdot 1\\ &= -(-1)^{p-1-r}P(\infty)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\cdot X^r = \frac{-1}{r!}\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\cdot X^r = (-1)^r(p-1-r)!\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\cdot X^r,\end{aligned}$$

where the last equality is by Wilson's theorem,

$$r!(p-1-r)! = (-1)^{p-1-r}(p-1)! = -(-1)^{p-1-r} = -(-1)^r.$$

□

Lemma (3.14). *The k_L vector space $R(W,\Pi(r,0,\chi))$ contains the relations R_0 and R_1.*

Lemma (3.15). *The $\mathscr{O}_L[G]$-module generated by the relations R_0, R_1 contains the sub-$\mathscr{O}_L$-module $(T_p\cdot I(W_{r,\chi}),0)\oplus(0,T_p\cdot I(W_{p-1-r,\chi\omega^r}))$ as a strict submodule.*

□

3. Definitions

I collect in this section definitions that Colmez uses, together with references to where they first appear.

- A (2.1, pg. 11) - The subgroup of diagonal matrices in G.
- A^+ (2.1, pg. 11) - The subgroup $\left(\begin{smallmatrix}F^*&0\\0&1\end{smallmatrix}\right)$ of G.
- A^- (2.1, pg. 11) - The subgroup $\left(\begin{smallmatrix}1&0\\0&F^*\end{smallmatrix}\right)$ of G.
- admissible (representation) (2.3, pg. 13) - A Λ-representation Π of G is admissible if Π^{K_n} is of finite type over Λ for each $n\in\mathbb{N}$.
- B (2.1, pg. 11) - The standard Borel of $\mathbf{GL}_2(F)$, ie $\left(\begin{smallmatrix}F^*&F\\0&F^*\end{smallmatrix}\right)$.
- $B(s,N)$ (2.7, pg. 22) - For $s\in\mathscr{T}$ and $N\in\mathbb{N}$ define $B(s,N)$ to be the set of vertices of $\mathscr{T}$ of distance at most N from s. Colmez doesn't explicitly define this notation on page 22.
- $B(\mathbb{Z}_p)$ (3.2, pg. 27) - The Borel with entries in $\mathbb{Z}_p$.
- $B(\delta_1,\delta_2)$(3.3, pg. 27) - For $\delta_1,\delta_2\in\widehat{\mathscr{T}}(k_L)$, define $B(\delta_1,\delta_2)$to be the vector space of functions with values in k_L, locally constant on $\mathbb{Q}_p$, such that $x\mapsto(\omega^{-1}\delta_1\delta_2^{-1})(x)\cdot\phi(1/x)$ extends to 0 and defines a locally constant function on $\mathbb{Q}_p$. Equal to $\mathrm{LC}_c(\mathbb{Q}_p,k_L)\oplus k_L\cdot\phi_\infty$. Define a right action of G by

$$(\phi\star_{\delta_1,\delta_2}\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right))(x) = (\omega\delta_1^{-1})(ad-bc)(\omega^{-1}\delta_1\delta_2^{-1})(cx+d)\phi(\frac{ax+b}{cx+d}),$$

and a left action by $g \cdot_{\delta_1,\delta_2} \phi = \phi \star_{\delta_1,\delta_2} g^{-1}$.

- central character (2.3, pg. 13) - If Π is a Λ-representation then we say that $\omega\colon Z \to \Lambda^*$ is a central character of Π if every $g \in Z$ acts by multiplication by $\omega(g)$.
- compact support (2.7, pg. 22) - We say that $\mu \in \Gamma(\mathscr{T}, \mathscr{F}(W,\Pi))$ is compactly supported in F if there exists $a \in \mathbb{N}$ such that μ is zero on $D(\infty, a)$, and compactly supported on F^* if it's zero on $D(0,a)$ and $D(\infty,a)$. It turns out that this notion is independent of W: $\mu \in \Pi^\vee$ is compactly supported in F if there is a $W \in \mathscr{W}^{(1)}(\Pi)$ such that μ, as an element of $\Gamma(\mathscr{T}, \mathscr{F}(W,\Pi))$, is compactly supported in F, and similarly for F^*.
- complexity $\leq n$ (2.4, pg. 15) - If $n \in \mathbb{N}$ we say that Π is of complexity $\leq n$ if there is a $W \in \mathscr{W}(\Pi)$ such that $R^{(n)}(W,\Pi)$ generates the $\mathscr{O}_L[G]$-module $R(W,\Pi)$.
- $d(s,s')$ (2.2, pg. 12) - If $s, s' \in \mathscr{I}$ and Λ is a representative for s, there is a unique Λ' representing s' with $\Lambda' \subset \Lambda$ and Λ/Λ' a cyclic $\mathscr{O}_F$-module, ie isomorphic to $\mathscr{O}_F/\pi^n\mathscr{O}_F$ for some n. Define $d(s,s') = n$.
- $D(a,n)$ (2.2, pg. 12) - An elementary open subset of $\mathbf{P}^1(F)$, given by $a + \pi^n\mathscr{O}_F$ for $a \in F$ and $n \in \mathbb{Z}$.
- $D(\infty,n)$ (2.2, pg. 12) - An elementary open subset of $\mathbf{P}^1(F)$, defined as the complement of $D(0,1-n)$. The image of $D(0,n)$ under w.
- $D_{[s,s']}$ (2.2, pg. 13) - See s_x for the definition of b. If $s' = s_\infty$ then define $D_{[s,s']}$ to be the elementary open of $\mathbf{P}^1(F)$given by the complement of $b + \pi^n\mathscr{O}_F$. If $s' = s_x$ for $x \in k_F$ then set $D_{[s,s']} = b + \pi^n\hat{x} + \pi^{n+1}\mathscr{O}_F$. Note that there is a typo (p in place of b in Colmez's definition).
- elementary open in $\mathbf{P}^1(F)$(2.2, pg. 12) - $D(a,n)$ or its complement for some $a \in F$ and $n \in \mathbb{Z}$.
- extremity (2.2, pg. 13) - If $\mathscr{T}'$ is a subtree of $\mathscr{T}$, we say an edge $[s_0,s_1]$ of $\mathscr{T}'$ is an extremity if $\mathscr{T}' - \{s_0\} \subset \mathscr{T}_{]s_0,s_1)}$.
- F (2.1, pg. 11) - a complete nonarchimedian local field. In section 3, F is set to be equal to $\mathbb{Q}_p$.
- $f(r,\chi)$ (3.5, pg. 30) - The vector $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot 1 \in \Pi(r,0,\chi)$.
- G (2.1, pg. 11) - $\mathbf{GL}_2(F)$.
- $\Gamma(\mathscr{T}', \mathscr{F}(W,\Pi))$ (2.7, pg. 21) - If Π is of complexity ≤ 1 and $W \in \mathscr{W}^{(1)}(\Pi)$ then define $\Gamma(\mathscr{T}', \mathscr{F}(W,\Pi))$ to be the set of $\mu \in \prod_{s\in\mathscr{T}'}[s,W]^\vee$ such that $\langle \mu, x\rangle = 0$ for all $x \in G \cdot R^{(1)}(W,\Pi)$ supported on $\mathscr{T}'$.
- H (2.5, pg. 16) - The subgroup of G generated by Z, $I^-(1)$ and the matrix $\left(\begin{smallmatrix} 0 & \pi \\ 1 & 0 \end{smallmatrix}\right)$.
- $\mathscr{I}$(2.2, pg. 12) - homothety classes of lattices in $Fe_1 \oplus Fe_2$, where a homothety is the action of a scalar matrix. Isomorphic to G/KZ.
- I_n (2.1, pg. 11) - The subgroup of K consisting of lower tringular matrices modulo π^n, where $n \geq 1$.
- $I^-(n)$ (2.5, pg. 16 (implicitly)) - The subgroup of K consisting of lower triangular matrices modulo π^n, where $n \geq 1$.
- $I^+(n)$ (2.5, pg. 15) - The subgroup of K consisting of upper triangular matrices modulo π^n, where $n \geq 1$.
- $I(W)$ (2.4, pg. 14) - For $W \in \mathscr{W}(\Pi)$, set $I(W) = \mathrm{Ind}_{KZ}^G W$.
- K (2.1, pg. 11) - $\mathbf{GL}_2(\mathscr{O}_F)$.
- K_n (2.1, pg. 11) - The subgroup of K consisting of matrices congruent to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ modulo π^n, where $n \in \mathbb{Z}$.

- L (0.1, pg. 2) - a finite extension of $\mathbb{Q}_p$.
- $\ell(I)$ (2.2, pg. 12) - For an oriented segment $I = [s, s']$, define the length of I to be $\ell(I) = d(s, s')$. The action of G on the oriented segments preserves the length, is transitive on segments of a given length, and the stabilizer of $[(e_1, e_2), (\pi^n e_1, e_2)]$ is I_n, and the G-set of oriented segments of length n is isomorphic to $G/I_n Z$.
- $\mathrm{LC}_c(\mathbb{Q}_p, k_L)$(3.2, pg. 26) - The $k_L$vectors space of locally constant functions with compact support in $\mathbb{Q}_p$and values in k_L.
- $\mathrm{LC}_c(\delta_1 \otimes \delta_2)$(3.2, pg. 26) - $\mathrm{LC}_c(\mathbb{Q}_p, k_L)$equipped with left and (and corresponding right) actions of B,

$$\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \cdot_{\delta_1\otimes\delta_2} \phi(x) = \delta_1(a)\delta_2(d)\phi(\frac{dx-b}{a}),$$

$$\phi \star_{\delta_1\otimes\delta_2} \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)(x) = \delta_1^{-1}(a)\delta_2^{-1}(d)\phi(\frac{ax+b}{d}).$$

- $\mathrm{LC}_c(\mathbb{Q}_p^*, k_L)$(3.3, pg. 29) - the vector space of locally constant functions on $\mathbb{Q}_p$with values in $k_L$and compactly supported in $\mathbb{Q}_p^*$.
- $\mathrm{LC}(\mathbf{P}^1(\mathbb{Q}_p), k_L)$(3.4, pg. 29) - $B(\omega, 1)$, the vector space of locally constant functions on $\mathbf{P}^1(\mathbb{Q}_p)$with a left action of G defined by $g \cdot \phi = \phi \star g^{-1}$ and $\phi \star \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(x) = \phi(\frac{ax+b}{cx+d})$.
- locally constant (representation) (2.3, pg. 13) - A representation Π of G is locally constant (or lisse) if the stabilizer of each element $v \in \Pi$ is open in G.
- $M_{\delta_1\otimes\delta_2}$ (2.8, pg. 24) - If M is a $k_L[B]$-module of finite length over k_L with central character ω_M and $\delta_1 \otimes \delta_2$ a character of A then denote by $M_{\delta_1\otimes\delta_2}$ the set of $x \in M$ such that there is a $k(x) \in \mathbb{N}$ with $(g - \delta_1 \otimes \delta_2(g))^{k(x)} \cdot x = 0$ for all $g \in B$. Note that $M_{\delta_1\otimes\delta_2} = 0$ if $\delta_1\delta_2 \neq \omega_M$.
- P (2.1, pg. 11) - The mirabolic subgroup of $\mathbf{GL}_2(F)$, ie $\left(\begin{smallmatrix} F^* & F \\ 0 & 1 \end{smallmatrix}\right)$.
- P (3.1, pg. 25) - used to represent an element of $\mathrm{Sym}^r k_L^2$ thought of as a polynomial in X of degree $\leq r$.
- $P(\infty)$ (3.1, pg. 26) - if $P \in W_{r,\chi}$ then $P(\infty)$ is the coefficient of X^r.
- P^+ (2.1, pg. 11; 2.7, pg. 21) - Defined on page 11 as the monoid $\left(\begin{smallmatrix} \mathscr{O}_F - \{0\} & \mathscr{O}_F \\ 0 & 1 \end{smallmatrix}\right)$. Redefined on page 21 as the monoid $\left(\begin{smallmatrix} \pi^{\mathbb{N}} & \mathscr{O}_F \\ 0 & 1 \end{smallmatrix}\right)$.
- P_r (3.5, pg. 30) - The polynomial of degree r given by $\frac{(-X+1)\cdots(-X+r)}{r!}$.
- principal series (3.1, pg. 26) - A representation of the form $\mathrm{Ind}_B^G \delta_1 \otimes \delta_2$ is called a principal series.
- $R(W, \Pi)$ (2.4, pg. 14) - The kernel of the morphism of G-modules from $I(W)$ to W defined by $\phi \mapsto \sum_{g\in G/KZ} g \cdot \phi(g^{-1})$.
- $R^{(0)}(W, \Pi)$ (2.4, pg. 15) - For $W \in \mathscr{W}(\Pi)$, define

$$R^{(0)}(W, \Pi) = \{[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), x] - [\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right), y] : y \in W \cap \left(\begin{smallmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot W, x = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot y\}$$

- $R^{(n)}(W, \Pi)$ (2.4, pg. 15) - For $n \geq 1$, the kernel of the natural map $\bigoplus_{d(s,\sigma_0)\leq n}[s, W] \to W^{[n]}$.
- $R(W, W', \iota)$ (2.5, pg. 16) - If W is a finite type $\mathscr{O}_L$-module with an action of KZ, W' a sub-$\mathscr{O}_L$-module stable under $I^+(1)$ and $\left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)$, and ι is an isomorphism $W' \to \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot W'$ with the property that $\iota(g \cdot x) = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right) g \left(\begin{smallmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \iota(x)$ for all $x \in W'$ and $g \in I^+(1)$ and the property that $\iota(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot \iota(v)) = \left(\begin{smallmatrix} 0 & \pi \\ \pi & 0 \end{smallmatrix}\right) \cdot v$ for all $v \in W'$, then define $R(W, W', \iota)$ to be the sub-$\mathscr{O}_L[G]$-module generated by $[\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right), v] - [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \iota(v)]$ as v ranges over W'.

- $R_{\delta_1,\delta_2,0}$(3.2, pg. 27 and 3.3, pg. 28) - Define
$$R_{\delta_1,\delta_2,0} = [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \phi_0] - \sum_{i \in \mathbb{Z}_p/p\mathbb{Z}_p} [\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), \delta_1(p)^{-1}\phi_i].$$
- $R_{\delta_1,\delta_2,\infty}$(3.3, pg. 28) Define
$$R_{\delta_1,\delta_2,\infty} = [\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), \delta_1(p)^{-1}\phi_\infty] - [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \phi_\infty] - \sum_{i \in (\mathbb{Z}_p/p\mathbb{Z}_p)^*} [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), (\omega^{-1}\delta_1\delta_2^{-1})(i)\phi_i].$$
- R_0 (3.5, pg. 30) - R_0 is the element of $I(W_{r,\chi}) \oplus I(W_{p-1-r,\chi\omega^r})$ given by
$$R_0 = [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), (0, Y^{p-1-r})] - [\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), (1,0)].$$
- R_1 (3.5, pg. 30) - R_1 is the element of $I(W_{r,\chi}) \oplus I(W_{p-1-r,\chi\omega^r})$ given by
$$R_0 = [\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), (0,1)] - (-1)^r\chi(p)^2[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), (X^r, 0)].$$
- ray from s (2.2, pg. 13) - If $s \in \mathscr{T}$, define a ray from s to be a nested union of oriented segments J_n with $\ell(J_n) \to \infty$ as $n \to \infty$.
- $\text{Rep}_{\mathscr{O}_L} G$(2.3, pg. 13) - The category of locally constant, admissible, finite length $\mathscr{O}_L$-representations of G admitting a central character.
- s, s', s_0, s_1 (2.2, pg. 12) - usually represent elements of $\mathscr{T}$.
- s_x (2.2, pg. 13) - Given $s \in \mathscr{T}$, there is a unique lattice Λ_s in the class of s such that the projection of Λ_s onto Fe_2 parallel to Fe_1 is $\mathscr{O}_F e_2$. $\Lambda_s \cap Fe_1$ will be of the form $\pi^n \mathscr{O}_F e_1$ and there will be a $b \in F$, uniquely defined up to $\pi^n \mathscr{O}_F$, such that Λ_s has $\mathscr{O}_F$-basis $\{\pi^n e_1, e_2 + be_1\}$. Fix a choice of b. For $x \in k_F$, define s_x to be the class of the lattice $(\pi^{n+1}e_1, e_2 + (b + \pi^n\hat{x})e_1)$ where $\hat{x} \in \mathscr{O}_F$ lifts x. Define s_∞ to be the class of the lattice $(\pi^{n-1}e_1, e_2 + be_1)$. The edges emanating from s are the set $\{[s, s_x] : x \in \mathbf{P}^1(k_F)\}$.
- St (3.4, pg. 29) - The quotient of $\text{LC}(\mathbf{P}^1(\mathbb{Q}_p), k_L)$by the subspace of constant functions.
- standard presentation (2.4, pg. 15) - We say that $I(W)/R(W,\Pi)$ is a standard presentation of Π if $R(W,\Pi)$ is generated as an $\mathscr{O}_L[G]$-module by $W \cap \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot W$. Equivalently, $R(W,\Pi)$ is generated by $R^{(0)}(W,\Pi)$.
- supersingular (3.1, pg. 26) A representation $\Pi(r, 0, \chi)$ is called a supersingular.
- support of x (2.4, pg. 15) - The smallest subtree $\mathscr{T}'$ of $\mathscr{T}$such that x is supported on $\mathscr{T}'$.
- supported on $\mathscr{T}'$ (2.4, pg. 14) - Since $I(W) = \bigoplus_{s \in \mathscr{T}} [s, W]$, so we can write $x \in I(W)$ as $x = \sum_{s \in \mathscr{T}} x_s$ with $x_s \in [s, W]$. If $\mathscr{T}'$ is a subtree of $\mathscr{T}$, then we say that x is supported on $\mathscr{T}'$ if $x_s = 0$ for $s \notin \mathscr{T}'$.
- T_p (3.1, pg. 26) - If $0 \le r \le p-1$, Barthel and Livné construct $T_p \colon I(W_{r,\chi}) \to I(W_{r,\chi})$, commuting with the action of G and such that, if $g \in G$ and $P \in W_{r,\chi}$,
$$T_p([g,P]) = \sum_{i=0}^{p-1} P(-i)[g\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right), 1] + P(\infty)[g\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), X^r].$$
- $\mathscr{T}$(2.2, pg. 12) - The tree (building) of $\mathbf{PGL}_2(F)$. The vertices of $\mathscr{T}$are the homothety classes of lattices in $Fe_1 \oplus Fe_2$. The oriented edges are pairs $[s, s']$ with $d(s, s') = 1$.
- $\widehat{\mathscr{T}}(\Lambda)$ (0.1, pg. 2) - the ring of continuous characters $\mathbb{Q}_p^* \to \Lambda^*$, where Λ is a topological ring.
- $\mathscr{T}_U$ (2.2, pg. 13) - If U is an elementary open of $\mathbf{P}^1(F)$then it corresponds to an edge $[s_0, s_1]$. Set $\mathscr{T}_U = \mathscr{T}_{]s_0,s_1)}$.

- $\mathscr{T}_{]s_0,s_1)}$ (2.2, pg.13) - The subtree issuing from $[s_0, s_1]$, the vertices of which are the vertices $s \in \mathscr{T}$ with $s_1 \in [s_0, s]$. Note that $s_0 \notin \mathscr{T}_{]s_0,s_1)}$ but $s_1 \in \mathscr{T}_{]s_0,s_1)}$.
- U^+ (2.1, pg. 11) - The subgroup of G consisting of upper triangular unipotent matrices, ie $\begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix}$.
- U^- (2.1, pg. 11) - The subgroup of G consisting of lower triangular unipotent matrices, ie $\begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix}$.
- $U^+(\pi^n \mathscr{O}_F)$ (2.1, pg. 11) - The subgroup $\begin{pmatrix} 1 & \pi^n \mathscr{O}_F \\ 0 & 1 \end{pmatrix}$ of G, where $n \in \mathbb{Z}$.
- $U^-(\pi^n \mathscr{O}_F)$ (2.1, pg. 11) - The subgroup $\begin{pmatrix} 1 & 0 \\ \pi^n \mathscr{O}_F & 1 \end{pmatrix}$ of G, where $n \in \mathbb{Z}$.
- W (2.4, pg. 14) - Through much of chapter 2, W is the symbol used for an element of $\mathscr{W}(\Pi)$or $\mathscr{W}^{(n)}(\Pi)$.
- W (3.5, pg. 30) - $W_{r,\chi} \oplus W_{p-1-r,\chi\omega^r}$. Represent an element as (P, Q) where P is a polynomial in X of degree $\leq r$ and Q a polynomial in Y of degree $\leq p-1-r$.
- w (2.1, pg. 11) - The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- $\mathscr{W}(\Pi)$(2.4, pg. 14) - For $\Pi \in \mathrm{Rep}_{\mathscr{O}_L} G$, denote by $\mathscr{W}(\Pi)$the set of finite type sub-$\mathscr{O}_L$-modules of Π that are stable under KZ and generate Π as a G-module.
- $\mathscr{W}^{(n)}(\Pi)$ (2.4, pg. 15) - For $n \in \mathbb{N}$, denote by $\mathscr{W}^{(n)}(\Pi)$ the set of $W \in \mathscr{W}(\Pi)$ such that $R^{(n)}(W, \Pi)$ generates the $\mathscr{O}_L[G]$-module $R(W, \Pi)$.
- $W^{[n]}$ (2.4, pg. 15) - If $W \subset \Pi$ is stable under K and $n \in \mathbb{N}$, set $W^{[n]}$ to be the image in Π of the submodule $\sum_{d(s,\sigma_0)\leq n} [s, W] \subset I(W)$.
- $W_{r,\chi}$ (3.1, pg. 25) - the KZ-module $(\mathrm{Sym}^r k_L^2) \otimes \chi \circ \det$, where the action of K factors through $\mathbf{GL}_2(\mathbb{F}_p)$.
- $W_0(\omega, 1)$ (3.4, pg. 29) - Defined by $W(\omega, 1)/k_L \cdot \mathbf{1}_{\mathbf{P}^1(\mathbb{Q}_p)}$.
- $W(\delta_1, \delta_2)$(3.3, pg. 28) - The k_L-subspace of $B(\delta_1, \delta_2)$generated by ϕ_∞ and the ϕ_i for $i \in \mathbb{Z}_p/p\mathbb{Z}_p$.
- $Y(\delta_1, \delta_2)$ (3.2, pg. 27) - The $k_L$vector space $\bigoplus_{i\in\mathbb{Z}_p/p\mathbb{Z}_p} k_L \cdot \phi_i$ with the action of $ZB(\mathbb{Z}_p)$ obtained by restriction from $\mathrm{LC}_c(\delta_1 \otimes \delta_2)$.
- Z (2.1, pg. 11) - The center of $\mathbf{GL}_2(F)$, ie $\left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in F^*\right\}$.
- zero on U (2.7, pg. 21) - We say that $\mu \in \Gamma(\mathscr{T}, \mathscr{F}(W, \Pi))$ is zero on U if the restriction to $[s, W]$ is identically zero for all $s \in \mathscr{T}_U$. Equivalently, if $g_U \in G$ sends $\mathscr{O}_F$to U then we require $\langle \mu, g_U h \cdot v \rangle = 0$.
- Δ (2.1, pg. 11) - The dihedral group generated by A and w.
- χ (3.1, pg. 25) - a character $\mathbb{Q}_p^* \to k_L^*$.
- ι (2.5, pg. 15) - Define $\iota \colon W \to \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \cdot W$ by $\iota(v) = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \cdot v$.
- σ_n (2.2, pg. 12) - the homothety class of the lattice $(\pi^n e_1, e_2)$.
- ϕ_v (3.3, pg. 27) - For $v \in \mathrm{Ind}_B^G \delta_1 \otimes \delta_2$, define $\phi_v \colon \mathbb{Q}_p \to k_L$ by $\phi_v(x) = v(\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix})$.
- ϕ_∞ (3.3, pg. 27) - The function on $\mathbb{Q}_p$defined by:

$$\phi_\infty(x) = \begin{cases} (\omega^{-1}\delta_1\delta_2^{-1})(x) & \text{if } x \notin \mathbb{Z}_p, \\ 0 & \text{if } x \in \mathbb{Z}_p. \end{cases}$$

- ϕ_i (3.2, pg. 27) - For $i \in \mathbb{Z}_p/p\mathbb{Z}_p$, set $\phi_i = \mathbf{1}_{i+p\mathbb{Z}_p} \in \mathrm{LC}_c(\mathbb{Q}_p, k_L)$.
- π (2.1, pg. 11) - a uniformizer for F.
- Π (2.3, pg. 13) - For a ring Λ, a Λ-representation Π of G is a Λ-module equipped with a left, Λ-linear action of G. We often implicitly set $\Lambda = \mathscr{O}_L$.

- $\Pi^\vee$ (2.7, pg. 21) - If Π is an $\mathscr{O}_L$-representation of G, define the dual of Π by $\Pi^\vee = \mathrm{Hom}(\Pi, L/\mathscr{O}_L)$. It is given the structure of a G module by $(g\cdot\mu)(v) = \mu(g^{-1}\cdot v)$. $\Pi^\vee$ is given the weak convergence topology, making it a compact $\mathscr{O}_L$-module.
- $\Pi_c^\vee$ (2.7, pg. 22) - The set of elements of $\Pi^\vee$ compactly suppported in F^*. See compact support.
- $\Pi(r,\lambda,\chi)$ (3.1, pg. 26) - For $\lambda \in k_L$, $0 \le r \le p-1$ and $\chi\colon \mathbb{Q}_p^* \to k_L^*$ define $\Pi(r,\lambda,\chi) = I(W_{r,\chi})/(T_p - \lambda)\cdot(I(W_{r,\chi}))$.
- ω (3.3, pg. 27) - Define $\omega\colon \mathbb{Q}_p^* \to \mathbb{F}_p^*$ to be the reduction modulo p of the character $x \mapsto x|x|$.
- ω_M (2.8, pg. 24) - The central character of a finite length $k_L[B]$-module M.
- $[s,s']$ (2.2, pg. 12) - When $s, s' \in \mathscr{T}$, this is an oriented edge or oriented segment of the tree $\mathscr{T}$.
- $[g,v]$ (2.4, pg. 14) - If $W \in \mathscr{W}(\Pi)$, $v \in W$ and $g \in G$ let $[g,v]$ be the element of $I(W)$ defined by
$$[g,v](h) = \begin{cases} hg\cdot v & \text{if } hg \in KZ, \\ 0 & \text{if } hg \notin KZ. \end{cases}$$
- $[g,W]$ (2.4, pg. 14) - If $W \in \mathscr{W}(\Pi)$ and $g \in G$, set $[g,W] = \{[g,v] : v \in W\}$. This is a submodule of $I(W)$ depending only on the class of g in $G/KZ \cong \mathscr{T}$. It's image under the map to Π is the translate $g\cdot W$.
- $[s,W]$ (2.4, pg. 14) - Since $[g,W]$ depends only on the class of g in $G/KZ \cong \mathscr{T}$ we can define $[s,W]$ in the natural way.
- $\langle\mu,v\rangle$ (2.7, pg. 21) - If $\mu \in \Pi^\vee$ and $v \in \Pi$ then $\langle\mu,v\rangle$ is the result of applying μ to v.
- $\cdot_{\delta_1\otimes\delta_2}$ (3.2, pg. 26) - see $\mathrm{LC}_c(\delta_1\otimes\delta_2)$and $B(\delta_1,\delta_2)$.
- $\star_{\delta_1\otimes\delta_2}$ (3.2, pg. 27) - see $\mathrm{LC}_c(\delta_1\otimes\delta_2)$and $B(\delta_1,\delta_2)$.