Hypergeometric L-functions in average polynomial time

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Hasse-Weil L-functions

Let X be a nice algebraic variety over $\mathbb{Q}$. For $w = 0, \ldots, 2\dim(X)$, we get an associated **(incomplete) Hasse-Weil L-function** built out of **Euler factors**:

$$L_w(X, s) = \prod_p L_p(X, p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad L_p(X, T) := \det(1 - T\,\mathrm{Frob}_p, H^w_{\mathrm{et}}(X, \mathbb{Q}_\ell)^{I_p}).$$

We can similarly define $L(M, s)$ for M a **motive** factor of $H^w(X)$; we refer to w as the **weight** of M and $r = \dim(M)$ as the **degree**. A prime p is **good** if I_p acts trivially, and **bad** otherwise. We have $\deg(L_p(X, T)) \leq r$, with equality iff p is good. One can define the conductor N as a certain product of powers of the bad primes, and the **completed L-function** $\Lambda(M, s)$ as the product of $L(M, s)$ with $N^{s/2}$ and a certain product of Gamma factors.

Goal

Gather numerical data for such L-functions: zero distribution, special values, analytic continuation and functional equation, murmurations. Want a diverse source of motives where $L(M, s)$ is computable, with varying weight, degree, and Hodge numbers.

Hypergeometric data

A **hypergeometric datum** over $\mathbb{Q}$ of degree r is defined by two disjoint tuples

$$(\alpha_1, \ldots, \alpha_r), (\beta_1, \ldots, \beta_r) \text{ over } \mathbb{Q} \cap [0, 1)$$

which are each **Galois-stable**: the multiplicity of any reduced fraction depends only on its denominator. For example

$$\alpha = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{4}),\ \beta = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}).$$

This datum defines a family of hypergeometric motives $M_z^{\alpha,\beta}$ over $z \in \mathbb{Q} \setminus \{0, 1\}$, and a family of degree r L-functions, where $F_p(T) := L_p\big(M_z^{\alpha,\beta}, T\big) = 1 - a_p T + \cdots \in \mathbb{Z}[T]$ is of degree at most r. The Hodge vector and motivic weight can be read from the **zigzag function**

$$Z_{\alpha,\beta}(x) := \#\{j : \alpha_j \leq x\} - \#\{j : \beta_j \leq x\}.$$

Hypergeometric families in the wild

- Legendre Family: $E_t\colon y^2 = x(1-x)(x-t)$

$$\mathrm{H}^1(E_t,\mathbb{Q}) \simeq M_t^{\alpha,\beta} \text{ where } \alpha = (\tfrac{1}{2},\tfrac{1}{2}),\ \beta = (1,1).$$

- Dwork family: $X_\lambda\colon x^4+y^4+z^4+w^4-4\lambda xyzw = 0 \subset \mathbb{P}^3$

$$\mathrm{H}^2(X_\lambda,\mathbb{Q}) = Pic(X_\lambda)\oplus T_\lambda \quad (22 = 19+3)$$
$$T_\lambda \simeq M_{\lambda^4}^{\alpha,\beta} \text{ where } \alpha = (\tfrac{1}{4},\tfrac{1}{2},\tfrac{3}{4}),\ \beta = (1,1,1).$$

- K3 family with Picard rank 16: $X_\lambda\colon x^3y+y^4+z^4+w^4-12\lambda xyzw = 0 \subset \mathbb{P}^3$

$$\mathrm{H}^2(X_\lambda,\mathbb{Q}) = Pic(X_\lambda)\oplus T_\lambda \qquad (22 = 16+6)$$
$$T_\lambda \simeq M_{2^{10}3^6\lambda^{12}}^{\alpha,\beta} \text{ where } \alpha = (\tfrac{1}{12},\tfrac{1}{6},\tfrac{5}{12},\tfrac{7}{12},\tfrac{5}{6},\tfrac{11}{12}),$$
$$\beta = (0,0,0,\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3}).$$

L-functions of hypergeometric motives

$$L(M_z^{\alpha,\beta}, s) = \prod_p F_p(p^{-s}) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

The primes p of **bad reduction** (i.e., $\deg F_p < r$) have the following forms.

- p is **wild** if $v_p(\gamma) < 0$ for some $\gamma \in \alpha \cup \beta$ (e.g., 2 and 3 in our last example).
- p is **tame** if it is not wild, and either $v_p(z) \neq 0$ or $v_p(z-1) \neq 0$.

Completing the L-function gives

$$\Lambda(s) := N^{s/2} \cdot \Gamma_{\alpha,\beta}(s) \cdot L\big(M_z^{\alpha,\beta}, s\big).$$

We expect Λ to satisfy the functional equation

$$\Lambda(s) = \pm\Lambda(w+1-s)$$

To numerically study the analytic properties of $\Lambda(s)$ and check its functional equation one needs to know

$$a_n \leq B, \text{ where } B = O(\sqrt{N}).$$

The Good, the Tame and the Wild

$$L(M_z^{\alpha,\beta}, s) = \prod_p F_p(p^{-s}) = \sum_{n\geq 1} \frac{a_n}{n^s} = L_{\mathrm{good}}(s)\cdot L_{\mathrm{tame}}(s)\cdot L_{\mathrm{wild}}(s)$$

We do not yet have formulas for F_p at the wild primes.
There is a recipe for F_p at the tame primes.
For p a good prime (neither wild nor tame), $F_p(T) = \det(1 - T\,\mathrm{Frob}_p \,|\, M_z^{\alpha,\beta})$ may be recovered from a trace formula of the shape

$$\mathrm{Tr}(\mathrm{Frob}_q) = H_q\left(\begin{matrix}\alpha\\ \beta\end{matrix}\middle| z\right) := \frac{1}{1-q}\sum_{m=0}^{q-2} \pm p^{\xi(m)}\left(\prod_{j=1}^{r}\frac{(\alpha_j)_m^*}{(\beta_j)_m^*}\right)[z]^m,$$

where $[z]$ is the multiplicative lift of $z \bmod p$ and $(\gamma)_m^*$ is a p-adic variant of the Pochhammer symbol $(\gamma)_m = \gamma(\gamma+1)\cdots(\gamma+m-1)$.

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$$a_p = H_p\left(\begin{matrix}\alpha\\ \beta\end{matrix}\middle| z\right) := \frac{1}{1-p}\sum_{m=0}^{p-2} \pm p^{\xi(m)} \left(\prod_{j=1}^{r} \frac{(\alpha_j)_m^*}{(\beta_j)_m^*}\right) [z]^m,$$

where $[z]$ is the multiplicative lift of $z \bmod p$ and $(\gamma)_m^*$ is a p-adic variant of the Pochhammer symbol $(\gamma)_m = \gamma(\gamma+1)\cdots(\gamma+m-1)$.

Theorem (Costa–Kedlaya–R, 2024)

We exhibit an algorithm to compute a_p for all primes $p \leq X$.
For fixed α, β, and z the time and space complexities are both $\widetilde{O}(X)$.

Our initial algorithm (2020) allowed computation of L-functions with motivic weight 1; the 2024 version dropped the restriction on weight.

`github.com/edgarcosta/amortizedHGM`

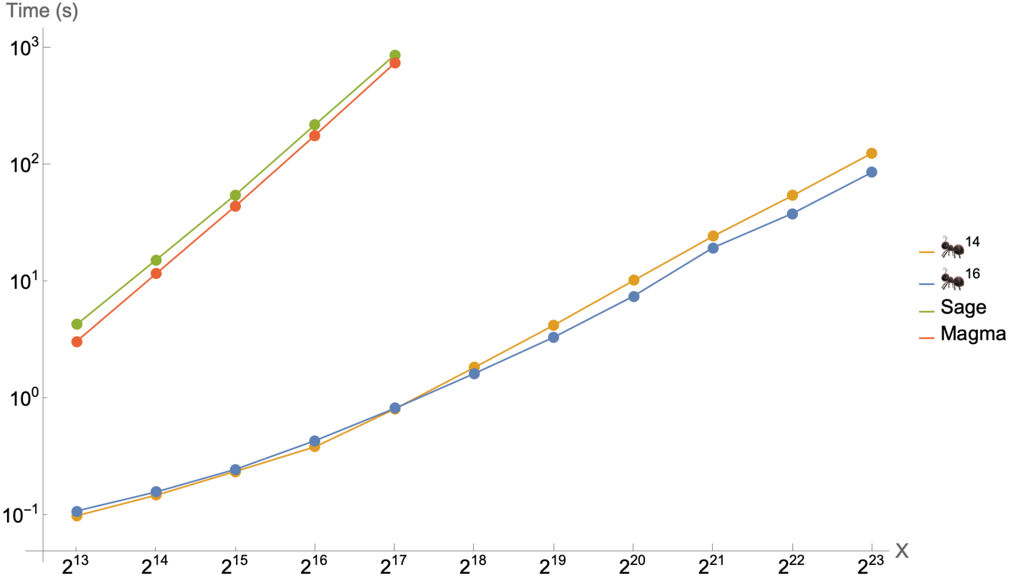
Amortization over primes

$$a_p = H_p\left(\begin{matrix}\alpha \\ \beta\end{matrix}\middle| z\right) := \frac{1}{1-p}\sum_{m=0}^{p-2} \pm p^{\xi(m)} \left(\prod_{j=1}^{r} \frac{(\alpha_j)_m^*}{(\beta_j)_m^*}\right) [z]^m,$$

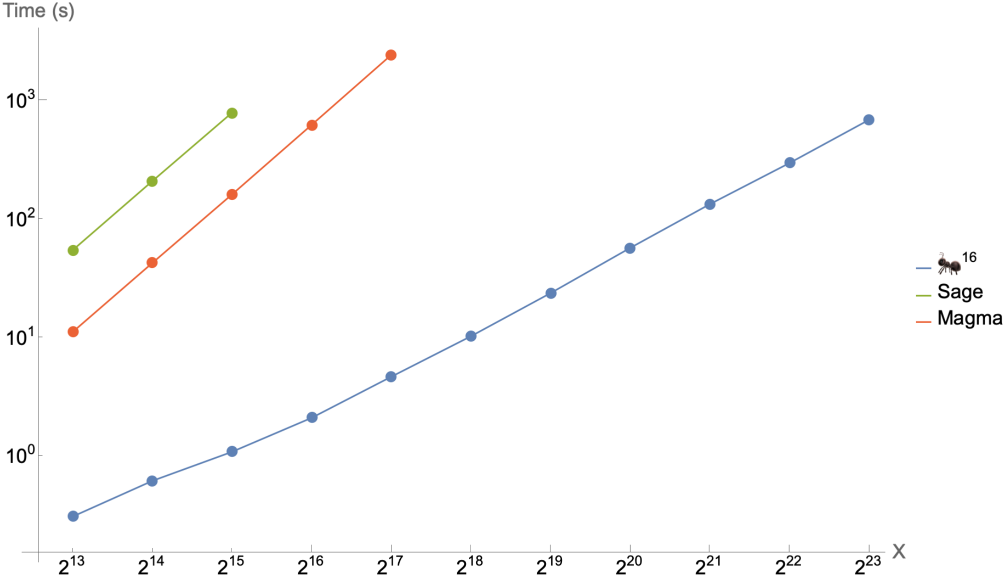
where $[z]$ is the multiplicative lift of $z \bmod p$ and $(\gamma)_m^*$ is a p-adic variant of the Pochhammer symbol $(\gamma)_m = \gamma(\gamma+1)\cdots(\gamma+m-1)$.

The implementations in `Magma` and `Sage` compute a_p one p at a time. Since the sum is over $O(p)$ terms, computing all prime Dirichlet coefficients up to X requires $\widetilde{O}(X^2)$ arithmetic operations.
The shape of the formula makes it feasible to amortize this complexity over p, thus requiring $\widetilde{O}(X)$ arithmetic operations.

Timings: working $\pmod{p^1}$, degree $= 4$, weight $= 1$



Timings: working $\pmod{p^3}$, degree $= 6$, weight $= 5$



Amortization ingredients

- Overall approach is via accumulating remainder trees, following Costa, Gerbicz, Harvey, and Sutherland;
- Use a generic prime, with computations in $\mathbb{Z}[P]/P^e$ amortized with polynomial arithmetic and then substituting $P = \frac{p}{1-p}$ afterward;
- Break amortization up by dividing $[0, p-2]$ into pieces based on locations of α and β;
- Divide primes into arithmetic progressions modulo the denominators from α and β;
- Precompute various p-adic gamma factors using a similar amortization method (independent of z, only mild dependence on α and β)
- Extract the desired sum from a product of block triangular matrices, using accumulating remainder trees to compute the product modulo all necessary p.

Accumulating remainder trees: Wilson primes

The Alhazen–Wilson theorem states that $(p-1)! \equiv -1 \pmod{p}$ for all primes p. A **Wilson prime** is one so that $(p-1)! \equiv -1 \pmod{p^2}$. The only known examples are $p = 5, 13, 563$.

Costa–Gerbicz–Harvey computed $(p-1)! + 1 \pmod{p^2}$ for all $p \le X = 2 \times 10^{13}$ using a new technique that reduced the overall complexity from $\widetilde{O}(X^2)$ to $\widetilde{O}(X)$.

The idea is to replace the separate computation of $(p-1)! + 1 \pmod{p^2}$ with the serial computation of

$$n! \pmod{\prod_{n<p\le X} p^2} \qquad \text{for } n = 0, \dots, X-1.$$

To make this work, this serialization must be balanced against making the moduli so large that they slow down the computation. Harvey–Sutherland generalized this process into **accumulating remainder trees**.

Accumulating remainder trees

Given integers (or matrices) $A_0, \ldots, A_{b-1}$ and integers $m_0, \ldots, m_{b-1}$, we want to compute simultaneously

$$C_j := A_0 \ldots A_{j-1} \pmod{m_j} \qquad (j = 0, \ldots, b-1).$$

For simplicity, assume $b = 2^\ell$. Form a complete binary tree of depth ℓ with nodes (i, j) where $i = 0, \ldots, \ell$ and $j = 0, \ldots, 2^i - 1$. By computing from the leaves to the root, we can compute products over diadic ranges:

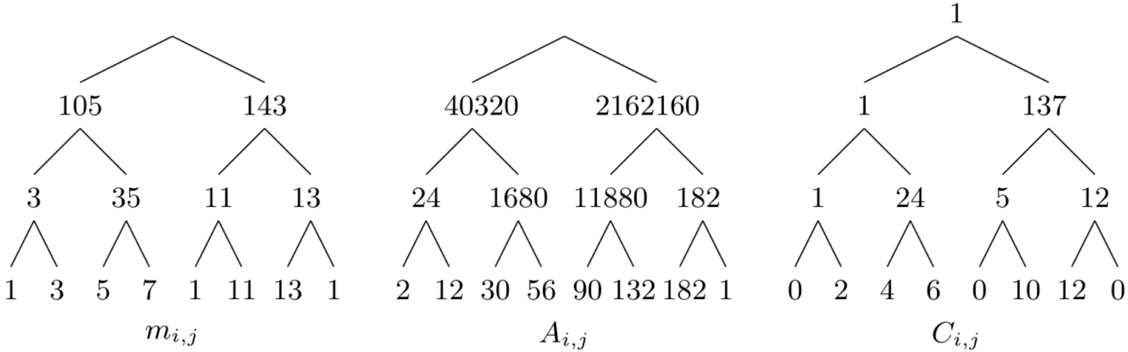
$$m_{i,j} := m_{j2^{\ell-i}} \ldots m_{(j+1)2^{\ell-i}-1},$$
$$A_{i,j} := A_{j2^{\ell-i}} \ldots A_{(j+1)2^{\ell-i}-1}.$$

Then from the root to the leaves, we compute the products $C_{i,j} := A_{i,0} \ldots A_{i,j-1} \pmod{m_{i,j}}$ by writing $C_{0,0} = 1$ and

$$C_{i,j} = \begin{cases} C_{i-1,\lfloor j/2 \rfloor} \pmod{m_{i,j}} & j \equiv 0 \pmod 2 \\ C_{i-1,\lfloor j/2 \rfloor} A_{i,j-1} \pmod{m_{i,j}} & j \equiv 1 \pmod 2. \end{cases}$$

Example: $(p-1)! \pmod{p}$ (Harvey–Sutherland 2014)

- Set $m_0, \ldots, m_7$ to be the first 8 odd numbers, with composites replaced by 1
- $A_i = (i+1)(i+2)$.
- $C_{i,j} = \begin{cases} C_{i-1,\lfloor j/2 \rfloor} \pmod{m_{i,j}} & j \equiv 0 \pmod 2 \\ C_{i-1,\lfloor j/2 \rfloor} A_{i,j-1} \pmod{m_{i,j}} & j \equiv 1 \pmod 2. \end{cases}$



Next steps

We want to add a bunch of interesting hypergeometric L-functions to the LMFDB. Two main obstacles:

1. We need to choose good specialization points z. In order to keep the conductor small, we want both z and $z-1$ to only have small prime factors. We can achieve this by solving S-unit equations, but this approach won't practically be able to ever guarantee all L-functions in a family below a certain conductor bound.
2. The wild L-factors are still not pinned down. Ongoing work by Roberts–Rodriguez Villegas may help, and we can numerically check possible factors using the functional equation.

Thank you!