Finite Groups and K3 surfaces in the LMFDB

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LMFDB: the L-functions and modular forms database

- The LMFDB provides examples of connections proven and predicted by the Langlands correspondence, in an online interface that allows for both exposition and searching.
- Paradigmatic example is

  [elliptic curves] $\leftrightarrow$ [L-functions] $\leftrightarrow$ [classical modular forms]

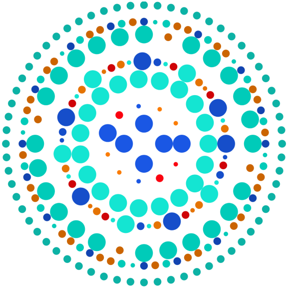
  but many other supporting objects (number fields, finite groups, Dirichlet characters...) play a role.
- Today: focus on finite groups and K3 surfaces.

Follow along at `beta.lmfdb.org`!

Finite groups

- Groups arise elsewhere in the LMFDB: Galois groups, Sato-Tate groups, automorphism groups of lattices and curves of larger genus, modular curves corresponding to subgroups of $\mathrm{GL}_2(\mathbb{Z}/N)$.
- Motivated by groupnames.org to build a reference site for finite groups.
- Connect multiple existing databases of groups (small, transitive, Lie type, perfect, sporadic, $\subseteq \mathrm{GL}_n(\mathbb{F}_q)$, $\subseteq S_{15}$, $\subseteq \mathrm{GL}_2(\mathbb{Z}/N)$)
- New features: searchable, online, invariants, subgroups and characters.
- Aim to be complete (with gaps) along several different ways of measuring the size of a group: cardinality, permutation degree, linear degree, composition factors.

Demo



https://beta.lmfdb.org/Groups/Abstract/

Hashing

The transitive group database classifies as permutation groups (conjugacy within S_n); to get abstract isomorphism classes, need a hash that is isomorphism invariant, fast and with few collisions.

Primary hash

1. If order is identifiable by GAP or Magma, use IdentifyGroup.
2. If abelian, use abelian invariants.
3. Otherwise, use the orders and EasyHash for the maximal subgroups (up to conjugacy), where
4. EasyHash is the multiset of (order, size) for conjugacy classes.
5. Combine into a 64 bit integer.

- Fast: computed hashes for the 408,641,062 groups of order 1536.
- Few collisions: 408,597,690 distinct values, maximum cluster size 72.

Conjugacy vs Automorphism

Made an effort to compute subgroups and elements up to automorphism, not just conjugacy.

- Sometimes made subgroups possible to store: there are only 11 subgroups of C_2^{10} up to automorphism, but 229,755,605 up to conjugacy.
- Compute up to automorphism either by working in the holomorph $G \rtimes \mathrm{Aut}(G)$ (slow in Magma when $|G|$ large) or filling out a graph using generators of $\mathrm{Aut}(G)$.

Example: 32.27

Challenges

- Code structured to accommodate timeouts and errors; this has added complexity.
- `IsIsomorphic` often slow for large 2-groups.
- Found (and reported) dozens of bugs in Magma, including silent wrong results and one over 30 years old.
- Want repeatability, so try to make labels deterministic.
- Substantial computation time (CPU centuries) makes rerunning computations after bugfixes costly

K3 surfaces

Definition

A *K3 surface* is a smooth, proper, geom. connected surface X with trivial canonical bundle ($\omega_X = \wedge^2 \Omega_X \simeq \mathcal{O}_X$) and with $H^1(X, \mathcal{O}_X) = 0$.

Neron-Severi lattice

Under the intersection pairing, $H^2(X, \mathbb{Z}) \cong \Lambda = -2E_8 \oplus 3U$. The intersection $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ is the *Neron-Severi lattice*, a primitive sublattice of rank $1 \leq \rho \leq 20$ generated by Chern classes of line bundles on X.

Lattice-polarized families

- Conversely, given a lattice L of signature $(1, \rho - 1)$ embedded in $\Lambda = -2E_8 \oplus 3U$, the moduli space of K3 surfaces whose Neron-Severi lattice contains L has dimension $20 - \rho$.
- When $\rho = 1$, $L = \mathbb{Z} \cdot u$ and $u^2 = d$ we recover the classical families of double covers of $\mathbb{P}^2$ branched along a sextic ($d = 2$), quartic hypersurfaces in $\mathbb{P}^3$ ($d = 4$), intersection of quadric and cubic in $\mathbb{P}^4$ ($d = 6$), intersection of 3 quadrics in $\mathbb{P}^5$ ($d = 8$)....
- When $\rho = 20$, the moduli space is zero dimensional, the CM K3 surfaces.
- Given L, elliptic fibrations correspond to isomorphisms $L \cong L_- \oplus U$ for negative definite L_- (L_- can vary over a genus of definite lattices).

Unpolarized K3 surfaces

Want a way to identify specific K3 surfaces up to isomorphism (without specifying a polarization or model).

Definition

The *period map* sends X to the image of the line $H^{2,0}(X) \subset H^2(X, \mathbb{C}) \cong \Lambda \otimes \mathbb{C}$ in $P(\Lambda_{\mathbb{C}}) \cong \mathbb{P}^{21}$.

This map is surjective over $\mathbb{C}$, and lets us encode surfaces as vectors of complex numbers. We upgrade this by including an embedding for the full transcendental lattice (the orthogonal complement of the Neron-Severi lattice).

Toric K3 surfaces

- In the physics literature, K3 surfaces arise as simpler versions of Calabi-Yau 3-folds, and of particular interest are those arising as generic hypersurfaces inside toric varieties associated to reflexive polytopes (lattice polytopes whose dual also has lattice vertices).
- There are 4319 3-dimensional reflexive polytopes, and these yield interesting lattice polarized families.
- Aim to connect these families of K3s with those arising in other ways.

Schema

Demo of our K3 schemas.

Currently in a private repo (sorry!); I will work on making a public version and updating posted slides.

MathBases

I am starting to work on `mathbases.org`, which will provide both an online index of databases across many areas of mathematics, as well as resources for mathematicians interested in creating their own databases. Part of a broader Code4Math community (join our Zulip server!).

Linear degrees: irreducible case

Goal

Given a finite group G and a ring R, find the smallest integer n so that G acts faithfully (and irreducibly) on a free rank-n R-module (ie an injection $G \hookrightarrow \mathrm{GL}_n(R)$).

- For irreducible representations and $R = \mathbb{C}$, solved by character table: character χ is *faithful* if $\chi(g) = \chi(1)$ implies $g = 1$.
- For $R = \mathbb{R}$ and $R = \mathbb{Q}$, need to modify the dimension by taking Galois orbits and considering Schur indexes (the number of copies of χ needed for an R-representation to exist, rather than just an R-valued character).
- Note that for some groups (e.g. if the center is not cyclic), there will be no faithful irreducible representations.

Linear degrees: arbitrary case

- Dropping the irreducibility requirement, we seek faithful linear combinations of characters with minimal degree.
- Naive attempt to minimize sum of degrees while ensuring that some row is nontrivial in each column takes exponential time.
- Instead: consider the poset of normal subgroups. Can carry out inclusion-exclusion using the Möbius function on the poset, ending up with a power series giving the number of faithful representations of each dimension.

Linear degrees: arbitrary case

Formula

- Define μ on the poset P of normal subgroups of G by setting $\mu(1) = 1$ and requiring that $\sum_{N \in I} \mu(N) = 0$ for intervals $I \subseteq P$.
- For a characteristic 0 field F and $N \trianglelefteq G$, write $\mathrm{Irr}_{F,N}(G)$ for the set of irreducible F-characters of G with kernel containing N.
- For an F-character χ, write $\mathrm{idx}_F(\chi)$ for the Schur index of χ over F: the minimal $m > 0$ so that χ^m may be realized as an F-representation.

The power series

$$\sum_{N \trianglelefteq G} \mu(N) \prod_{\chi \in \mathrm{Irr}_{F,N}(G)} (1 - X^{\chi(1) \cdot \mathrm{idx}_F(\chi)})^{-1}.$$

is a generating function, where the coefficient of X^a gives the number of faithful F-representations of dimension a.