A function–sheaf dictionary for tori over local fields

David Roe
joint with Clifton Cunningham

Department of Mathematics
University of Calgary/PIMS

Canadian Number Theory Association Meetings
June 17, 2014

Objective

Goal

Bring a *geometric, categorical* perspective to the study of the local Langlands correspondence for *p-adic groups*.

We approach this task from two directions:

1. Find geometric avatars for objects in local Langlands,
2. Find p-adic analogues of objects in geometric Langlands.

Key Idea

The representation theory of $G(\mathbb{Q}_p)$ depends on its structure as a topological group, not as a scheme. A scheme $\mathfrak{G}$ over $\mathbb{F}_p$ with $\mathfrak{G}(\mathbb{F}_p) = G(\mathbb{Q}_p)$ offers a new perspective.

Quasicharacters to sheaves

K – a non-archimedean local field with residue field k,
R – the ring of integers of K with uniformizer π,
T – an algebraic torus over K,
X^* – for a group X, notation for $\mathrm{Hom}(X, \overline{\mathbb{Q}}_\ell^\times)$.

1. We construct a commutative group scheme $\mathfrak{T}$ over k with $\mathfrak{T}(k) \cong T(K)$.
2. For any smooth commutative group scheme G over k we define a category $\mathcal{QC}(G)$ of quasicharacter sheaves on G and show

Theorem (Cunningham-R.)

Trace of Frobenius defines an isomorphism of groups

$$\mathcal{QC}(G)_{/iso} \cong G(k)^*.$$

The Néron model of a torus

The Néron model T_R of T is a separated, smooth commutative group scheme over R so that

Néron mapping property

For any smooth R-scheme Z and morphism $f : Z_K \to T$, f extends uniquely to $Z \to T_R$.

As a consequence,

$$T_R(R) = T(K).$$

Note that T_R is not necessarily finite type.

Examples of Néron models

Example ($\mathbb{G}_m$)

If $T = \mathbb{G}_m$, then the Néron model for T is

$$T_R = \bigcup_{n \in \mathbb{Z}} \mathbb{G}_{m,R},$$

with gluing along generic fibers:

$$\begin{tikzcd} \mathbb{G}_{m,R} & \mathbb{G}_{m,R} \\ \mathbb{G}_m \arrow[u] \arrow[r, "\cong"] & \mathbb{G}_m \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} R[x_0, x_0^{-1}] \arrow[d] & R[x_n, x_n^{-1}] \arrow[d] \\ K[x_0, x_0^{-1}] & K[x_n, x_n^{-1}] \arrow[l, "\text{iso}"'] \end{tikzcd}$$

given by: $\pi^n x_0 \longleftarrow\!\shortmid\, x_n$

Examples of Néron models

Example (SO_2)

Let $T = \mathrm{SO}_2$ over K, split over $E = K(\sqrt{\pi})$. Then

$$K[T] = K[x, y]/(x^2 - \pi y^2 - 1).$$

The Néron model for T is given by

$$R[T_R] = R[x, y]/(x^2 - \pi y^2 - 1).$$

Here T_R is finite type, but not connected: the special fiber T_k of T_R is given by

$$k[T_k] = k[x, y]/(x^2 - 1),$$

two disjoint lines.

The Greenberg functor

Greenberg defines a functor

$$(\mathrm{Sch}\,/R) \to (\mathrm{Sch}\,/k).$$
$$X \to \mathrm{Gr}(X)$$

Proposition (Greenberg)

- *If X is separated and locally of finite type then*

$$\mathrm{Gr}(X)(k) = X(R).$$

- *This functor respects open and closed immersions, étale and smooth morphisms and geometric components.*
- *There are finite level Greenberg functors Gr_n with $\mathrm{Gr}(X) = \lim_{\leftarrow} \mathrm{Gr}_n(X)$.*

Greenberg of Néron

Definition

$$\mathfrak{T} := \mathrm{Gr}(T_R).$$

Proposition

1. $\mathfrak{T}(k) = T(K)$
2. $\mathfrak{T}$ *is a smooth commutative group scheme over* k
3. $\pi_0(\mathfrak{T}) = X_*(T)_{\mathcal{I}}$

Greenberg of Néron for $\mathbb{G}_m$

Set $\mathbb{W}_k^\times$ as the group of units in the Witt ring scheme $\mathbb{W}_k$.

Example

If $T = \mathbb{G}_m$, then

$$\mathfrak{T} = \coprod_{n\in\mathbb{Z}} \mathbb{W}_k^\times.$$

The component group for $\mathfrak{T}$ is

$$X_*(T)_{\mathcal{I}} = \mathbb{Z},$$

with the trivial $\mathrm{Gal}(\bar{k}/k)$ action.

Local Systems

From now on, G will denote a smooth, commutative group scheme over k. We will write $m : G \times G \to G$ for multiplication.

Definition (Local System)

An *ℓ-adic local system* on G is a constructible sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces on the étale site of G, locally constant on each connected component.

Rigid Quasicharacter Sheaves

Definition (Rigid quasicharacter sheaf)

A *rigid quasicharacter sheaf* on G is a triple $\mathcal{L} := (\bar{\mathcal{L}}, \mu, \phi)$.

1. $\bar{\mathcal{L}}$ is a rank-one local system on $\bar{G}$,
2. $\mu : \bar{m}^*\bar{\mathcal{L}} \to \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G} \times \bar{G}$, satisfying an associativity diagram.
3. $\phi : \mathrm{F}_G^*\bar{\mathcal{L}} \to \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G}$ compatible with μ.

A morphism of quasicharacter sheaves is a morphism of constructible ℓ-adic sheaves on $\bar{G}$ commuting with μ and ϕ.

Tensor product makes $\mathcal{QC}_{rig}(G)$ into a rigid monoidal category and $\mathcal{QC}_{rig}(G)_{/iso}$ into a group.

Bounded and Finite Rigid Quasicharacter Sheaves

Definition

- A *bounded rigid quasicharacter sheaf* on G is a pair $(\mathcal{L}_0, \mu_0)$, where $\mathcal{L}_0$ is a rank-one local system on G and μ_0 is as before.
- A *finite rigid quasicharacter sheaf* on G is a pair (f, ψ), where $f : H \to G$ is a finite, surjective, étale morphism of group schemes and $\psi : \ker f \to \overline{\mathbb{Q}}_\ell^\times$.

- Have full and faithful functors $\mathcal{QC}_f(G) \to \mathcal{QC}_0(G) \to \mathcal{QC}_{rig}(G)$,
- these are equivalences when G is connected,
- Bounded rigid quasicharacter sheaves will correspond to bounded characters, and finite to ones with finite image.

Étale Group Schemes

$\mathcal{L} \leftrightsquigarrow$ (stalks $\bar{\mathcal{L}}_x$ and indexed isomorphisms $\mu_{x,y}$ and ϕ_x).
Choice of basis for $\bar{\mathcal{L}}_x \rightsquigarrow a \in C^2(\bar{G}, \overline{\mathbb{Q}}_\ell^\times)$ and $b \in C^1(\bar{G}, \overline{\mathbb{Q}}_\ell^\times)$.

$$\begin{tikzcd}
\bar{\mathcal{L}}_{x+y+z} \arrow[r, "\mu_{x+y,z}"] \arrow[d, "\mu_{x,y+z}"'] & \bar{\mathcal{L}}_{x+y} \otimes \bar{\mathcal{L}}_z \arrow[d, "\mu_{x,y}\otimes \mathrm{id}"] \\
\bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_{y+z} \arrow[r, "\mathrm{id} \otimes \mu_{y,z}"] & \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y \otimes \bar{\mathcal{L}}_z
\end{tikzcd} \Rightarrow a \in Z^2(\bar{G}, \overline{\mathbb{Q}}_\ell^\times)$$

$$\begin{tikzcd}
\bar{\mathcal{L}}_{\mathrm{F}(x)+\mathrm{F}(y)} \arrow[r, "\mu_{\mathrm{F}(x),\mathrm{F}(y)}"] \arrow[d, "\phi_{x+y}"'] & \bar{\mathcal{L}}_{\mathrm{F}(x)} \otimes \bar{\mathcal{L}}_{\mathrm{F}(y)} \arrow[d, "\phi_x\otimes\phi_y"] \\
\bar{\mathcal{L}}_{x+y} \arrow[r, "\mu_{x,y}"] & \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y
\end{tikzcd} \Rightarrow \frac{a(\mathrm{F}(x),\mathrm{F}(y))}{a(x,y)} = \frac{b(x+y)}{b(x)b(y)}$$

Hochschild-Serre Spectral Sequence

$$\begin{aligned} \mathcal{W} & \text{ – the Weil group of } k, \\ a & \rightsquigarrow \alpha \in C^0(\mathcal{W}, Z^2(\bar{G}, \overline{\mathbb{Q}}_\ell^\times)), \\ b & \rightsquigarrow \beta \in Z^1(\mathcal{W}, C^1(\bar{G}, \overline{\mathbb{Q}}_\ell^\times)) \text{ with } \beta(\mathrm{F}) = b, \\ E_0^{i,j} & = C^i(\mathcal{W}, C^j(\bar{G}, \overline{\mathbb{Q}}_\ell^\times)). \end{aligned}$$

Proposition

- *The map* $\mathcal{QC}_{rig}(G)_{/iso} \to \mathrm{H}^2(E_0^\bullet)$ *to the cohomology of the total complex given by* $\mathcal{L} \mapsto (\alpha, \beta, 0)$ *is an isomorphism.*
- *The spectral sequence yields an exact sequence*

$$1 \to \mathrm{H}^0(\mathcal{W}, \mathrm{H}^2(\bar{G}, \overline{\mathbb{Q}}_\ell^\times)) \to \mathrm{H}^2(E_0^\bullet) \to \mathrm{H}^1(\mathcal{W}, \mathrm{H}^1(\bar{G}, \overline{\mathbb{Q}}_\ell^\times)) \to 1.$$

- $$\mathrm{H}^1(\mathcal{W}, \mathrm{H}^1(\bar{G}, \overline{\mathbb{Q}}_\ell^\times)) \to (G(\bar{k})^*)_{\mathcal{W}} \to G(k)^*$$

  is an isomorphism compatible with trace of Frobenius.

Quasicharacter Sheaves

So $\mathcal{QC}_{rig}(G)_{/iso} \twoheadrightarrow G(k)^*$ has kernel $\mathrm{H}^2(G(\bar{k}), \overline{\mathbb{Q}}_\ell^\times)^{\mathrm{F}}$ for étale G.

Definition (Quasicharacter sheaf)

For any smooth, commutative, group scheme G, a *quasicharacter sheaf* on G is a Weil sheaf $\mathcal{L} := (\bar{\mathcal{L}}, \phi)$ so that $(\bar{\mathcal{L}}, \mu, \phi)$ is a rigid quasicharacter sheaf for some μ.

Proposition

For étale G, trace of Frobenius induces an isomorphism

$$\mathcal{QC}(G)_{/iso} \to G(k)^*.$$

Snake Lemma

For any G, trace of Frobenius defines a map

$$t_G : \mathcal{QC}(G)_{/iso} \to G(k)^*.$$

Pullback then gives the rows of

$$\begin{tikzcd} & \mathcal{QC}(\pi_0(G))_{/iso} \arrow[r] \arrow[d] & \mathcal{QC}(G)_{/iso} \arrow[r] \arrow[d] & \mathcal{QC}(G^\circ)_{/iso} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & (\pi_0(G))(k)^* \arrow[r] & G(k)^* \arrow[r] & G^\circ(k)^* \arrow[r] & 1 \end{tikzcd}$$

- t_{G° is an isomorphism by the classic function–sheaf dictionary (Deligne),
- $t_{\pi_0(G)}$ is an isomorphism as above,
- the snake lemma finishes the job.

Transfer of quasicharacter sheaves

Suppose T and T' are tori over local fields K and K'. We say that T and T' are *N-congruent* if there are isomorphisms

$$\alpha : \mathcal{O}_L/\pi_K^N\mathcal{O}_L \to \mathcal{O}_{L'}/\pi_{K'}^N\mathcal{O}_{L'},$$
$$\beta : \mathrm{Gal}(L/K) \to \mathrm{Gal}(L'/K'),$$
$$\phi : X^*(T) \to X^*(T'),$$

satisfying natural conditions. If T and T' are N-congruent then $\mathrm{Hom}_{<N}(T(K), \overline{\mathbb{Q}}_\ell^\times) \cong \mathrm{Hom}_{<N}(T'(K'), \overline{\mathbb{Q}}_\ell^\times)$.

- Chai and Yu give an isomorphism of group schemes $T_n \cong T'_n$, for n depending on N.
- This isomorphism induces an equivalence of categories $\mathcal{QC}(\mathfrak{T}_n) \to \mathcal{QC}(\mathfrak{T}'_n)$.

Class Field Theory

We have constructed the diagram

$$\begin{tikzcd} & \mathcal{QC}(\mathfrak{T})_{/iso} \arrow[dl, "{}^{t}\mathfrak{T}"'] \arrow[dr, dashed] & \\ \mathrm{Hom}(T(K), \overline{\mathbb{Q}}_\ell^\times) \arrow[rr, "\mathrm{rec}_T"'] & & \mathrm{H}^1(K, \hat{\mathbf{T}}_\ell) \end{tikzcd}$$

We are working with Takashi Suzuki to construct Langlands parameters directly from quasicharacter sheaves, which would give a different construction of the reciprocity map.

Non-commutative groups

If $\mathbf{G}$ is a connected reductive group over K, no Néron model. Instead, parahorics correspond to facets in the Bruhat-Tits building and give models for $\mathbf{G}$ over $\mathcal{O}_K$. After taking the Greenberg transform, we can glue the resulting k-schemes and try to build sheaves on the resulting space using some form of Lusztig induction from quasicharacter sheaves on a maximal torus. This work is still in progress.

Affine Grassmanians and Flag Varieties

- *K equal characteristic*
  Starting with **G** over k, the affine Grassmanian $\mathbf{G}(K)/\mathbf{G}(\mathcal{O}_K)$ and affine flag variety $\mathbf{G}(K)/\mathbf{I}$ (**I** is the Iwahori) are ind-schemes over k. They play a large role in the geometric Langlands program.
- *K mixed characteristic*
  Now we need to start with a **G** defined over K, and can no longer construct these directly as quotients. Martin Kreidl considers representability of $\mathbf{G}(K)/\mathbf{G}(\mathcal{O}_K)$ for $\mathbf{G} = \mathrm{SL}_n$ but runs into complications with non-perfect rings. Again with Takashi Suzuki, we are working on representing this functor in a slightly modified category.