A function–sheaf dictionary for tori over local fields

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Outline

1. Introduction
2. Greenberg of Néron
3. Quasicharacter Sheaves
4. Applications and Further Work

Objective

K – a non-archimedean local field,
$\mathbf{T}$ – an algebraic torus over K,
ℓ – a prime different from p,
X^* – for a group X, notation for $\mathrm{Hom}(X, \overline{\mathbb{Q}}_\ell^\times)$.

Goal

Attach a space $\mathfrak{T}$ to $\mathbf{T}$ and find a dictionary that translates

$$\{\text{characters of } \mathbf{T}(K)\} \leftrightarrow \{\text{sheaves on } \mathfrak{T}\}.$$

- Try to push characters forward along maps such as $\mathbf{T} \hookrightarrow \mathbf{G}$;
- Deligne-Lusztig representations $\Longrightarrow$ character sheaves;
- Give a new perspective on class field theory.

Approach

1. For commutative group schemes G, locally of finite type over the residue field k of K we define a category $\mathcal{QC}(G)$ of quasicharacter sheaves on G.
2. We show

Main Result over k

$$\mathcal{QC}(G)_{/iso} \cong G(k)^*.$$

3. Given a torus T over K we construct a commutative group scheme $\mathfrak{T}$ over k with $T(K) \cong \mathfrak{T}(k)$.

Main Result for Tori

Theorem (Cunningham & R., [CR13])

For every torus $\mathbf{T}$ *over* K*, there is a pro-algebraic group*

$$\mathfrak{T}/k \qquad \textit{with} \qquad \mathfrak{T}(k) = T(K)$$

and a monoidal category

$$\mathcal{QC}(\mathfrak{T})$$

of Weil local systems on $\mathfrak{T}$ *so that*

$$\mathcal{QC}(\mathfrak{T})_{/iso} \cong \mathbf{T}(K)^*.$$

The Néron model of a torus

Let R be the ring of integers of K with uniformizer π. The Néron model $\mathbf{T}_R$ of $\mathbf{T}$ is a separated, smooth commutative group scheme over R, locally of finite type with the Néron mapping property:

For

As a consequence,

$$\mathbf{T}_R(R) = \mathbf{T}(K).$$

Examples of Néron models

Example ($\mathbb{G}_m$)

If $\mathbf{T} = \mathbb{G}_m$, then the Néron model for $\mathbf{T}$ is

$$\mathbf{T}_R = \bigcup_{n \in \mathbb{Z}} \mathbb{G}_{m,R},$$

with gluing along generic fibers:

$$\begin{tikzcd} \mathbb{G}_{m,R} & \mathbb{G}_{m,R} \\ \mathbb{G}_m \arrow[u] \arrow[r, "\cong"] & \mathbb{G}_m \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} R[x_0, x_0^{-1}] \arrow[d] & R[x_n, x_n^{-1}] \arrow[d] \\ K[x_0, x_0^{-1}] & K[x_n, x_n^{-1}] \arrow[l, "\text{iso}"'] \end{tikzcd}$$

given by: $\pi^n x_0 \longleftarrow\!\shortmid\; x_n$

Examples of Néron models

Example (SO_2)

Let $\mathbf{T} = SO_2$ over K, split over $E = K(\sqrt{\pi})$. Then

$$K[\mathbf{T}] = K[x, y]/(x^2 - \pi y^2 - 1).$$

The Néron model for $\mathbf{T}$ is given by

$$R[\mathbf{T}_R] = R[x, y]/(x^2 - \pi y^2 - 1).$$

Here $\mathbf{T}_R$ is finite type, but not connected: the special fiber $\mathbf{T}_k$ of $\mathbf{T}_R$ is given by

$$k[\mathbf{T}_k] = k[x, y]/(x^2 - 1),$$

two disjoint lines.

The Greenberg functor

Proposition ([DG70, V, §4, no. 1; BLR80, Ch. 9, §6; SN08, §2.2; AC13, §5])

The Greenberg functor

$$\begin{aligned}(\mathrm{Sch}\,/R) &\to (\mathrm{Sch}\,/k)\\ X &\to \mathrm{Gr}(X)\end{aligned}$$

has the property that, if X is separated and locally of finite type then

$$\mathrm{Gr}(X)(k) = X(R).$$

Greenberg of Néron

Definition

$$\mathfrak{T} := \mathrm{Gr}(\mathbf{T}_R).$$

Proposition

1. $\mathfrak{T}(k) = \mathbf{T}(K)$
2. $\mathfrak{T}$ *is a smooth commutative group scheme over* k
3. $\mathfrak{T}$ *is locally of finite type over* k
4. $\pi_0(\mathfrak{T}) = X_*(\mathbf{T})_{\mathcal{I}}$

Greenberg of Néron for $\mathbb{G}_m$

Set $\mathbb{W}_k^\times$ as the group of units in the Witt ring scheme $\mathbb{W}_k$.

Example

If $\mathbf{T} = \mathbb{G}_m$, then

$$\mathfrak{T} = \coprod_{n \in \mathbb{Z}} \mathbb{W}_k^\times.$$

The component group for $\mathfrak{T}$ is

$$X_*(\mathbf{T})_{\mathcal{I}} = \mathbb{Z},$$

with the trivial $\mathrm{Gal}(\bar{k}/k)$ action.

Local Systems

From now on, G will denote a smooth, commutative group scheme, locally of finite type over k with finitely generated geometric component group. We will write $m : G \times G \to G$ for multiplication.

Definition (Local System)

An ℓ*-adic local system* on G is a constructible sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces on the étale site of G, locally constant on each connected component.

Quasicharacter Sheaves

Definition (Quasicharacter sheaf)

A *quasicharacter sheaf* on G is a triple $\mathcal{L} := (\bar{\mathcal{L}}, \mu, \phi)$, where

1. $\bar{\mathcal{L}}$ is a rank-one local system on $\bar{G}$,
2. $\mu : \bar{m}^* \bar{\mathcal{L}} \to \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G} \times \bar{G}$, satisfying an associativity diagram.
3. $\phi : \mathrm{F}_G^* \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G}$ compatible with μ.

A morphism of quasicharacter sheaves is a morphism of constructible ℓ-adic sheaves on $\bar{G}$ commuting with μ and ϕ.

Tensor product makes $\mathcal{QC}(G)$ into a rigid monoidal category and $\mathcal{QC}(G)_{/iso}$ into a group.

Bounded Quasicharacter Sheaves

Definition (Bounded Quasicharacter Sheaf)

A *bounded quasicharacter sheaf* on G is a pair $(\mathcal{L}_0, \mu_0)$, where

1. $\mathcal{L}_0$ is a rank-one local system on G,
2. $\mu_0 : m^*\mathcal{L}_0 \to \mathcal{L}_0 \boxtimes \mathcal{L}_0$ is an isomorphism of sheaves on $G \times G$, satisfying the same associativity diagram.

A morphism is a morphism of constructible sheaves on G commuting with μ_0. Write $\mathcal{QC}_0(G)$ for this category.

- Base change defines a full and faithful functor $B_G : \mathcal{QC}_0(G) \to \mathcal{QC}(G)$,
- B_G is an equivalence when G is connected.
- Under the isomorphism $\mathcal{QC}(G)_{/iso} \cong G(k)^*$, bounded quasicharacter sheaves correspond to bounded characters.

Discrete Isogenies

Definition (Discrete Isogeny)

A *discrete isogeny* is a finite, surjective, étale morphism of group schemes $f : H \to G$ so that $\mathrm{Gal}(\bar{k}/k)$ acts trivially on the kernel of f.

Write $C(G)$ for the category whose objects are pairs (f, ψ), where

1. $f : H \to G$ is a discrete isogeny,
2. $\psi : \ker f \to \mathrm{Aut}(V)$ is a representation on a $\overline{\mathbb{Q}}_\ell$-vector space.

A morphism $(f, \psi) \to (f', \psi')$ is a pair (g, T), where

1. $g : H' \to H$ is a morphism with $f' = f \circ g$,
2. $T : V \to V'$ is a linear transformation, equivariant for ψ' and $\psi \circ g$.

Finite Quasicharacter Sheaves

Let $C_1(G)$ be the subcategory where V is one-dimensional.

Definition (Finite Quasicharacter Sheaf)

The category $\mathcal{QC}_f(G)$ of *finite quasicharacter sheaves* is the localization of $C_1(G)$ at morphisms where g is surjective and T is an isomorphism.

Write V_H for the constant sheaf V on H.

- Taking the ψ-isotypic component of $f_* V_H$ defines a full and faithful functor $L_G : \mathcal{QC}_f(G) \to \mathcal{QC}_0(G)$.
- L_G is an equivalence when G is connected.
- Under the isomorphism $\mathcal{QC}(G)_{/iso} \cong G(k)^*$, finite quasicharacter sheaves correspond to characters with finite image.

Sketch of Main Result

For any G, trace of Frobenius defines a map

$$t_G : \mathcal{QC}(G)_{/iso} \to G(k)^*.$$

Pullback then gives the rows of

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{QC}(\pi_0(G))_{/iso} \arrow[r] \arrow[d] & \mathcal{QC}(G)_{/iso} \arrow[r] \arrow[d] & \mathcal{QC}(G^\circ)_{/iso} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & (\pi_0(G))(k)^* \arrow[r] & G(k)^* \arrow[r] & G^\circ(k)^* \arrow[r] & 1
\end{tikzcd}$$

- t_{G° is an isomorphism by [Del77]: the classic function–sheaf dictionary,
- one can build an isomorphism by hand for étale group schemes using stalks,
- the snake lemma finishes the job.

Transfer of quasi-character sheaves

Suppose T and T' are tori over local fields K and K'. We say that T and T' are *N-congruent* if there are isomorphisms

$$\alpha : \mathcal{O}_L/\pi_K^N\mathcal{O}_L \to \mathcal{O}_{L'}/\pi_{K'}^N\mathcal{O}_{L'},$$
$$\beta : \mathrm{Gal}(L/K) \to \mathrm{Gal}(L'/K'),$$
$$\phi : X^*(T) \to X^*(T'),$$

satisfying natural conditions. If T and T' are N-congruent then $\mathrm{Hom}_{<N}(T(K), \overline{\mathbb{Q}}_\ell^\times) \cong \mathrm{Hom}_{<N}(T'(K'), \overline{\mathbb{Q}}_\ell^\times)$.

- Chai and Yu give an isomorphism of group schemes $\mathbf{T}_n \cong \mathbf{T}'_n$, for n depending on N.
- This isomorphism induces an equivalence of categories $\mathcal{QC}(\mathfrak{T}_n) \to \mathcal{QC}(\mathfrak{T}'_n)$.

Class Field Theory

We have constructed the diagram

$$\begin{tikzcd} & \mathcal{QC}(\mathfrak{T})_{/iso} \arrow[dl, "{}^{t}\mathfrak{T}"'] \arrow[dr, dashed] & \\ \mathrm{Hom}(T(K), \overline{\mathbb{Q}}_\ell^\times) \arrow[rr, "\mathrm{rec}_T"'] & & \mathrm{H}^1(K, \hat{\mathbf{T}}_\ell) \end{tikzcd}$$

We hope to be able to construct Langlands parameters directly from quasicharacter sheaves, which would give a different construction of the reciprocity map.

Non-commutative groups

If $\mathbf{G}$ is a connected reductive group over K, no Néron model. Instead, parahorics correspond to facets in the Bruhat-Tits building and give models for $\mathbf{G}$ over $\mathcal{O}_K$. After taking the Greenberg transform, we can glue the resulting k-schemes and try to build sheaves on the resulting space using some form of Lusztig induction from quasicharacter sheaves on a maximal torus. This work is still in progress.

Affine Grassmanians and Geometric Satake Transforms

- *K equal characteristic*
  The affine Grassmanian $\mathbf{G}(K)/\mathbf{G}(\mathcal{O}_K)$ and affine flag variety $\mathbf{G}(K)/\mathbf{I}$ (here $\mathbf{I}$ is the Iwahori) are ind-schemes over k. They play a large role in the geometric Langlands program.
- *K mixed characteristic*
  Can no longer construct these directly as quotients. We are working on defining analogues via gluing Schubert cells. As a test of the construction, we hope to give a geometric Satake transform in mixed characteristic.

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