Quasicharacter Sheaves for Tori

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Outline

Objective

K – a finite extension of $\mathbb{Q}_p$,
$\mathbf{T}$ – an algebraic torus over K (e.g. $\mathbb{G}_m$),
ℓ – a prime different from p,
X^* – for a group X, notation for $\mathrm{Hom}(X, \overline{\mathbb{Q}}_\ell^\times)$.

Goal

Construct "geometric avatars" for characters in

$$\mathbf{T}(K)^* :$$

sheaves on some space functorially associated to $\mathbf{T}$.

- Try to push characters forward along maps such as $\mathbf{T} \hookrightarrow \mathbf{G}$;
- Deligne-Lusztig representations $\Longrightarrow$ character sheaves;
- Give a new perspective on class field theory.

Approach

1. For commutative group schemes G, locally of finite type over the residue field k of K we define a category $\mathcal{QC}(G)$ of quasicharacter sheaves on G.
2. We show

Main Result

$$\mathcal{QC}(G)_{/iso} \cong G(k)^*.$$

3. Given a torus T over K we construct a commutative group scheme $\mathfrak{T}$ over k with $T(K) \cong \mathfrak{T}(k)$.

Local Systems

From now on, G will denote a smooth, commutative group scheme, locally of finite type over k with finitely generated geometric component group. We will write $m : G \times G \to G$ for multiplication.

Definition (Local System)

An ℓ*-adic local system* on G is a constructible sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces on the étale site of G, locally constant on each connected component.

Quasicharacter Sheaves

Definition (Quasicharacter sheaf)

A *quasicharacter sheaf* on G is a triple $\mathcal{L} := (\bar{\mathcal{L}}, \mu, \phi)$, where

1. $\bar{\mathcal{L}}$ is a rank-one local system on $\bar{G}$,
2. $\mu : \bar{m}^* \bar{\mathcal{L}} \to \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G} \times \bar{G}$, satisfying an associativity diagram.
3. $\phi : \mathrm{F}_G^* \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G}$ compatible with μ.

A morphism of quasicharacter sheaves is a morphism of constructible ℓ-adic sheaves on $\bar{G}$ commuting with μ and ϕ.

Tensor product makes $\mathcal{QC}(G)$ into a rigid monoidal category and $\mathcal{QC}(G)_{/iso}$ into a group.

Bounded Quasicharacter Sheaves

Definition (Bounded Quasicharacter Sheaf)

A *bounded quasicharacter sheaf* on G is a pair $(\mathcal{L}_0, \mu_0)$, where

1. $\mathcal{L}_0$ is a rank-one local system on G,
2. $\mu_0 : m^*\mathcal{L}_0 \to \mathcal{L}_0 \boxtimes \mathcal{L}_0$ is an isomorphism of sheaves on $G \times G$, satisfying the same associativity diagram.

A morphism is a morphism of constructible sheaves on G commuting with μ_0. Write $\mathcal{QC}_0(G)$ for this category.

- Base change defines a full and faithful functor $B_G : \mathcal{QC}_0(G) \to \mathcal{QC}(G)$,
- B_G is an equivalence when G is connected.
- Under the isomorphism $\mathcal{QC}(G)_{/iso} \cong G(k)^*$, bounded quasicharacter sheaves correspond to bounded characters.

Discrete Isogenies

Definition (Discrete Isogeny)

A *discrete isogeny* is a finite, surjective, étale morphism of group schemes $f : H \to G$ so that $\mathrm{Gal}(\bar{k}/k)$ acts trivially on the kernel of f.

Write $C(G)$ for the category whose objects are pairs (f, ψ), where

1. $f : H \to G$ is a discrete isogeny,
2. $\psi : \ker f \to \mathrm{Aut}(V)$ is a representation on a $\overline{\mathbb{Q}}_\ell$-vector space.

A morphism $(f, \psi) \to (f', \psi')$ is a pair (g, T), where

1. $g : H' \to H$ is a morphism with $f' = f \circ g$,
2. $T : V \to V'$ is a linear transformation, equivariant for ψ' and $\psi \circ g$.

Finite Quasicharacter Sheaves

Let $C_1(G)$ be the subcategory where V is one-dimensional.

Definition (Finite Quasicharacter Sheaf)

The category $\mathcal{QC}_f(G)$ of *finite quasicharacter sheaves* is the localization of $C_1(G)$ at morphisms where g is surjective and T is an isomorphism.

Write V_H for the constant sheaf V on H.

- Taking the ψ-isotypic component of $f_* V_H$ defines a full and faithful functor $L_G : \mathcal{QC}_f(G) \to \mathcal{QC}_0(G)$.
- L_G is an equivalence when G is connected.
- Under the isomorphism $\mathcal{QC}(G)_{/iso} \cong G(k)^*$, finite quasicharacter sheaves correspond to characters with finite image.

Characters in the connected case

- Suppose $\mathcal{L}$ is a quasicharacter sheaf on G. Define a character $t_{\mathcal{L}}$ of $G(k)$ by

$$t_{\mathcal{L}}(g) = \mathrm{Tr}(\phi_{\bar{g}}, \bar{\mathcal{L}}_{\bar{g}})$$

  for $g \in G(k)$.
- Suppose χ is a character of $G(k)$. Define a quasicharacter sheaf on G using the Lang isogeny $L(\bar{g}) = \bar{g}^{-1}\,\mathrm{Fr}_q(\bar{g})$,

$$1 \to G(k) \to G \xrightarrow{L} G \to 1,$$

  together with the character χ of $G(k)$.

Theorem (Deligne, SGA 4.5)

The maps defined above are mutually inverse isomorphisms between quasicharacter sheaves on G and $G(k)^*$.

Trace of Frobenius

For any G, trace of Frobenius defines a map

$$\mathcal{QC}(G)_{/iso} \to G(k)^*.$$

Pullback then gives a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{QC}(G/G^\circ)_{/iso} & \longrightarrow & \mathcal{QC}(G)_{/iso} & \longrightarrow & \mathcal{QC}(G^\circ)_{/iso} & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & (G/G^\circ)(k)^* & \longrightarrow & G(k)^* & \longrightarrow & G^\circ(k)^* & \longrightarrow & 1 \end{array}$$

Extending quasicharacter sheaves

Theorem

Every quasicharacter sheaf on G° extends to a (finite) quasicharacter sheaf on G.

Proof.

We will fit any discrete isogeny $\pi : B \to G^\circ$ into

$$\begin{array}{ccccc}
A & = & A & & \\
\downarrow & & \downarrow & & \\
B & \longrightarrow & H & \longrightarrow & \pi_0(H) \\
\pi \downarrow & & f \downarrow & & \pi_0(f) \downarrow \wr \\
G^0 & \longrightarrow & G & \longrightarrow & \pi_0(G)
\end{array}$$

To build H, we first show that $H(\bar{k})$ exists as a $\mathbb{Z}[\mathcal{W}]$-module.

Extending quasicharacter sheaves

Proof.

On extension classes, this map is the first in

$$\mathrm{Ext}^1_{\mathbb{Z}[\mathcal{W}]}(G, A) \to \mathrm{Ext}^1_{\mathbb{Z}[\mathcal{W}]}(G^\circ, A) \to \mathrm{Ext}^2_{\mathbb{Z}[\mathcal{W}]}(G/G^\circ, A).$$

Since $\mathcal{W} \simeq \mathbb{Z}$ has cohomological dimension 1, $\mathrm{Ext}^2_{\mathbb{Z}[\mathcal{W}]}(G/G^\circ, A)$ vanishes. So $\mathrm{Ext}^1_{\mathbb{Z}[\mathcal{W}]}(G, A) \to \mathrm{Ext}^1_{\mathbb{Z}[\mathcal{W}]}(G^\circ, A)$ is surjective. Thus the diagram exists at the level of $\bar{k}$-points, and we can transport the structure of a group scheme from B to H. □

Trace of Frobenius Diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \mathcal{QC}(G/G^\circ)_{/iso} & \longrightarrow & \mathcal{QC}(G)_{/iso} & \longrightarrow & \mathcal{QC}(G^\circ)_{/iso} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & (G/G^\circ)(k)^* & \longrightarrow & G(k)^* & \longrightarrow & G^\circ(k)^* & \longrightarrow & 1 \\
 & & & & & & \downarrow & & \\
 & & & & & & 1 & &
\end{array}
$$

Quasicharacter Sheaves (G étale)

- The category of étale k-group schemes is equivalent to the category of groups with with Galois action through the functor $G \mapsto G(\bar{k})$.
- A quasicharacter sheaf on an étale group scheme G is a collection of 1-dimensional $\overline{\mathbb{Q}}_\ell$-vectors spaces $\bar{\mathcal{L}}_x$ for $x \in G(\bar{k})$ together with $\phi_x : \bar{\mathcal{L}}_{\mathrm{F}(x)} \xrightarrow{\sim} \bar{\mathcal{L}}_x$ and $\mu_{x,y} : \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y \xrightarrow{\sim} \bar{\mathcal{L}}_{x+y}$.

Proposition

Suppose that G is an étale commutative group scheme and $G(\bar{k})$ is finitely generated. Then there is a canonical isomorphism

$$\mathcal{QC}(G)_{/iso} \cong \mathrm{H}^1(\mathcal{W}, G(\bar{k})^*).$$

$$\mathcal{QC}(G)_{/iso} \cong \mathrm{H}^1(\mathcal{W}_k, G(\bar{k})^*)$$

Proof.

A *global section* of $\mathcal{L}$ is a function $s : G(\bar{k}) \to \coprod_{x \in G(\bar{k})} \bar{\mathcal{L}}_x$ with $s(x) \in \bar{\mathcal{L}}_x$ and

$$\mu_{x,y}(s(x+y)) = s(x) \otimes s(y).$$

Using a choice of global section, we define a cocycle $\tau_{\mathcal{L}}$ by

$$\phi_x(s(\mathrm{F}(x))) = \tau_{\mathcal{L}}(\mathrm{F})(x)s(x),$$

where $\phi_x : \bar{\mathcal{L}}_{\mathrm{F}(x)} \to \bar{\mathcal{L}}_x$ is determined by $\mathcal{L}$. One then checks that everything is well-defined and independent of s. □

A Galois cohomology result

Lemma

If X is an abelian group with an action of $\mathcal{W}$, then

$$(X^*)_{\mathsf{F}} \to (X^{\mathsf{F}})^*$$
$$[f] \mapsto f|_{X^{\mathsf{F}}}$$

is an isomorphism.

Proof.

Note that X^{F} is the kernel of $X \xrightarrow{\mathsf{F}-1} X$; let Y be the image. We have

$$0 \to Y^* \to X^* \to (X^{\mathsf{F}})^* \to \mathrm{Ext}^1_{\mathcal{W}}(Y, \overline{\mathbb{Q}}_\ell^\times).$$

Since the Ext-group vanishes, we get an isomorphism between the cokernel of $Y^* \xrightarrow{\mathsf{F}-1} X^*$ to $(X^{\mathsf{F}})^*$. □

Trace of Frobenius for étale group schemes

Since $\mathcal{W}$ is cyclic, $\mathrm{H}^1(\mathcal{W}, G(\bar{k})^*) \cong (G(\bar{k})^*)_{\mathcal{W}}$. We thus see that trace of Frobenius is an isomorphism for étale group schemes. For general G, we use the snake lemma:

Corollary

If G is a commutative group scheme with finitely generated component group then trace of Frobenius gives an isomorphism

$$\mathcal{QC}(G)_{/iso} \cong G(k)^*.$$

The Néron model of a torus

R – ring of integers of K with uniformizer π

R_d – $R/\pi^{d+1}R$

$\mathbf{T}_R$ – The Néron model of $\mathbf{T}$: a separated, smooth commutative group scheme over R, locally of finite type with the Néron mapping property.

$$\mathbf{T}_R(R) = \mathbf{T}(K)$$

In the $\mathbb{G}_m$ case the Néron model is a union of copies of $\mathbb{G}_m/R$, glued along the generic fiber.

$\mathbf{T}_d$ – $\mathbf{T}_R \times_R R_d$.

The Greenberg functor

The Greenberg functor Gr takes a group scheme over an Artinian local ring A (locally of finite type) and produces a group scheme over the residue field k whose k points are canonically identified with the A-points of the original scheme. We set

$$\mathfrak{T}_d = \mathrm{Gr}(\mathbf{T}_d)$$

and

$$\mathfrak{T} = \varprojlim \mathfrak{T}_d.$$

$\mathfrak{T}$ is a commutative group scheme over k with

$$\mathfrak{T}(k) = \mathbf{T}(K).$$

Quasicharacter sheaves on $\mathfrak{T}$

We write $\mathcal{QC}(\mathfrak{T})$ for the projective limit of the categories $\mathcal{QC}(\mathfrak{T}_d)$.

Theorem

$$T(K)^* \cong \mathcal{QC}(\mathfrak{T})_{/iso}$$

and this isomorphism preserves depth.

Transfer of character sheaves

Suppose T and T' are tori over local fields K and K'. We say that T and T' are *N-congruent* if there are isomorphisms

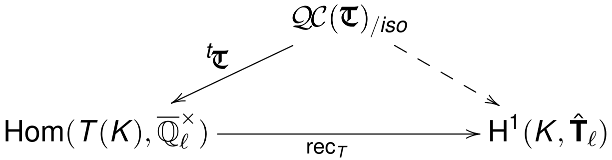
$$\alpha : \mathcal{O}_L/\pi_K^N\mathcal{O}_L \to \mathcal{O}L'/\pi_{K'}^N\mathcal{O}L',$$
$$\beta : \mathrm{Gal}(L/K) \to \mathrm{Gal}(L'/K'),$$
$$\phi : X^*(T) \to X^*(T'),$$

satisfying natural conditions. If T and T' are N-congruent then $\mathrm{Hom}_{<N}(T(K), \overline{\mathbb{Q}}_\ell^\times) \cong \mathrm{Hom}_{<N}(T'(K'), \overline{\mathbb{Q}}_\ell^\times)$.

- Chai and Yu give an isomorphism of group schemes $\mathbf{T}_n \cong \mathbf{T}'_n$, for n depending on N.
- This isomorphism induces an equivalence of categories $\mathcal{QC}(\mathfrak{T}_n) \to \mathcal{QC}(\mathfrak{T}'_n)$.

Class Field Theory

We have constructed the diagram



We hope to be able to construct Langlands parameters directly from quasicharacter sheaves, which would give a different construction of the reciprocity map.

Non-commutative groups

If **G** is a connected reductive group over K, no Néron model. Instead, parahorics correspond to facets in the Bruhat-Tits building and give models for **G** over $\mathcal{O}_K$. After taking the Greenberg transform, we can glue the resulting k-schemes and try to build sheaves on the resulting space using some form of Lusztig induction from quasicharacter sheaves on a maximal torus. This work is still in progress.

Affine Grassmanians and Geometric Satake Transforms

When K has equal characteristic, one may put the structure of an ind-scheme over k on $\mathbf{G}(K)/\mathbf{G}(\mathcal{O}_K)$ and on $\mathbf{G}(K)/\mathbf{I}$, where $\mathbf{I}$ is the Iwahori. The standard constructions rely on the fact that K is a k-algebra, and recover a decomposition into affine Schubert cells afterward. Clifton and I are working to create analogues of these spaces when K has mixed characteristic. Our first test case will be trying to geometrize the Satake transform, but we're not there yet.

Thank you.