Geometrizing Characters of Tori

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Outline

Objective

K – a finite extension of $\mathbb{Q}_p$,
$\mathbf{T}$ – an algebraic torus over K (e.g. $\mathbb{G}_m$),
ℓ – a prime different from p.

Goal

Construct "geometric avatars" for characters in

$$\mathrm{Hom}(\mathbf{T}(K), \overline{\mathbb{Q}}_\ell^\times):$$

sheaves on some space functorially associated to $\mathbf{T}$.

- Try to push characters forward along maps such as $\mathbf{T} \hookrightarrow \mathbf{G}$;
- Deligne-Lusztig representations $\implies$ character sheaves;
- Give a new perspective on class field theory.

Approach

1. Associate to $\mathbf{T}$ a projective system $\mathfrak{T}$ of commutative group schemes $\mathfrak{T}_d$ over the residue field k of K.
2. Define *character sheaves* on $\mathfrak{T}$ following Deligne.
3. Map from character sheaves on $\mathfrak{T}$ to characters on $T(K)$.

The Néron model of $\mathbb{G}_m$

R – ring of integers of K with uniformizer π

R_d – $R/\pi^{d+1}R$

$\mathbf{T}_R$ – The Néron model of $\mathbf{T}$: a separated, smooth commutative group scheme over R, locally of finite type with the Néron mapping property.

$$\mathbf{T}_R(R) = \mathbf{T}(K)$$

In the $\mathbb{G}_m$ case the Néron model is a union of copies of $\mathbb{G}_m/R$, glued along the generic fiber.

$\mathbf{T}_d$ – $\mathbf{T}_R \times_R R_d$.

Components

- The geometric component group of $\mathbf{T}_R$ is $X_*(\mathbf{T})_{\mathcal{I}_K}$, where $\mathcal{I}_K$ is the inertia group of K.
- $\pi_0(\mathbf{T}_R)$ is a constant group scheme after base change to the maximal unramified extension of K, but Frobenius may act nontrivially.
- The sequence of commutative R-group schemes

$$1 \to \mathbf{T}_R^\circ \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to 1$$

splits if $\mathbf{T}$ is unramified.

The Greenberg functor

The Greenberg functor Gr takes a group scheme over an Artinian local ring A (locally of finite type) and produces a group scheme over the residue field k whose k points are canonically identified with the A-points of the original scheme. We set

$$\mathfrak{T}_d = \mathrm{Gr}(\mathbf{T}_d)$$

and

$$\mathfrak{T} = \lim_{\leftarrow} \mathfrak{T}_d.$$

$\mathfrak{T}$ is a commutative group scheme over k with

$$\mathfrak{T}(k) = \mathbf{T}(K).$$

Character Sheaves

Two definitions of character sheaves for a connected (commutative) algebraic groups G over k.

Definition

- An *ℓ-adic local system* is a constructible sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces on the étale site of G that becomes trivial after pulling back along a finite étale map $H \to G$.
- A *character sheaf* on G is an ℓ-adic local system $\mathcal{E}^\circ$ on G equipped with an isomorphism $m^*\mathcal{E}^\circ \cong \mathcal{E}^\circ \boxtimes \mathcal{E}^\circ$, where $m \colon G \times G \to G$ is multiplication.

Character Sheaves (definition 2)

Definition

Alternatively, a character sheaf on G is a short exact sequence

$$1 \to A \to H \to G \to 1$$

together with a character $A \to \overline{\mathbb{Q}}_\ell^\times$, so that

1. $H \to G$ is a finite étale cover,
2. Fr_q acts trivially on A.

Remark

One can replace the character of A with a higher dimensional representation. This adds little for tori, but may prove useful when considering other algebraic groups.

Frobenius

Base change to $\overline{k}$ yields a pair $(\overline{\mathcal{E}}^{\circ}, \mathrm{Fr}_{\mathcal{E}^{\circ}})$, where $\overline{\mathcal{E}}^{\circ}$ is a character sheaf on $\overline{G}$ and $\mathrm{Fr}_{\mathcal{E}^{\circ}} : \mathrm{Fr}_q^* \overline{\mathcal{E}}^{\circ} \xrightarrow{\sim} \overline{\mathcal{E}}^{\circ}$.

Proposition

When $G = \mathfrak{T}_d^{\circ}$, base change defines an equivalence of categories

$$\left\{ \begin{array}{c} \textit{character sheaves} \\ \textit{on } \mathfrak{T}_d^{\circ} \end{array} \right\} \to \left\{ \textit{pairs } (\overline{\mathcal{E}}^{\circ}, \mathrm{Fr}_{\mathcal{E}^{\circ}}) \right\}$$

Characters of the k-rational points

- Suppose $(\overline{\mathcal{E}}^{\circ}, \mathrm{Fr}_{\mathcal{E}^{\circ}})$ is a character sheaf on G. Define a character $\chi^{\circ}_{\mathcal{E}}$ of $G(k)$ by

  $$\chi^{\circ}_{\mathcal{E}}(x) = \mathrm{Tr}(\mathrm{Fr}_{\mathcal{E}^{\circ}}, \overline{\mathcal{E}}^{\circ}_{x})$$

  for $x \in G(k)$.
- Suppose χ is a character of $G(k)$. Define a character sheaf on G using the Lang isogeny $L(x) = x^{-1}\,\mathrm{Fr}_q(x)$,

  $$1 \to G(k) \to G \xrightarrow{L} G \to 1,$$

  together with the character χ of $G(k)$.

Theorem (Deligne, SGA 4.5)

The maps defined above are mutually inverse isomorphisms between character sheaves on G and $\mathrm{Hom}(G(k), \overline{\mathbb{Q}}_\ell^{\times})$.

Character Sheaves on $\mathfrak{T}_d$

Definition

A character sheaf $\mathcal{E}$ on $\mathfrak{T}_d$ is a character sheaf

$$\mathcal{E}^\circ = (\overline{\mathcal{E}}^\circ, \mathrm{Fr}_{\mathcal{E}^\circ})$$

on $\mathfrak{T}_d^\circ$ plus an action of

$$X_*(\mathbf{T})_{\mathcal{I}_K} \rtimes W_k$$

on $\overline{\mathcal{E}}^\circ$, compatible with $\mathrm{Fr}_{\mathcal{E}^\circ}$.

Characters of $\mathfrak{T}_d(k)$

Suppose now that $\mathbf{T}$ is unramified so that

$$1 \to \mathbf{T}_R^\circ \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to 1$$

splits. A splitting defines an extension of $\chi_{\mathcal{E}}^\circ$ from $\mathfrak{T}_d^\circ(k)$ to

$$\mathfrak{T}_d(k) = \mathbf{T}(K)/\mathbf{T}_R(R)_{d+}.$$

From the action of $X_*(\mathbf{T})_{\mathcal{I}_K}$ on $\mathcal{E}^\circ$ one can produce a character of

$$(X_*(\mathbf{T})_{\mathcal{I}_K})^{W_k} = \mathbf{T}(K)/\mathbf{T}_R^\circ(R).$$

Thus we may associate to $\mathcal{E}$ a depth d character $\chi_{\mathcal{E}}$ of $\mathbf{T}(K)$: the product of these two.

Characters of $\pi_0(\mathfrak{T}_d)$

$$\pi_0(\mathfrak{T}_d) = \coprod_{a \in X_*(\mathbf{T})_{\mathcal{I}_K}} \mathrm{Spec}(\bar{k}) + \mathrm{Gal}(\bar{k}/k)\text{-action.}$$

Given a character sheaf $\mathcal{E}$ on $\mathfrak{T}_d$, we construct a Weil sheaf $\mathcal{E}_0$ on $\pi_0(\mathfrak{T}_d)$ by setting the stalk at any geometric point to be the stalk of $\overline{\mathcal{E}}^{\circ}$ at the identity, and defining an action of Frobenius via the action of $X_*(\mathbf{T})_{\mathcal{I}_K} \rtimes W_k$ given with $\mathcal{E}$.
Trace of Frobenius then defines a character of

$$\pi_0(\mathfrak{T}_d)(k) = X_*(\mathbf{T})_{\mathcal{I}_K}^{W_k}.$$

Invisible Character Sheaves

The orbits of W_k on $\pi_0(\mathfrak{T}_d)(\bar{k})$ are given by $(X_*(\mathbf{T})_{\mathcal{I}_K})_{W_k}$ and character sheaves on $\pi_0(\mathfrak{T}_d)$ correspond to characters of $(X_*(\mathbf{T})_{\mathcal{I}_K})_{W_k}$. The passage to characters is then given by pullback along the composition

$$X_*(\mathbf{T})_{\mathcal{I}_K}^{W_k} \hookrightarrow X_*(\mathbf{T})_{\mathcal{I}_K} \twoheadrightarrow (X_*(\mathbf{T})_{\mathcal{I}_K})_{W_k}.$$

Definition

We say that a character sheaf $\mathcal{E}_0$ on $\pi_0(\mathfrak{T}_d)$ is *invisible* if the corresponding character is trivial.

Existence of Invisible Character Sheaves

Let Y be the cokernel of the composition

$$(X_*(\mathbf{T})_{\mathcal{I}_K})^{W_k} \hookrightarrow X_*(\mathbf{T})_{\mathcal{I}_K} \twoheadrightarrow (X_*(\mathbf{T})_{\mathcal{I}_K})_{W_k}.$$

A character sheaf on $\pi_0(\mathfrak{T}_d)$ is invisible if and only if it factors through Y.

Remark

Y is trivial if $\mathbf{T}$ is split or totally ramified. But $Y \simeq \mathbb{Z}/2\mathbb{Z}$ when $\mathbf{T}$ is an unramified U_1 for example.

Characters of $\mathbf{T}(K)$

We define a character sheaf on $\mathfrak{T}$ as the pullback of a character sheaf on $\mathfrak{T}_d$ under the projection $\mathfrak{T} \to \mathfrak{T}_d$ for some d.
A character of $\mathbf{T}(K)$ is *smooth* if it has depth d for some d: it factors through the quotient $\mathbf{T}(K)/\mathbf{T}(K)_{d+}$.

Theorem

The map

$$\{\textit{character sheaves on } \mathfrak{T}\} \to \mathrm{Hom}_{\mathrm{sm}}(\mathbf{T}(K), \overline{\mathbb{Q}}_\ell^\times)$$

is surjective with fibers parameterized by $\mathrm{Hom}(Y, \overline{\mathbb{Q}}_\ell^\times)$.

Local class field theory

Suppose that L/K is a totally ramified abelian extension of local fields and we're given a character of $\mathrm{Gal}(L/K)$. The Artin reciprocity map gives a character of $K^\times$ vanishing on $\mathrm{Nm}_{L/K}(L^\times)$. We'd like to give a different description of this map, passing through character sheaves. Let $\mathbf{T} = \mathbb{G}_m$ and $\mathfrak{T}$ the Greenberg transform of $\mathbf{T}_R$.

An Isogeny

- U_K – the connected Néron model of $\mathbb{G}_m$.
- U_L – the connected Néron model of $\mathrm{Res}_{L/K}\,\mathbb{G}_m$.
- H – the kernel of $\mathrm{Nm}_{L/K}\colon U_L \to U_K$.
- H_0 – the subgroup of H generated by $\frac{\sigma(u)}{u}$ for $\sigma \in \mathrm{Gal}(L/K)$ and $u \in U_L$.

$$\begin{tikzcd}
 & H_0 \arrow[d] & H_0 \arrow[d] & & \\
1 \arrow[r] & H \arrow[r] \arrow[d] & U_L \arrow[r] \arrow[d] & U_K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H/H_0 \arrow[r] & U_L/H_0 \arrow[r] & U_K \arrow[r] & 1
\end{tikzcd}$$

A Character of $\mathcal{O}_K^\times$

The Greenberg transform is exact on commutative algebraic groups, so we get a finite étale cover of $\mathfrak{T}^\circ$. Write $\mathfrak{T}_L^\circ$ for the Greenberg transform of U_L/H_0, and note that $H/H_0 \cong \mathrm{Gal}(L/K)$. Then the sequence

$$1 \to \mathrm{Gal}(L/K) \to \mathfrak{T}_L^\circ \to \mathfrak{T}^\circ \to 1,$$

together with a character of $\mathrm{Gal}(L/K)$, yields a character sheaf on $\mathfrak{T}^\circ$. From this character sheaf, we can recover a character of $\mathcal{O}_K^\times$.

Local Langlands

$\mathbf{G}$ – connected quasi-split reductive group over K
E – splitting field of $\mathbf{G}$
$\hat{\mathbf{G}}$ – dual group over $\overline{\mathbb{Q}}_\ell$
${}^L\mathbf{G}$ – $\hat{\mathbf{G}} \rtimes \mathrm{Gal}(E/K)$
φ – a tame discrete Langlands parameter $W_K \to {}^L\mathbf{G}$

A construction of DeBacker and Reeder produces from φ an unramified anisotropic torus $\mathbf{T}$ in $\mathbf{G}$ and a depth 0 character χ of $\mathbf{T}(K)$. They then describe supercuspidal representations of $\mathbf{G}(K)$ as compact inductions of Deligne-Lusztig representations determined by $\mathbf{T}$ and χ.

Geometrizing Local Langlands

In contrast to the Néron model of $\mathbf{T}$, there's no canonical integral model of $\mathbf{G}$. Instead there are many models, parameterized by the Bruhat-Tits building of $\mathbf{G}$. We hope to obtain "representation sheaves" on the Greenberg transforms of these models from character sheaves on $\mathfrak{T}$ by an analogue of Lusztig induction. Ideally, this process would allow

- the generalization of DeBacker and Reeder's methods beyond the depth 0 case,
- better understanding of the functoriality of the local Langlands correspondence,
- new descriptions of L-packets.

Clifton and I are currently pursuing these questions.

Questions

- Is
$$1 \to \mathbf{T}_R^\circ \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to 1$$
split for ramified tori? Is there a natural description of the splittings?
- Do you have questions for me?