The Local Langlands Correspondence and character sheaves

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Outline

What is the Langlands Correspondence?

- A generalization of class field theory to non-abelian extensions.
- A tool for studying L-functions.
- A correspondence between representations of Galois groups and representations of algebraic groups.

Local Class Field Theory

Irreducible 1-dimensional representations of $\mathcal{W}_{\mathbb{Q}_p}$

$$\updownarrow$$

Irreducible representations of $\mathrm{GL}_1(\mathbb{Q}_p)$

The 1-dimensional case of local Langlands is local class field theory.

Conjecture

Irreducible n-dimensional representations of $\mathcal{W}_{\mathbb{Q}_p}$

$$\updownarrow$$

Irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$

In order to make this conjecture precise, we need to modify both sides a bit.

Smooth Representations

For $n > 1$, the representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ that appear are usually infinite dimensional.

Definition

A *smooth* $\mathbb{C}$*-representation* of $\mathrm{GL}_n(\mathbb{Q}_p)$ is a pair (π, V), where

- V is a $\mathbb{C}$-vector space (possibly infinite dimensional),
- $\pi\colon\ \mathrm{GL}_n(\mathbb{Q}_p) \to \mathrm{GL}(V)$ is a homomorphism,
- The stabilizer of each $v \in V$ is open in $\mathrm{GL}_n(\mathbb{Q}_p)$.

The only finite-dimensional irreducible smooth π are

$$g \mapsto \chi(\det(g))$$

for some character $\chi\colon \mathbb{Q}_p^\times \to \mathbb{C}^\times$.

Langlands Parameters

We also need to clarify what kinds of representations of $\mathcal{W}_{\mathbb{Q}_p}$ to focus on.

Definition

A *Langlands parameter* is a pair (φ, V) with

$$\varphi \colon \mathcal{W}_{\mathbb{Q}_p} \to \mathrm{GL}(V) \qquad \dim_{\mathbb{C}} V = n$$

such that φ is continuous and semisimple.

Parabolic Subgroups

Given a number of Langlands parameters $\varphi_i : \mathbf{W}_{\mathbb{Q}_p} \to \mathrm{GL}(V_i)$, one can form their direct sum. There should be a corresponding operation on the $\mathrm{GL}_n(\mathbb{Q}_p)$ side.

Definition

A *parabolic subgroup* of GL_n is a subgroup P conjugate to one consisting of block triangular matrices of a given pattern. For example:

$$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}$$

Such a subgroup has a Levi decomposition $P = M \ltimes N$, where M is conjugate to the corresponding subgroup of block diagonal matrices, and N consists of the subgroup of P with identity blocks on the diagonal.

Parabolic Induction

Since each Levi subgroup M is just a direct product of GL_{n_i}, a collection of representations $\pi_i \colon \mathrm{GL}_{n_i}(\mathbb{Q}_p) \to \mathrm{GL}(V_i)$ yields a representation $\boxtimes_i \pi_i$ of M. We can pull this back to P and then induce to obtain

$$\pi = \mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{Q}_p)} \boxtimes_i \pi_i.$$

Definition

We say that π is the *parabolic induction* of the π_i. We say that π is *supercuspidal* if π is not parabolically induced from any proper parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$.

The Weil-Deligne Group

There is a natural bijection

Supercuspidal representations of $\mathrm{GL}_n(\mathbb{Q}_p)$	$\leftrightarrow$	n-dimensional irreducible representations of $\mathcal{W}_{\mathbb{Q}_p}$.

But the parabolic induction of irreducible representations does not always remain irreducible. To extend this bijection from supercuspidal representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ to all smooth irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$, one enlarges the right hand side using the following group:

$$\mathrm{WD}_{\mathbb{Q}_p} := \mathcal{W}_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}).$$

Theorem (Local Langlands for GL_n: Harris-Taylor, Henniart)

There is a unique system of bijections

$$\left\{\textit{Irreducible representations of } \mathrm{GL}_n(\mathbb{Q}_p)\right\} \xrightarrow{\mathrm{rec}_n} \left\{\textit{n-dimensional irreducible representations of } \mathrm{WD}_{\mathbb{Q}_p}\right\}$$

- rec_1 *is induced by the Artin map of local class field theory.*
- rec_n *is compatible with* 1*-dimensional characters:* $\mathrm{rec}_n(\pi \otimes \chi \circ \det) = \mathrm{rec}_n(\pi) \otimes \mathrm{rec}_1(\chi)$.
- *The central character* ω_π *of* π *corresponds to* $\det \circ \mathrm{rec}_n$*:* $\mathrm{rec}_1(\omega_\pi) = \det(\mathrm{rec}_n(\pi))$.
- $\mathrm{rec}_n(\pi^\vee) = \mathrm{rec}_n(\pi)^\vee$
- rec_n *respects natural invariants associated to each side, namely L-factors and* ϵ*-factors of pairs.*

A First Guess

Now suppose **G** is some other connected reductive group defined over $\mathbb{Q}_p$, such as SO_n, Sp_n or U_n. We'd like to use a Langlands correspondence to understand representations of $\mathbf{G}(\mathbb{Q}_p)$ in terms of Galois representations. Something like

Homomorphisms $\varphi\colon \mathrm{WD}_{\mathbb{Q}_p} \to \mathbf{G}(\mathbb{C})$	$\leftrightarrow$	Irreducible representations of $\mathbf{G}(\mathbb{Q}_p)$.

We need to modify this guess in two ways:

- change $\mathbf{G}(\mathbb{C})$ to a related group, ${}^L\mathbf{G}(\mathbb{C})$,
- and account for the fact that our correspondence is no longer a bijection.

Root Data

Reductive groups over algebraically closed fields are classified by root data

$$(X^*(\mathbf{S}), \Phi(\mathbf{G}, \mathbf{S}), X_*(\mathbf{S}), \Phi^\vee(\mathbf{G}, \mathbf{S})),$$

where

- $\mathbf{S} \subset \mathbf{G}$ is a maximal torus,
- $X^*(\mathbf{S})$ is the lattice of characters $\chi \colon \mathbf{S} \to \mathbb{G}_m$,
- $X_*(\mathbf{S})$ is the lattice of cocharacters $\lambda \colon \mathbb{G}_m \to \mathbf{S}$,
- $\Phi(\mathbf{G}, \mathbf{S})$ is the set of roots (eigenvalues of the adjoint action of $\mathbf{S}$ on $\mathfrak{g}$),
- $\Phi^\vee(\mathbf{G}, \mathbf{S})$ is the set of coroots ($\langle \alpha, \alpha^\vee \rangle = 2$).

Connected Langlands Dual

Given $\mathbf{G} \supset \mathbf{S}$, the connected Langlands dual group $\hat{\mathbf{G}}$ is defined to be the algebraic group over $\mathbb{C}$ with root datum

$$(X_*(\mathbf{S}), \Phi^\vee(\mathbf{G}, \mathbf{S}), X^*(\mathbf{S}), \Phi(\mathbf{G}, \mathbf{S})).$$

For semisimple groups, this has the effect of exchanging the long and short roots (as well as interchanging the simply connected and adjoint forms).

$\mathbf{G}$	GL_n	SL_n	PGL_n	Sp_{2n}	SO_{2n}	U_n
$\hat{\mathbf{G}}$	GL_n	PGL_n	SL_n	SO_{2n+1}	SO_{2n}	GL_n

Langlands Dual Group

For non-split **G**, such as U_n, we need to work a little harder. Suppose that **G** is quasi-split with Borel $\mathbf{B} \supset \mathbf{S}$, splitting over a finite extension $E/\mathbb{Q}_p$. The fact that **B** is defined over $\mathbb{Q}_p$ implies that $\mathrm{Gal}(E/\mathbb{Q}_p)$ acts on the root datum. The connected dual group $\hat{\mathbf{G}}$ comes equipped with maximal torus $\hat{\mathbf{S}}$ canonically dual to **S**. By choosing basis vectors for each (1-dimensional) root space in the Lie algebra of $\hat{\mathbf{G}}$, we can extend the action of $\mathrm{Gal}(E/\mathbb{Q}_p)$ from the root datum to an action on $\hat{\mathbf{G}}$.
Define

$$^L\mathbf{G} := \hat{G} \rtimes \mathrm{Gal}(E/\mathbb{Q}_p),$$

the L-group of **G**.

Unitary Groups

A unitary group over $\mathbb{Q}_p$ is specified by the following data:

- $E/\mathbb{Q}_p$ a quadratic extension (so for $p \neq 2$ there are three possibilities),
- set $\tau \in \mathrm{Gal}(E/\mathbb{Q}_p)$ the nontrivial element,
- V an n-dimensional E-vector space,
- Non-degenerate Hermitian form $\langle , \rangle$ (so $\langle x, y \rangle = \tau\langle y, x \rangle$).

Then $\mathrm{U}(V)$ is the group of automorphisms of V preserving $\langle , \rangle$. Over $\bar{\mathbb{Q}}_p$, U becomes isomorphic to GL_n, so $\hat{\mathrm{U}}_n$ is GL_n, but ${}^L\mathbf{G}$ is non-connected: τ acts on $\mathrm{GL}_n(\mathbb{C})$ by the outer automorphism

$$g \mapsto (g^{-1})^{\mathrm{T}}.$$

Langlands Parameters

A Langlands parameter is now an equivalence class of homomorphisms

$$\varphi\colon \mathrm{WD}_{\mathbb{Q}_p} \to {}^L\mathbf{G}.$$

- We require that the composition of φ with the projection ${}^L\mathbf{G} \to \mathrm{Gal}(E/\mathbb{Q}_p)$ agrees with the standard projection $\mathcal{W}_{\mathbb{Q}_p} \to \mathrm{Gal}(E/\mathbb{Q}_p)$.
- We consider two parameters to be equivalent they are conjugate by an element of $\hat{\mathbf{G}}$. This definition of equivalence is chosen to match up with the notion of isomorphic representations on the $\mathbf{G}(\mathbb{Q}_p)$ side.

A Map

Conjecture

There is a natural map

Irreducible representations of $\mathbf{G}$ $\rightarrow$ Langlands parameters $\varphi\colon\ \mathrm{WD}_{\mathbb{Q}_p} \rightarrow {}^L\mathbf{G}$

It is surjective and finite-to-one; the fibers are called *L-packets*.

L-packets

Moreover, we can naturally parameterize these fibers. Given a Langlands parameter φ, let $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)$ be the centralizer in $\hat{\mathbf{G}}$ of φ, and let ${}^LZ$ be the center of ${}^L\mathbf{G}$. Define

$$A_\varphi = \pi_0(\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)/{}^LZ).$$

The fibers should be in bijection with

$$A_\varphi^\vee = \{\text{irreducible representations of } A_\varphi\}.$$

So we get a natural bijection

Irreducible representations of $\mathbf{G}$	$\leftrightarrow$	(φ, ρ) with $\varphi\colon \mathrm{WD}_{\mathbb{Q}_p} \to {}^L\mathbf{G}$ and $\rho \in A_\varphi^\vee$

Approaches to Local Langlands

- One approach to proving the local Langlands correspondence for general $\mathbf{G}$ is to try to reduce to the GL_n case: the recent book of Jim Arthur for example.
- Another approach is that of Stephen DeBacker and Mark Reeder, outlined below.

Assumptions

- Let **G** be a connected reductive group defined over $\mathbb{Q}_p$, and assume that **G** splits over an unramified extension $E/\mathbb{Q}_p$.
- Let φ be a Langlands parameter vanishing on $SL_2(\mathbb{C})$.
- Assume that φ is *tame*: it vanishes on wild inertia.
- Assume that φ is *discrete*: the centralizer of φ in $\hat{\mathbf{G}}$ is finite modulo the center of ${}^L\mathbf{G}$.
- Assume that φ is *regular*: the image of inertia is generated by a semisimple element of $\hat{\mathbf{G}}$ whose centralizer is a maximal torus $\hat{\mathbf{S}}$.

DeBacker-Reeder produce an L-packet that satisfies many of the properties expected of the local Langlands correspondence.

DeBacker and Reeder's approach

For each $\lambda \in X^*(\hat{\mathbf{S}})$ they construct

- F_λ, a twisted action of Frobenius on $\mathbf{G}$, and
- π_λ, a representation of $\mathbf{G}^{\mathrm{F}_\lambda}$, the $\mathbb{Q}_p$-points of the pure inner form of $\mathbf{G}$ determined by F_λ.

They define an equivalence relation on such pairs, and prove that the equivalence class of $(\pi_\lambda, \mathrm{F}_\lambda)$ depends only on the class of λ in

$$X^*(\hat{\mathbf{S}})/(1 - w\theta)X^*(\hat{\mathbf{S}}) \cong A_\varphi^\vee$$

where $w\theta$ is the automorphism of $X^*(\hat{\mathbf{S}})$ induced by $\varphi(\mathrm{F})$. They thus obtain an L-packet as the set of such equivalence classes for a fixed φ.

The Construction of π_λ

- Let t_λ be translation by λ in the apartment $\mathcal{A}$ associated to $\mathbf{S}$ in the Bruhat-Tits building of $\mathbf{G}$. By the discreteness of φ, the automorphism $t_\lambda w\theta$ has a unique fixed point x_λ in $\mathcal{A}$.
- Find another decomposition

$$t_\lambda w\theta = w_\lambda y_\lambda \theta,$$

  where w_λ lies in the "parahoric subgroup" of the affine Weyl group at x_λ, and $y_\lambda\theta$ fixes an alcove with closure containing x_λ.
- From y_λ define a 1-cocycle u_λ, from which $\mathrm{F}_\lambda = \mathrm{Ad}(u_\lambda) \circ \mathrm{F}$. Note that x_λ is a vertex of $\mathcal{B}(\mathbf{G}^{\mathrm{F}_\lambda})$.
- From w_λ define an anisotropic torus $\mathbf{T}_\lambda$ of $\mathbf{G}$ with $\mathbf{T}_\lambda^{\mathrm{F}_\lambda} \subset \mathbf{G}_\lambda$.

The Construction of π_λ (cont.)

- Apply a canonical modification to φ so that the image lies in a group isomorphic to ${}^L\mathbf{T}_\lambda$.
- Obtain a character of $\mathbf{T}_\lambda(\mathbb{F}_p)$ using the (depth-preserving) local Langlands correspondence for tori.
- Use Deligne-Lusztig theory to produce an irreducible representation of $\mathbf{G}_\lambda$.
- Compactly induce to $\mathbf{G}(\mathbb{Q}_p)$, yielding a depth zero supercuspidal representation π_λ.

L-packets

They then prove that $\mathbf{G}(\mathbb{Q}_p)$ acts on the pairs $(\mathrm{F}_\lambda, \pi_\lambda)$, and the orbit of a given pair is independent of all choices. Moreover, two such pairs are equivalent if and only if the two λs represent the same class in $A_\varphi^\vee$. Much of their paper is then devoted to proving that this construction yields L-packets with desirable properties:

- The ratio of formal degrees $\deg(\pi_\lambda)/\deg(\mathrm{St}_\lambda)$ is independent of λ.
- Generic representations in the L-packet correspond to hyperspecial vertices in the building.
- Their L-packet yields a stable class function on the set of strongly regular semisimple elements of $\mathbf{G}(\mathbb{Q}_p)$.

Restrictions on φ

From now on we fix a totally ramified quadratic extension $E/\mathbb{Q}_p$ and set $\mathbf{G} = \mathrm{U}(V)$ for V a quasi-split Hermitian space over E. We say that a Langlands parameter φ is

- *discrete* if $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)$ is finite,
- *tame* if φ factors through the maximal tame quotient (and thus $p \neq 2$).
- *regular* if $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau}))$ is connected and minimum dimensional (here $\tilde{\tau}$ is a procyclic generator of tame inertia).

We will construct an L-packet of supercuspidal representations of pure inner forms of $\mathbf{G}(\mathbb{Q}_p)$ given a tame, discrete regular parameter.

Filtrations

$\mathbf{G}(\mathbb{Q}_p)$ acts on the Bruhat-Tits building $\mathcal{B}(\mathbf{G})$, and we can classify the compact subgroups of $\mathbf{G}(\mathbb{Q}_p)$ as stabilizers of convex subsets of $\mathcal{B}(\mathbf{G})$

- Each such compact $\mathbf{H}$ has the structure of a $\mathbb{Z}_p$-scheme.
- There is a decreasing filtration on each $\mathbf{H}$.
- $\mathbf{H}^0$ is just the connected component of the identity (as a $\mathbb{Z}_p$-scheme) and is of finite index in $\mathbf{H}$.
- The special fiber $\mathbf{H}(\mathbb{F}_p)$ is given by $\mathbf{H}/\mathbf{H}^{0+}$.
- The filtration on $\mathbf{T}$ is the one given by Moy and Prasad, coming from the filtration on $\mathbb{Q}_p^\times$.

We can thus obtain representations of compact subgroups of $\mathbf{G}$ by pulling back representations of reductive groups over finite fields.

Outline

Our plan for constructing an L-packet from φ is as follows. We construct:

- A maximal unramified anisotropic torus **T**, which embeds into **G** in various ways,
- A character χ_φ on $\mathbf{T}^0$ that vanishes on $\mathbf{T}^{0+}$,
- For each $\rho \in A_\varphi^\vee$, an embedding of **T** into a maximal compact subgroup $\mathbf{H} \subset \mathbf{G}$.
- We get a Deligne-Lusztig representation of $\mathbf{H}^0(\mathbb{F}_p) = \mathbf{H}^0/\mathbf{H}^{0+}$ associated to the torus $\mathbf{T}^0(\mathbb{F}_p) = \mathbf{T}^0/\mathbf{T}^{0+}$ and the character χ_φ.
- We induce this representation up to a representation of **G**.

Structure of a Tame Parameter

The tame Weil group is topologically generated by two elements: an (arithmetic) Frobenius F and a generator $\tilde{\tau}$ of the procyclic group

$$\mathcal{I}_{\mathbb{Q}_p} = \mathrm{Gal}(\varinjlim \tilde{K}(p^{1/m})/\tilde{K}) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell.$$

- The assumption that $E/\mathbb{Q}_p$ is totally ramified implies that $\varphi(\mathrm{F}) \in \hat{\mathbf{G}}$, while $\varphi(\tilde{\tau}) \in {}^L\mathbf{G}$ projects to $\tau \in \mathrm{Gal}(E/\mathbb{Q}_p)$.
- Recall that we have a specified maximal torus $\hat{\mathbf{S}}$ in ${}^L\mathbf{G}$. As Langlands parameters are defined only up to conjugacy, we may conjugate so that $\varphi(\tilde{\tau}) \in \hat{\mathbf{S}}^\tau \rtimes \mathrm{Gal}(E/\mathbb{Q}_p)$.

A Twisted Torus

- The equality
$$\mathrm{F}\,\tilde{\tau}\,\mathrm{F} = \tilde{\tau}^p$$
implies that $\varphi(\mathrm{F})$ lies in the normalizer of $\varphi(\tilde{\tau})$, and thus in the normalizer of $\hat{\mathbf{S}}$.
- Composing with the projection onto the Weyl group, we get a cocycle in
$$\mathrm{H}^1(\langle \mathrm{F} \rangle, W^{\mathcal{I}}) \hookrightarrow \mathrm{H}^1(\mathbb{Q}_p, W).$$
- Such a cocycle is precisely the data needed to define a torus over $\mathbb{Q}_p$ as a twist of $\mathbf{S}$: here we've identified the Weyl groups of $\mathbf{S}$ and $\hat{\mathbf{S}}$. Write $\mathbf{T}$ for this torus.

Unramified and Anisotropic

- $\mathbf{T}$ cannot literally be unramified, since no torus in $\mathbf{G}$ splits over an unramified extension. But it does become isomorphic to the canonical torus $\mathbf{S}$ after an unramified extension: we will call such tori in $\mathbf{G}$ *unramified*.
- A torus $\mathbf{T}$ is called *anisotropic* if $X_*(\mathbf{T})^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} = 0$, or equivalently if $\mathbf{T}(\mathbb{Q}_p)$ is compact. The action of inertia on $\mathbf{T}$ is the same as on $\hat{\mathbf{S}}$, so any invariants in $X_*(\mathbf{T})$ would yield invariants in $X_*(\mathbf{S}^\tau)$ under the action of $\varphi(\mathrm{F})$. But any such invariants would contradict our assumption that φ is discrete, since
$$(\hat{\mathfrak{g}}^{\mathcal{I}})^{\mathrm{F}} = 0.$$
Thus $\mathbf{T}$ is anisotropic.

Image of a Parameter

- Since the tame Weil group is topologically generated by F and $\tilde{\tau}$, the image of φ is contained in $\mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{S}}) \rtimes \mathrm{Gal}(E/\mathbb{Q}_p)$. In fact, it is contained in the subgroup D of ${}^L\mathbf{G}$ generated by $\hat{\mathbf{S}} \rtimes \mathrm{Gal}(E/\mathbb{Q}_p)$ and $\varphi(\mathrm{F})$.
- The minimal splitting field $M = \mathbb{Q}_{p^s} \cdot E$ of $\mathbf{T}$ has Galois group

$$\mathrm{Gal}(M/\mathbb{Q}_p) \cong \mathrm{Gal}(E/\mathbb{Q}_p) \times \langle w \rangle,$$

where $w \in \mathbf{W}^{\mathcal{I}}$ is the image of $\varphi(\mathrm{F})$. Thus D fits into an exact sequence

$$1 \to \hat{\mathbf{S}} \to D \to \mathrm{Gal}(M/\mathbb{Q}_p) \to 1.$$

A Character

- Suppose that this sequence split and $D \cong \hat{\mathbf{T}} \rtimes \mathrm{Gal}(M/\mathbb{Q}_p)$. Then φ would yield an element of $\mathrm{H}^1(\mathbb{Q}_p, \hat{\mathbf{T}})$, and the local Langlands correspondence for tori would give us a character of $\mathbf{T}(\mathbb{Q}_p)$:

$$\mathrm{H}^1(\mathbb{Q}_p, \hat{\mathbf{T}}) \cong \mathrm{Hom}(\mathbf{T}(\mathbb{Q}_p), \mathbb{C}^\times).$$

- In general the sequence for D does not split. So our next task is to modify the Langlands correspondence for tori to obtain a character in the non-split case. We will obtain a character χ_φ of $\mathbf{T}^0(\mathbb{Q}_p)$, where $\mathbf{T}^0$ is the connected component in the Néron model of $\mathbf{T}$.

Restriction to $\mathrm{Gal}(\mathbb{Q}_{p^s}/\mathbb{Q}_p)$

- Let $P_K(D, \mathbf{T})$ be the set of homomorphisms from $\mathrm{Gal}(\bar{K}/K)$ to D that project correctly onto $\mathrm{Gal}(M/\mathbb{Q}_p)$, modulo conjugacy by $\hat{\mathbf{T}}$. If D were a semidirect product then we would have $P_K(D, \mathbf{T}) \cong \mathrm{H}^1(\mathbb{Q}_p, \hat{\mathbf{T}})$.
- Set D_s as the preimage in D of $\mathrm{Gal}(M/\mathbb{Q}_{p^s})$ and let $\Gamma = \mathrm{Gal}(\mathbb{Q}_{p^s}/\mathbb{Q}_p)$. The splitting of ${}^L\mathbf{G} = \hat{\mathbf{G}} \rtimes \mathrm{Gal}(E/\mathbb{Q}_p)$ yields a splitting of
$$1 \to \hat{\mathbf{S}} \to D_s \to \mathrm{Gal}(M/\mathbb{Q}_{p^s}) \to 1.$$
- The restriction map of group cohomology
$$\mathrm{H}^1(\mathbb{Q}_p, \hat{\mathbf{T}}) \to \mathrm{H}^1(\mathbb{Q}_{p^s}, \hat{\mathbf{T}})^\Gamma$$
generalizes to a map
$$P_{\mathbb{Q}_p}(D, \mathbf{T}) \to P_{\mathbb{Q}_{p^s}}(D_s, \mathbf{T})^\Gamma$$

Descending back to $\mathbb{Q}_p$

- We can now obtain a character χ_φ as the image of φ under the composition

$$P_{\mathbb{Q}_p}(D, \mathbf{T}) \xrightarrow{\text{res}} P_{\mathbb{Q}_{p^s}}(D_s, \mathbf{T})^\Gamma \cong \mathrm{H}^1(\mathbb{Q}_{p^s}, \hat{\mathbf{T}})^\Gamma$$
$$\cong \mathrm{Hom}(\mathbf{T}(\mathbb{Q}_{p^s})_\Gamma, \mathbb{C}^\times).$$

- From Tate cohomology we have

$$1 \to \hat{\mathrm{H}}^{-1}(\Gamma, \mathbf{T}) \to \mathbf{T}(\mathbb{Q}_{p^s})_\Gamma \to \mathbf{T}(\mathbb{Q}_p) \to \hat{\mathrm{H}}^0(\Gamma, \mathbf{T}) \to 1$$

When the Néron model of $\mathbf{T}$ is not connected, these outer groups can be nontrivial. We get around this issue by restricting χ_φ to $\mathbf{T}^0(\mathbb{Q}_p)$, a finite index subgroup of $\mathbf{T}(\mathbb{Q}_p)$.

Depth of Character

- Using Lang's theorem on the cohomology of connected algebraic groups over finite fields, the corresponding outer terms for $\mathbf{T}^0$ vanish. The isomorphism $\mathbf{T}^0(\mathbb{Q}_{p^s})_\Gamma \cong \mathbf{T}^0(\mathbb{Q}_p)$ associates to φ a character of $\mathbf{T}^0(\mathbb{Q}_p)$, which we will also denote by χ_φ.
- Since φ vanished on wild inertia, the depth-preservation properties of the local Langlands correspondence for tori imply that χ_φ vanishes on $\mathbf{T}^{0+}(\mathbb{Q}_p)$, and thus induces a character of $\mathbf{T}^0(\mathbb{F}_p)$.
- The regularity of φ implies that χ_φ is not fixed by any element of $\mathbf{W}^{\mathcal{I}}$: it is in "general position."

Summary

From a Langlands parameter φ we've produced:

- An anisotropic unramified torus **T**. Note that **T** is not yet provided with an embedding into **G**.
- A character χ_φ of $\mathbf{T}^0(\mathbb{F}_p)$.

In order to produce representations of $\mathbf{G}(\mathbb{Q}_p)$ we need to understand the embeddings of **T** into **G**.

Basic Tori

We classify unramified anisotropic twists of the "quasi-split" torus $\mathbf{S}$. For each $s = 2r$, define $\mathbf{T}_s = \{x \in E_s : \mathrm{Nm}_{E_s/L_r} x = 1\}$,

$$
\begin{tikzcd}
 & & E_s \arrow[dl, dash, "\mathrm{Gal}(E_s/E_r)=\langle \sigma_s^r \rangle"'] \arrow[d, dash, "\langle \eta_s \rangle"] \arrow[dr, dash, "\mathrm{Gal}(E_s/\mathbb{Q}_{p^s})=\langle \tau_s \rangle"] & \\
 & E_r \arrow[dl, dash] \arrow[dr, dash] & L_r \arrow[d, dash] & \mathbb{Q}_{p^s} \arrow[dl, dash] \\
E \arrow[dr, dash] & & \mathbb{Q}_{p^r} \arrow[dl, dash] & \\
 & \mathbb{Q}_p & &
\end{tikzcd}
$$

Every anisotropic unramified torus in $\mathbf{G}$ is a product of such basic tori, together with at most one copy of U_1.

Embeddings of Basic Tori

In order to get Deligne-Lustig representations, we need to embed **T** into maximal compacts of **G**. We do so by building a Hermitian space around each basic torus in the product decomposition of **T**.

For each $\kappa \in L_r^\times$, we define a Hermitian product on E_s

$$\phi_\kappa(x, y) = \mathrm{Tr}_{E_s/E}\left(\frac{\kappa}{\pi_L} x \cdot \eta_s(y)\right)$$

This Hermitian space is quasi-split if and only if $v_L(\kappa)$ is even. By the definition of $\mathbf{T}_s$ we have an embedding of $\mathbf{T}_s$ into $\mathrm{U}(E_s, \phi_\kappa)$.

Embeddings of General Tori

In general, we choose a κ_i for each basic torus in the decomposition of $\mathbf{T}$. This choice corresponds to a choice of $\rho \in A_\varphi^\vee$ as long as the sum of the valuations of the κ_i is even.

We prove $\mathbf{T}$ fixes a unique point on the building $\mathcal{B}(\mathbf{G})$ and thus embeds in a unique maximal compact $\mathbf{H} \subset \mathbf{G}$.
The reduction of $\mathbf{H}$ is

$$\mathrm{O}(m) \times \mathrm{Sp}(m'),$$

where m is the sum of the dimensions of basic tori whose κ_i has even valuation and m' is the sum of those with $v(\kappa_i)$ odd.

Constructing a representation of $\mathbf{G}(\mathbb{Q}_p)$

Modulo p, we have a maximal torus $\mathbf{T}^0(\mathbb{F}_p)$ sitting in a connected reductive group $\mathbf{H}^0(\mathbb{F}_p)$ and a character χ_φ of $\mathbf{T}^0(\mathbb{F}_p)$. This situation was studied by Deligne and Lusztig, and they produce a representation of $\mathbf{H}^0(\mathbb{F}_p)$ using étale cohomology. The irreducibility of this representation follows from the regularity condition on φ.

We pull back to $\mathbf{H}^0$ and the only wrinkle in the induction process occurs between $\mathbf{H}^0$ and $\mathbf{H}$. Once we have a representation of $\mathbf{H}$, we define a representation on all of $\mathbf{G}(\mathbb{Q}_p)$ by compact induction.

A Finite Induction

There are three cases for the induction from $\mathbf{H}^0$ to $\mathbf{H}$.

- n even, $\mathbf{H}(\mathbb{F}_p) = \mathrm{Sp}(n)$. Here $\mathbf{H} = \mathbf{H}^0$ and there is no induction.
- n even, otherwise. The fact that the normalizer of $\mathbf{T}^0(\mathbb{F}_p)$ in $\mathbf{H}(\mathbb{F}_p)$ contains the normalizer in $\mathbf{H}^0(\mathbb{F}_p)$ with index 2 implies that the induction remains irreducible.
- n odd. Now the induction from $\mathbf{H}^0$ to $\mathbf{H}$ splits into two irreducible components. We can pick one using a recipe for the central character, together with the fact that in the case that n is odd the center of $\mathrm{O}(m)$ is not contained in $\mathrm{SO}(m)$.

Two Paths

$$
\begin{tikzcd}
& \mathbf{T}^0(\mathbb{Q}_p) \xrightarrow{\chi_\varphi} \mathbb{C}^\times \arrow[dl, "\text{Deligne-Lustzig representation}"'] \arrow[dr, "\text{sheaf-function dictionary}"] & \\
\mathbf{H}^0(\mathbb{F}_p) \circlearrowright \mathrm{DL}(\chi_\varphi, \mathbf{T}^0/\mathbb{F}_p) \arrow[d, "\text{inflation}"'] & & \text{character sheaf on } \mathfrak{T}/\mathbb{F}_p \arrow[dd, "\text{Deligne-Lusztig induction}"] \\
\mathbf{H}^0(\mathbb{Q}_p) \circlearrowright \mathrm{DL}(\chi_\varphi, \mathbf{T}^0/\mathbb{F}_p) \arrow[d, "\text{finite induction}"'] & & \\
\mathbf{H}(\mathbb{Q}_p) \circlearrowright \mathrm{Ind}\,\mathrm{DL}(\chi_\varphi, \mathbf{T}^0/\mathbb{F}_p) \arrow[d, "\text{compact induction}"'] & & \text{perverse sheaf on } \mathfrak{G}/\mathbb{F}_p \arrow[d, "\text{nearby cycles or trace}"] \\
\mathbf{G}(\mathbb{Q}_p) \circlearrowright \mathrm{cInd}\,\mathrm{DL}(\chi_\varphi, \mathbf{T}^0/\mathbb{F}_p) \arrow[rr] & & \text{distribution on } \mathbf{G}(\mathbb{Q}_p)
\end{tikzcd}
$$

Current work

The remainder of this talk is

- joint with Clifton Cunningham
- a summary of work in progress.

The right hand side of the diagram outlines an alternate construction of a distribution on $\mathbf{G}(\mathbb{Q}_p)$ from a depth zero character on $\mathbf{T}^0(\mathbb{Q}_p)$ and an embedding $\mathbf{T} \hookrightarrow \mathbf{G}$.

Warning: no step on the right side is complete

For the remainder of this talk I will discuss the first arrow: the passage from a depth zero character of $\mathbf{T}$ to a character sheaf on a related scheme $\mathfrak{T}$.

The Néron model of $\mathbb{G}_m$

Now let $\mathbf{T} = \mathbb{G}_m$. The Néron model of $\mathbf{T}$ is a separated, smooth commutative group scheme $\mathbf{T}_{\mathbb{Z}_p}$ locally of finite type over $\mathbb{Z}_p$ with the Néron mapping property. In particular,

$$\mathbf{T}_{\mathbb{Z}_p}(\mathbb{Z}_p) = \mathbf{T}(\mathbb{Q}_p) = \mathbb{Q}_p^\times.$$

The earlier $\mathbf{T}^0$ is just the identity component of the Néron model, and in the $\mathbb{G}_m$ case the Néron model is a union of copies of $\mathbb{G}_m/\mathbb{Z}_p$, glued along the generic fiber.
Set $\mathbf{T}_d = \mathbf{T}_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} (\mathbb{Z}/p^{d+1}\mathbb{Z})$.

The Greenberg functor

The Greenberg functor Gr takes an affine group scheme over an Artinian local ring A and produces an affine group scheme over the residue field k whose k points are canonically identified with the A-points of the original scheme. We set

$$\mathfrak{T}_d = \mathrm{Gr}(\mathbf{T}_d)$$

and

$$\mathfrak{T} = \varprojlim \mathfrak{T}_d.$$

$\mathfrak{T}$ is a commutative group scheme over $\mathbb{F}_p$ with $\mathfrak{T}(\mathbb{F}_p) = \mathbb{Q}_p^\times$, but it is neither connected nor locally of finite type.

Character Sheaves

- An *ℓ-adic Weil local system* on a scheme X over K is a pair $(\bar{\mathcal{L}}, \phi_{\mathcal{L}})$, where $\bar{\mathcal{L}}$ is an ℓ-adic local system on the étale site of $X_{\bar{K}}$ and $\phi_{\mathcal{L}}$ is an action of $\mathrm{Gal}(\bar{K}/K)$ on $\bar{\mathcal{L}}$ compatible with the action on $X_{\bar{K}}$.
- An *ℓ-adic Weil character sheaf* on a group scheme G is an ℓ-adic Weil local system $\mathcal{L}$ on G satisfying

$$m^*(\mathcal{L}) \cong \mathcal{L} \boxtimes \mathcal{L}.$$

- An ℓ-adic Weil character sheaf on $\mathfrak{T}$ is *smooth of depth d* if it arises as the pullback from $\mathfrak{T}_d$ of an ℓ-adic Weil character sheaf (with d minimal).

Characters and Character Sheaves

Theorem

There is a canonical, depth preserving isomorphism between smooth characters of $\mathbf{T}(\mathbb{Q}_p) = \mathbb{Q}_p^\times$ *and smooth* ℓ*-adic Weil character sheaves on* $\mathfrak{T}$.

Summary

$$\begin{tikzcd}
\mathcal{W}_{\mathbb{Q}_p} \xrightarrow{\varphi} D \subset {}^L\mathbf{G} \arrow[r] \arrow[d] & \mathcal{W}_{\mathbb{Q}_p^r} \xrightarrow{\varphi} \hat{\mathbf{S}} \rtimes \mathrm{Gal}(E/\mathbb{Q}_p) \arrow[d] & \\
\text{a twist } \mathbf{T} \text{ of } \mathbf{S} \arrow[r] & \mathbf{T}^0(\mathbb{Q}_p) \xrightarrow{\chi_\varphi} \mathbb{C}^\times \arrow[dl] \arrow[dr] & \\
\mathbf{H}^0(\mathbb{F}_p) \circlearrowright \mathrm{DL}(\chi_\varphi, \mathbf{T}^0/\mathbb{F}_p) \arrow[d] & \mathbf{T} \subset \mathbf{G} \arrow[l] \arrow[dr, bend right] & \text{character sheaf on } \mathfrak{T}/\mathbb{F}_p \arrow[d] \\
\mathbf{H}^0(\mathbb{Q}_p) \circlearrowright \mathrm{DL}(\chi_\varphi, \mathbf{T}^0/\mathbb{F}_p) \arrow[d] & & \text{perverse sheaf on } \mathfrak{G}/\mathbb{F}_p \arrow[dd] \\
\mathbf{H}(\mathbb{Q}_p) \circlearrowright \mathrm{Ind}\,\mathrm{DL}(\chi_\varphi, \mathbf{T}^0/\mathbb{F}_p) \arrow[d] & & \\
\mathbf{G}(\mathbb{Q}_p) \circlearrowright \mathrm{cInd}\,\mathrm{DL}(\chi_\varphi, \mathbf{T}^0/\mathbb{F}_p) \arrow[rr] & & \text{distribution on } \mathbf{G}(\mathbb{Q}_p)
\end{tikzcd}$$

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