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The 3-adic eigencurve at the boundary of weight space

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This paper generalizes work of Buzzard and Kilford to the case $p = 3$, giving an explicit bound for the overconvergence of the quotient $E_\kappa/V(E_\kappa)$ and using this bound to prove that the eigencurve is a union of countably many annuli over the boundary of weight space.

Keywords: eigencurve; slopes of modular forms; overconvergent modular forms.

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1. Introduction

This paper grew out of Kevin Buzzard's course A Concrete Introduction to p -adic Modular Forms [2], part of the eigenvarieties semester at Harvard in spring 2006. It generalizes the results of Buzzard and Kilford [4] from the case $p = 2$ to $p = 3$.

The eigencurve, first constructed by Coleman and Mazur [5], parameterizes eigenvalues of the compact operator U on the space of overconvergent modular forms. In this paper, we prove that the 3-adic eigencurve consists of a countable disjoint union of annuli near the boundary of weight space, and compute the eigenvalues of U on these components of the eigencurve explicitly. Let \mathcal{W}° be the open disc over \mathbb{C}_3 with center 0 and radius 1, write $v(w_0)$ for the 3-adic valuation of $w_0 \in \mathcal{W}^\circ$ and set $|w_0| = 3^{-v(w_0)}$. To each $w_0 \in \mathcal{W}^\circ$ we may associate a unique weight $\kappa \in \text{Hom}(\mathbb{Z}_3^\times, \mathbb{C}_3^\times)$ with $\kappa(-1) = 1$ and $\kappa(4) = 1 + w_0$.

Theorem 1.1. *If κ is a weight corresponding to $w_0 \in \mathcal{W}^\circ$ with $1/3 < |w_0| < 1$, and if $v = v(w_0)$, then the slopes of U acting on overconvergent modular forms of weight κ are the arithmetic progression $0, v, 2v, 3v, 4v, \dots$, each appearing with multiplicity 1.*

In Section 2, we introduce notation that we will need, including definitions of the operators U and V and definitions of the modular forms that will play a crucial role in what follows. In Lemma 2.4, we prove fundamental relationships between the modular forms just defined. In the proof of this lemma we used the q -expansion principle, GP/PARI [1] and Sage [15] in order to obviate the need for a detailed analysis of the poles and zeroes of

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the various modular forms involved. The results stated in Lemma 2.4 constitute the part of the paper most likely to fail for other p . Conversely, if such results can be proved for other p , most of the rest of the paper would follow. Section 2 concludes with a corollary giving the action of U and V on various power series rings.

Section 3 begins the analysis of families of modular forms. We analyze \mathbf{T} , a family given by powers of a theta series, in order to gather information about the Eisenstein family. Using the results of the previous section, we consider various quotients of \mathbf{T} , $U(\mathbf{T})$, $V(\mathbf{T})$ and $VU(\mathbf{T})$ and prove that these quotients have specific degrees of overconvergence.

We consider the overconvergence of $\mathbf{E}/V(\mathbf{E})$ in Section 4, where \mathbf{E} is the Eisenstein family. In order to find the degree of overconvergence of $\mathbf{E}/V(\mathbf{E})$, we use a technique suggested by Buzzard that eliminates the need for some of the arguments in [4, §4-5]. From Coleman and Mazur [5], we know that $\mathbf{E}/V(\mathbf{E})$ is at least slightly overconvergent. We use the fact that U increases overconvergence, together with the explicit overconvergence bounds for the family \mathbf{T} found in Section 3, to show that $\mathbf{E}/V(\mathbf{E})$ extends an explicit distance into the supersingular discs.

In Section 5 we consider specializations of $\mathbf{E}/V(\mathbf{E})$ to weights κ near the boundary of weight space. If we expand $E_\kappa/V(E_\kappa)$ as a power series in y (a specific parameter on $X_0(9)$ defined in Section 2), then reduce modulo the maximal ideal, the resulting power series over a finite field does not depend on κ .

In Section 6 we find a description for the action of U on the 3-adic Banach space of overconvergent modular forms of weight κ . In particular, if $|c|$ is sufficiently close to 1 then $V(E_\kappa)(cy)^n$ forms a basis for this Banach space as n ranges over non-negative integers. We find a generating function that gives us the matrix of U with respect to this basis.

In Section 7 we find the valuations of the coefficients of the characteristic power series of U . The coefficients are given by determinants of submatrices of the matrix of U . We use the generating function from Section 6 to find a lower bound on the valuation of the coefficients. Finally, we use the results of Section 5 to prove that this inequality is actually an equality by showing that a certain determinant is a 3-adic unit. Knowledge of the valuations of the coefficients then gives us the proof of the main theorem.

Finally, in Section 8 we summarize some related work, both for $p = 3$ and larger p .

2. Preliminaries

Throughout this paper, we will conflate modular forms with their q -expansions in order to make the grammar easier to follow. Let ω be a primitive cube root of unity, and define $K = \mathbb{Q}_3(\omega)$. Set \mathcal{O}_K to be the ring of integers of K , let $\pi = \omega - 1$ be a uniformizer for \mathcal{O}_K , and let v_3 be the extension of the standard valuation on \mathbb{Q}_3 to K (ie $v_3(3) = 1$). Define u such that $3 = u\pi^2$. Let \mathbb{C}_3 be the completion of an algebraic closure of \mathbb{Q}_3 and $\mathcal{O}_{\mathbb{C}_3}$ be the ring of integers of \mathbb{C}_3 . On all of these fields, we have the norm $|x| = 3^{-v_3(x)}$.

All rings are commutative with unity, and if R is a ring we define two R -module homomorphisms U and $V: R[[q]] \rightarrow R[[q]]$ by:

$$U\left(\sum_{n=0}^{\infty} r_n q^n\right) = \sum_{n=0}^{\infty} r_{3n} q^n,$$

and

$$V\left(\sum_{n=0}^{\infty} r_n q^n\right) = \sum_{n=0}^{\infty} r_n q^{3n}.$$

One can easily check that V is an R -algebra homomorphism.

Lemma 2.1. *For all $g, h \in R[[q]]$, we have $U(gV(h)) = hU(g)$.*

Proof. This follows from a straightforward computation. □

Corollary 2.2. *If $h \in R[[q]]^\times$, then $V(h)$ is too, and $U(g/V(h)) = U(g)/h$.*

Proof. Apply Lemma 2.1 to g and h^{-1} and note that V is a ring homomorphism. □

We now define modular forms that will serve as analogues of those in [4, §2] for the $p = 3$ case.

For $k \geq 2$ an even integer, define

$$E_k := 1 + \frac{2}{(1 - 3^{k-1})\zeta(1 - k)} \sum_{n=1}^{\infty} \left(\sum_{\substack{0 < d|n \\ 3 \nmid d}} d^{k-1} \right) q^n,$$

where $\zeta(s)$ is the Riemann zeta function. Then E_k is a modular form of level 3 and weight k obtained, for $k \geq 4$, from the standard level 1 Eisenstein form of weight k by dropping an Euler factor. The form E_k is an eigenform for U .

The function

$$\Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

is a standard level 1 weight 12 modular form. Set

$$f = \sqrt{\frac{\Delta(q^3)}{\Delta(q)}} = q + 12q^2 + 90q^3 + 508q^4 + \dots,$$

a level 3 modular function giving an isomorphism $X_0(3) \rightarrow \mathbb{P}^1$ (this fact follows from the observation that $f = q \prod_{3 \nmid n} (1 - q^n)^{-12}$ has a simple zero at the cusp ∞ and no other zeroes).

Define

$$\theta := \sum_{(a,b) \in \mathbb{Z}^2} q^{a^2+ab+b^2} = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + \dots,$$

a level 3 weight 1 modular form that will play the rôle that the form E_2 (with $p = 2$) did in [4].

Proposition 2.3. *θ and θ^2 are eigenforms for the U operator with eigenvalue 1.*

Proof. Since $\mathcal{M}_2(\Gamma_0(3))$ is 1 dimensional [7, Thm. 3.5.1], and the square of any element of $\mathcal{M}_1(\Gamma_1(3))$ lies in $\mathcal{M}_2(\Gamma_0(3))$, $\mathcal{M}_1(\Gamma_1(3))$ is at most one dimensional. Thus θ and θ^2 are

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both eigenforms. The computation of the eigenvalue follows from the first few terms of the q -expansion. \square

Finally, define

$$y = \frac{\frac{\theta}{V(\theta)} - 1}{6} = q - 5q^4 + 32q^7 - 198q^{10} + 1214q^{13} - \dots,$$

a level 9 modular function giving an isomorphism $X_0(9) \rightarrow \mathbb{P}^1$.

We encapsulate the crucial facts about these modular forms in the following lemma. Using this lemma, we will then be able to proceed in the same fashion as Buzzard and Kilford in [4].

Lemma 2.4.

- (1) $U(y) = U(y^2) = 0$ and $U(y^3) = \frac{y(1+3y+9y^2)}{(1+6y)^3}$.
- (2) For $m \in \mathbb{Z}_{\geq 0}$ we have $U(y^{3m+1}) = U(y^{3m+2}) = 0$ and $U(y^{3m}) = \left(\frac{y(1+3y+9y^2)}{(1+6y)^3}\right)^m$.
- (3) $f = \frac{y(1+3y+9y^2)}{(1-3y)^3}$, and $U(f) = 10 \cdot 3^2 f + 4 \cdot 3^7 f^2 + 3^{11} f^3$ and $V(f) = \frac{y^3}{1-27y^3}$.

Proof.

- (1) By Proposition 2.3, θ and θ^2 are both eigenforms for U . Putting this together with Corollary 2.2 and the definition of y , we have that

$$6U(y) = U\left(\frac{\theta}{V(\theta)}\right) - 1 = \frac{U(\theta)}{\theta} - 1 = 0$$

and

$$\begin{aligned} 36U(y^2) &= U\left(\left(\frac{\theta}{V(\theta)} - 1\right)^2\right) \\ &= \frac{U(\theta^2)}{\theta^2} - 2\frac{U(\theta)}{\theta} + 1 \\ &= 0. \end{aligned}$$

In order to show that $U(y^3) = \frac{y(1+3y+9y^2)}{(1+6y)^3}$, one could analyze the zeroes and poles of $U(y^3)$. But both are meromorphic functions on $X_0(9)$ with at most nine poles, and thus it suffices to check that the first 100 terms of their q -expansions agree, which is easily performed on a computer.

- (2) The fact that $U(y) = 0$ and $U(y^2) = 0$ implies that $y = qV(F)$ for some $F \in \mathbb{Z}[[q]]$. Applying Lemma 2.1 we thus have $U(y^n) = U(q^n V(F)^n) = U(q^n)F^n$, which easily implies $U(y^{3m+1}) = U(y^{3m+2}) = 0$. On the other hand, $U(y^3) = U(q^3 V(F)^3) = qF^3$, so $U(y^{3m}) = U(q^{3m} V(F)^{3m}) = q^m F^{3m} = U(y^3)^m$.
- (3) As in (1), these results follow by a comparison of q -expansions.

□

Using this lemma, we are able to deduce the following corollary, specifying the image under U and V of various subsets of power series rings.

Corollary 2.5. *Let R be a commutative ring containing O_K , let $r \in O_K$ satisfy $v_3(r) \leq 1$, and let $R[[ry]]$ denote the subring of $R[[q]]$ consisting of elements of the form $a_0 + a_1(ry) + a_2(ry)^2 + \dots$. Then*

- (1) $R[[rf]] = R[[ry]]$ and $rfR[[rf]] = ryR[[ry]]$.
- (2) $V(R[[r^3f]]) = R[[r^3y^3]] \subseteq R[[rf]]$ and $V(r^3fR[[r^3f]]) = r^3y^3R[[r^3y^3]] \subseteq rfR[[rf]]$.
- (3) $U(R[[rf]]) \subseteq R[[r^3f]]$ and $U(rfR[[rf]]) \subseteq r^3fR[[r^3f]]$.

Proof.

- (1) Lemma 2.4(3) gives $3f = \frac{3y(1+3y+9y^2)}{(1-3y)^3}$ and thus $rf = ry + \dots \in ryR[[ry]]$. As a power series in ry , we can invert this equation and find ry as a power series in rf , giving the desired equality.
- (2) Since V is an R -algebra homomorphism, continuous with respect to the q -adic topology, and so, again by Lemma 2.4(3), we have

$$V(R[[r^3f]]) = R[[V(r^3f)]] = R[[r^3V(f)]] = R[[r^3y^3]].$$

In addition, $R[[r^3y^3]] \subseteq R[[ry]] = R[[rf]]$. Finally, if $\alpha \in r^3fR[[r^3f]]$ has no constant term, then neither does $V(\alpha)$.

- (3) By part (1), we have that $R[[rf]] = R[[ry]]$. But

$$R[[ry]] = R[[r^3y^3]] \oplus ryR[[r^3y^3]] \oplus r^2y^2R[[r^3y^3]]$$

as R -modules, so one can write $g \in R[[ry]]$ as $g = g_0 + g_1 + g_2$ with $g_i \in r^i y^i R[[r^3y^3]]$. Then by Lemma 2.4(1) and Lemma 2.1, $U(g_1) = U(g_2) = 0$. Write $g_0 = V(h)$ with $h \in R[[r^3f]]$ using part (2). Then $U(g) = U(g_0) = UV(h) = h$, so we have $U(R[[rf]]) \subseteq R[[r^3f]]$. If $g \in rfR[[rf]] = ryR[[ry]]$ then $g_0 \in r^3y^3R[[r^3y^3]]$ and thus we can choose $h \in r^3fR[[r^3f]]$, again by part (2). □

3. Families of Modular Forms

We use weight space to 3-adically interpolate between modular forms of integral weight. Define $\mathcal{W} = \text{Hom}(\mathbb{Z}_3^\times, \mathbb{C}_3^\times)$, and let \mathcal{W}° be the open disc over \mathbb{C}_3 with center 0 and radius 1. For each $w \in \mathcal{W}^\circ$ there is a unique $\kappa \in \mathcal{W}$ with $\kappa(4) = 1 + w$ and $\kappa(-1) = 1$. This map $\mathcal{W}^\circ \rightarrow \mathcal{W}$ identifies \mathcal{W}° with one of the two components of weight space. As in Buzzard and Kilford [4], all of the weights appearing in this paper will lie in this identity component \mathcal{W}° .

For $k \in \mathbb{C}_3$ with $|k| < 1$, we can think of k as the weight $x \mapsto x^k$. In this case, $4^k = w + 1$ and thus $\frac{w}{3} \in k\mathbb{Z}_3[[k]]$. Therefore, we have

$$\mathbb{Z}_3[[w]] \subset \mathbb{Z}_3[[w/3]] \rightarrow \mathbb{Z}_3[[k]],$$

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where the inclusion is the natural one and the map on the right is the isomorphism sending $w/3$ to $(4^k - 1)/3 = k + \dots \in k\mathbb{Z}_3[[k]]$.

We shall use italics to denote modular forms of fixed weight, and bold face to denote families of modular forms. We shall consider two families: first \mathbf{T} , defined below, and then $\mathbf{E}/V(\mathbf{E})$, defined in the next section. We will use \mathbf{T} to study $\mathbf{E}/V(\mathbf{E})$, our ultimate object of interest.

Define

$$\mathbf{T} = \theta^k,$$

that is, \mathbf{T} is the element θ^k of $1 + 3kq\mathbb{Z}_3[[k, q]] \subset \mathbb{Z}_3[[k, q]]^\times$. One constructs \mathbf{T} explicitly using the binomial theorem. In addition, we have the following application of the binomial theorem that will be used repeatedly in what follows:

Lemma 3.1. *Let R be a commutative ring containing \mathcal{O}_K , let $r \in \mathcal{O}_K$ be arbitrary, let ξ be an indeterminate, and let $g \in R[[r\xi]]$. Then $(1 + r\pi\xi g)^k \in 1 + r\pi k\xi R[[k, r\xi]]$.*

Proof. First note that $v_3(n!) \leq (n - 1)/2$. We now use the binomial theorem to conclude that

$$\begin{aligned} (1 + r\pi\xi g)^k &= 1 + r\pi k\xi g \left(1 + \frac{k-1}{2!} (r\pi\xi g) + \frac{(k-1)(k-2)}{3!} (r\pi\xi g)^2 + \dots \right) \\ &= 1 + r\pi k\xi g \left(1 + \frac{\pi(k-1)}{2!} (r\xi g) + \frac{\pi^2(k-1)(k-2)}{3!} (r\xi g)^2 + \dots \right) \\ &\in 1 + r\pi k\xi R[[k, r\xi]]. \quad \square \end{aligned}$$

We use Lemma 3.1 to get information about the overconvergence of the family \mathbf{T} .

Lemma 3.2. *We have the following containments:*

- (1) $\theta/V(\theta) \in 1 + 3f\mathcal{O}_K[[3f]]$.
- (2) $\mathbf{T}/V(\mathbf{T}) \in 1 + 3kf\mathcal{O}_K[[k, \pi f]]$.
- (3) $U(\mathbf{T})/\mathbf{T} \in 1 + 9kf\mathcal{O}_K[[k, 3\pi f]]$.
- (4) Let σ denote the $\mathcal{O}_K[[k]]$ algebra automorphism of $\mathcal{O}_K[[k, q]]$ sending q to ωq . Then we have $\sigma(\mathbf{T})/\mathbf{T} \in 1 + 3\pi ky\mathcal{O}_K[[k, 3y]]$ and $\sigma^2(\mathbf{T})/\mathbf{T} \in 1 + 3\pi ky\mathcal{O}_K[[k, 3y]]$.
- (5) $VU(\mathbf{T})/\mathbf{T} \in 1 + \pi ky\mathcal{O}_K[[k, 3y]]$.
- (6) $U(\mathbf{T})/VU(\mathbf{T}) \in 1 + 3ky\mathbb{Z}_3[[k, 3y]]$.

Proof.

- (1) By the definition of y we have that $\theta/V(\theta) = 1 + 6y$. But we know from Corollary 2.5 (i) that $3y \in 3f\mathcal{O}_K[[3f]]$, and thus $\theta/V(\theta) \in 1 + 3f\mathcal{O}_K[[3f]]$.
- (2) Write $\theta/V(\theta) = 1 + 3fg$ for $g \in \mathcal{O}_K[[3f]]$. Applying Lemma 3.1, we have that $\mathbf{T}/V(\mathbf{T}) \in 1 + 3kf\mathcal{O}_K[[k, \pi f]]$.
- (3) Applying Corollary 2.5 (iii) with $R = \mathcal{O}_K[[k]]$ and $r = \pi$ gives the containment $U(\mathbf{T})/\mathbf{T} \in 1 + 9kf\mathcal{O}_K[[k, 3\pi f]]$.

- (4) Note that σ fixes the image of V , so $\frac{\sigma(\mathbf{T})}{\mathbf{T}} = \frac{\sigma(\mathbf{T}/V(\mathbf{T}))}{\mathbf{T}/V(\mathbf{T})}$. Now, since the power series of y in terms of q contains only exponents congruent to 1 modulo 3, $\sigma(y) = \omega y$. Therefore $\sigma(\mathbf{T}/V(\mathbf{T})) = \sigma((1 + 6y)^k) = (1 + 6\omega y)^k$, and thus

$$\frac{\sigma(\mathbf{T})}{\mathbf{T}} = \left(\frac{1 + 6\omega y}{1 + 6y} \right)^k = \left(1 + \frac{6\pi y}{1 + 6y} \right)^k.$$

We now apply Lemma 3.1, yielding $\frac{\sigma(\mathbf{T})}{\mathbf{T}} \in 1 + 3\pi k y \mathcal{O}_K[[k, 3y]]$.

The same argument works with σ replaced by σ^2 , noting that $\omega^2 - 1 = \pi(\omega + 1)$.

- (5) Since $q^i + \sigma(q^i) + \sigma^2(q^i)$ equals 0 if $i \not\equiv 0 \pmod{3}$ and equals 3 if $i \equiv 0 \pmod{3}$, we have that $3VU(\mathbf{T}) = \mathbf{T} + \sigma(\mathbf{T}) + \sigma^2(\mathbf{T})$. Thus $3VU(\mathbf{T})/\mathbf{T} \in 3 + 3\pi k y \mathcal{O}_K[[k, 3y]]$, which yields the desired result after division by 3.
- (6) Part (iii) gives $U(\mathbf{T})/\mathbf{T} \in 1 + 9k f \mathcal{O}_K[[k, 3\pi f]] \subset 1 + 3k y \mathcal{O}_K[[1, 3y]]$. Putting this together with part (v) and dividing yields $U(\mathbf{T})/VU(\mathbf{T}) \in 1 + \pi k y \mathcal{O}_K[[k, 3y]]$. But $U(\mathbf{T})/VU(\mathbf{T})$ is clearly an element of $\mathbb{Z}_3[[k, y]]$, and since

$$\mathbb{Z}_3[[k, y]] \cap 1 + \pi k y \mathcal{O}_K[[k, 3y]] = 1 + 3k y \mathbb{Z}_3[[k, 3y]],$$

we have the desired conclusion. □

4. The Family $\mathbf{E}/V(\mathbf{E})$

In this section we will prove a result about the degree of overconvergence of the family of modular functions $\mathbf{E}/V(\mathbf{E})$. General expositions on families of overconvergent modular functions and overconvergent modular forms can be found in [3, Appendix; 5, §2.1, 2.4]. For our purposes, however, we may remain at the level of rings, using only one result from the more general expositions above. Specifically, we can rephrase Proposition 2.2.7 of Coleman and Mazur for our purposes in the following proposition:

Proposition 4.1. *For all weights k , the p -adic modular function $E_k/V(E_k) \in \mathcal{O}_{\mathbb{C}_3}[[rf]]$ for some $r \in \mathbb{C}_3$ with $|r| < 1$.*

Using the knowledge that $E_k/V(E_k)$ overconverges, we get the following explicit result on how far $\mathbf{E}/V(\mathbf{E})$ overconverges. Recall from Section 3 that $w = 4^k - 1 \in \mathcal{W}^\circ$.

Theorem 4.2. *The ratio $\mathbf{E}/V(\mathbf{E})$ belongs to $\mathbb{Z}_3[[w/3, 3y]]$.*

Proof. The key idea is to use the fact that U increases overconvergence to prove that something that we know overconverges to a small extent actually overconverges to a much greater degree. For the moment fix a weight k . Define a map $\tilde{U} : \mathcal{O}_{\mathbb{C}_3}[[f]] \rightarrow \mathcal{O}_{\mathbb{C}_3}[[f]]$ by

$$\tilde{U}(\alpha) = U\left(\alpha \frac{U(\theta^k)}{VU(\theta^k)}\right).$$

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Note that

$$\begin{aligned}\tilde{U}\left(\frac{E_k}{U(\theta^k)}\right) &= U\left(\frac{E_k}{U(\theta^k)}\frac{U(\theta^k)}{VU(\theta^k)}\right) \\ &= U\left(\frac{E_k}{VU(\theta^k)}\right) \\ &= \frac{E_k}{U(\theta^k)}.\end{aligned}$$

Now, if we knew that $E_k/U(\theta^k)$ were an element of $\mathcal{O}_{\mathbb{C}_3}\llbracket rf \rrbracket$ with $\frac{1}{3} \leq |r| < 1$ and $U(\theta^k)/VU(\theta^k) \in \mathcal{O}_{\mathbb{C}_3}\llbracket 3f \rrbracket$ then we could conclude using Corollary 2.5 (iii) that

$$E_k/U(\theta^k) \in \mathcal{O}_{\mathbb{C}_3}\llbracket r^3 f \rrbracket$$

and thus $E_k/U(\theta^k) \in \mathcal{O}_{\mathbb{C}_3}\llbracket 27f \rrbracket$ by repeated application of \tilde{U} . So we need to demonstrate the two assumptions above. Lemma 3.2 (vi) gives $U(\mathbf{T})/VU(\mathbf{T}) \in 1 + 3ky\mathbb{Z}_3\llbracket k, 3y \rrbracket$. Specializing to weight k and using Corollary 2.5 (i) yields $U(\theta^k)/VU(\theta^k) \in \mathcal{O}_{\mathbb{C}_3}\llbracket 3f \rrbracket$. In addition, by Proposition 4.1 and Lemma 3.2 we know that both $E_k/V(E_k)$ and $U(\theta^k)/VU(\theta^k)$ are in $\mathcal{O}_{\mathbb{C}_3}\llbracket rf \rrbracket$ for some r with $|r| < 1$, and thus so is their quotient, $\frac{E_k/U(\theta^k)}{V(E_k/U(\theta^k))}$. Therefore, so is $E_k/U(\theta^k)$ and thus we have by the argument above that $E_k/U(\theta^k)$ actually belongs to $\mathcal{O}_{\mathbb{C}_3}\llbracket 27f \rrbracket$. Corollary 2.5 (ii) now implies that $V(E_k)/VU(\theta^k) \in \mathcal{O}_{\mathbb{C}_3}\llbracket 3f \rrbracket$.

Putting all of the previous containments together yields

$$\frac{E_k}{V(E_k)} = \frac{E_k}{U(\theta^k)} \frac{U(\theta^k)}{VU(\theta^k)} \frac{VU(\theta^k)}{V(E_k)} \in \mathcal{O}_{\mathbb{C}_3}\llbracket 3f \rrbracket = \mathcal{O}_{\mathbb{C}_3}\llbracket 3y \rrbracket.$$

We now need to work over all weights k simultaneously. We know that

$$\mathbf{E}/V(\mathbf{E}) \in \mathbb{Z}_3\llbracket k, y \rrbracket = \mathbb{Z}_3\llbracket w/3, y \rrbracket.$$

Say $\mathbf{E}/V(\mathbf{E}) = \sum_{i,j \geq 0} \alpha_{i,j}(w/3)^i y^j$. Suppose for the sake of contradiction that for some i and j , $v_3(\alpha_{i,j}) < j$. Among such, choose one with minimal i and let w be a weight with $0 < v_3(w/3) < \frac{j - v_3(\alpha_{i,j})}{i}$. Consider the valuation of the coefficient of y^j in the expansion of $\mathbf{E}/V(\mathbf{E})$:

$$v_3\left(\sum_{m \geq 0} \alpha_{m,j}(w/3)^m\right).$$

Note that $v_3(\alpha_{i,j}(w/3)^i) < j$, so the only way that the whole sum could have valuation at least j would be if two terms with low valuation had exactly the same valuation. But for $m > \frac{j}{v_3(w/3)}$, we have $v_3(\alpha_{m,j}(w/3)^m) > j$, so by adjusting w slightly without changing this threshold value of m we can ensure that the minimum valuation occurring in the sum does not appear twice. This gives a contradiction, since we know that for each weight, $E_k/V(E_k) \in \mathcal{O}_{\mathbb{C}_3}\llbracket 3y \rrbracket$. \square

Corollary 4.3. *If we write $\mathbf{E}/V(\mathbf{E}) = \sum a_{i,j}w^i y^j$ then $3^{j-i} | a_{i,j}$ for $j \geq i \geq 0$.*

Proof. By the theorem and the fact that $\mathbf{E}/V(\mathbf{E}) \in \mathbb{Z}_3\llbracket w, q \rrbracket = \mathbb{Z}_3\llbracket w, y \rrbracket$ we can write $\mathbf{E}/V(\mathbf{E}) = \sum a_{i,j}w^i y^j = \sum b_{i,j}(w/3)^i (3y)^j$ with $a_{i,j}, b_{i,j} \in \mathbb{Z}_3$. Thus $a_{i,j} = 3^{j-i} b_{i,j}$ and the result follows. \square

5. Reduction of the Eisenstein Family Near the Boundary of Weight Space

Let \mathbb{F} denote the residue field of \mathcal{O}_K . As before, write $\mathbf{E}/V(\mathbf{E}) = \sum_{i,j} a_{i,j} w^i y^j$. Now specialize to some weight $w_0 \in \mathcal{O}_{\mathbb{C}_3}$ satisfying $1/3 < |w_0| < 1$, and let κ denote the corresponding character. We deduce that $E_\kappa/V(E_\kappa) \in \mathcal{O}_K[[w_0 y]]$. Write $E_\kappa/V(E_\kappa) = g_\kappa(w_0 y)$ with $g_\kappa \in \mathcal{O}_K[[X]]$. Let $\bar{g}_\kappa \in \mathbb{F}[[X]]$ denote the reduction of g_κ modulo the maximal ideal of \mathcal{O}_K .

Define $r(X) \in \mathbb{F}[[X]]$ by $r(X) = \sum_{m \geq 0} X^{3^m}$.

Lemma 5.1. *We have $\bar{g}_\kappa(X) = 1 - X^{-1}r(X^3) - X^{-2}(r(X^3) - r(X^3)^2)$. In particular, \bar{g}_κ is independent of κ (for κ corresponding to $w_0 \in \mathcal{W}^\circ$ with $1/3 < |w_0| < 1$).*

Proof. Fix κ and say $g_\kappa = \sum c_n X^n$, with $c_n = c_n(\kappa) \in \mathcal{O}_K$. Specializing the quotient $\mathbf{E}/V(\mathbf{E}) = \sum_{i,j} a_{i,j} w^i y^j$ to weight w_0 we have $c_j w_0^j = \sum_i a_{i,j} w_0^i$ and thus

$$c_j = \sum_i a_{i,j} w_0^{i-j}.$$

Since $|w_0| > 1/3$, Corollary 4.3 implies that $a_{i,j} w_0^{i-j}$ is in the maximal ideal of $\mathcal{O}_{\mathbb{C}_3}$ if $j > i$. But $a_{i,j} w_0^{i-j}$ is also in the maximal ideal of $\mathcal{O}_{\mathbb{C}_3}$ if $j < i$ since $a_{i,j} \in \mathbb{Z}_3$ and $|w_0| < 1$. Therefore,

$$\bar{c}_n = \bar{a}_{n,n} \in \mathbb{F}.$$

In particular, \bar{c}_n is independent of the choice of κ and thus \bar{g}_κ is as well. Thus to finish the proof of the lemma, we need only verify the formula for \bar{g}_κ for a particular choice of κ . Let κ_0 be the Dirichlet character of conductor 9 given by $\kappa_0(2) = \omega + 1$ where ω is a primitive cube root of unity. The weight corresponding to κ_0 is $\kappa_0(4) - 1 = \omega - 1$ which satisfies $1/3 < |\omega - 1| < 1$. The corresponding Eisenstein series is

$$\begin{aligned} E_{\kappa_0} &= 1 - \left(\frac{1}{18} \sum_{m=1}^8 m \kappa_0(m) \right)^{-1} \sum_{n>0} \left(\sum_{\substack{0 < d|n \\ 3 \nmid d}} \kappa_0(d) \right) q^n \\ &= 1 + (1 - \omega)q + 3q^2 + (1 - \omega)q^3 + (4 + 2\omega)q^4 + \dots \end{aligned}$$

and the corresponding ratio

$$f_0 := E_{\kappa_0}/V(E_{\kappa_0}) = 1 + (1 - \omega)q + 3q^2 + (4 + 5\omega)q^4 + \dots$$

is a function on $X_0(27)$ which can be checked to satisfy the equation

$$\begin{aligned} 9y^3 f_0^3 + (-27y^3 - 9y^2 - 3y) f_0^2 + ((27 - 27\omega)y^3 + 27y^2 + 9y + (2 + \omega)) f_0 \\ + ((-27 + 27\omega)y^3 - 27y^2 - 9y - (2 + \omega)) = 0. \end{aligned}$$

If we consider f_0 as an element of $\mathcal{O}_K[[y]]$ then this last equation is an identity in $\mathcal{O}_K[[y]]$. Dividing the whole equation by $-1 - 2\omega$ and setting $X = (-1 + \omega)y = w_0 y$, we deduce that the equation

$$\begin{aligned} X^3 g_{\kappa_0}(X)^3 + (-3X^3 + (1 - \omega)X^2 + \omega X) g_{\kappa_0}(X)^2 + ((3 - 3\omega)X^3 - (3 - 3\omega)X^2 - 3\omega X + \omega) g_{\kappa_0}(X) \\ + ((-3 + 3\omega)X^3 + (3 - 3\omega)X^2 + 3\omega X - \omega) = 0 \end{aligned}$$

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is an identity in $\mathcal{O}_K[[X]]$. Reducing modulo the maximal ideal we find that

$$X^3 \bar{g}_{\kappa_0}(X)^3 + X \bar{g}_{\kappa_0}(X)^2 + \bar{g}_{\kappa_0}(X) - 1 = 0$$

in $\mathbb{F}[[X]]$. Using the identity $r(X) - r(X)^3 = X$, which holds in $\mathbb{F}[[X]]$, it is straightforward to check that $\bar{g}_{\kappa_0}(X) = 1 - X^{-1}r(X^3) - X^{-2}(r(X^3) - r(X^3)^2)$ is the unique solution to this equation in $\mathbb{F}[[X]]$. \square

6. Generating Function for the Matrix of the U -operator Near the Boundary of Weight Space.

In this section we begin the computation of the characteristic power series of U acting on overconvergent forms of weight κ , where κ corresponds to a point w_0 in weight space with $1/3 < |w_0| < 1$. In particular, we give an expression for the coefficients of the matrix of U with respect to a certain basis using generating functions.

Almost by definition, $V(E_\kappa)$ is an overconvergent modular form of weight κ [5, Prop. 2.2.7]. Corollary 2.5 (i) implies that if $c \in \mathbb{C}_3$ with $1 > |c| > 1/3$ then the region of $X_0(9)$ defined by $|cy| \leq 1$ is isomorphic to the region of $X_0(3)$ defined by $|cf| \leq 1$ and thus the powers of cy can be taken as a Banach basis of a 3-adic Banach space \mathbf{M}_0 of weight 0 overconvergent modular forms (this space depends on c , but we will suppress this choice in our notation). For $|c|$ sufficiently close to 1, the space $V(E_\kappa)\mathbf{M}_0$ of overconvergent weight κ modular forms will be closed under the action of the standard Hecke operators, and the operator U will be compact. This space has a Banach basis $\{V(E_\kappa)(cy)^n : n = 0, 1, 2, \dots\}$ and we shall prove results about the U operator by analyzing its matrix with respect to this basis. Define $m_{i,j} \in \mathbb{C}_3$ for $i, j \geq 0$ by

$$U(V(E_\kappa)(cy)^j) = V(E_\kappa) \sum_i m_{i,j}(cy)^i. \quad (6.1)$$

Lemma 6.1. *The generating function $\sum_{i,j \geq 0} m_{i,j} X^i Y^j$ is equal to*

$$\frac{g_\kappa\left(\frac{w_0}{c}X\right)\left(1 + \frac{6}{c}X\right)^3}{\left(1 + \frac{6}{c}X\right)^3 - Y^3(c^2X + 3cX^2 + 9X^3)}.$$

Proof. A rearrangement of equation 6.1 gives

$$\sum_i m_{i,j}(cy)^i = (E_\kappa/V(E_\kappa))U((cy)^j).$$

By Lemma 2.4, $U(y^j) = 0$ if j is not a multiple of 3, so $m_{i,j} = 0$ in that case. For $j = 3t$, we have $U(y^j) = (y(1 + 3y + 9y^2)/(1 + 6y)^3)^t$ and thus

$$\sum_i m_{i,j}(cy)^i = g_\kappa(w_0y) \left(\frac{c^3y(1 + 3y + 9y^2)}{(1 + 6y)^3} \right)^t.$$

This is an identity in $\mathbb{C}_3[[y]]$, so substituting X for cy gives

$$\sum_i m_{i,j} X^i = g_\kappa(w_0X/c) \left(\frac{c^2X + 3cX^2 + 9X^3}{(1 + 6X/c)^3} \right)^t.$$

Multiplying by Y^j and summing over j gives

$$\sum_{i,j} m_{i,j} X^i Y^j = g_\kappa(w_0 X/c) \sum_{t \geq 0} \left(\frac{(c^2 X + 3cX^2 + 9X^3) Y^3}{(1 + 6X/c)^3} \right)^t,$$

and summing the geometric series on the right hand side gives the result. \square

Since the $m_{i,j}$ are just the matrix coefficients of U operating on the space of weight κ overconvergent modular forms, we can read off the well known result that U is compact for $|c| < 1$ sufficiently close to 1 by noting that if $|c| > |w_0|$ then the coefficients of g_κ are integral and $\frac{w_0}{c}, \frac{6}{c}$ and c all have norm less than 1.

7. The Characteristic Power Series of U Near the Boundary of Weight Space

As in the previous section, let w_0 satisfy $1/3 < |w_0| < 1$ and let κ be the corresponding weight. In this section we compute the characteristic power series for various compact operators on p -adic Banach spaces; see Serre [13] for the definitions and basic theorems. Our goal in this section is to determine the valuations of the roots of the characteristic power series of U . In order to do so, we compute the valuations of the coefficients of the characteristic polynomial in Proposition 7.4, then read off the valuations of the roots using Newton polygons. Since y has the property that $U(y^j) = 0$ when j is not a multiple of three, our matrix is only nonzero on every third row. In addition, since U is compact we know that the valuations of the rows are increasing. In Lemma 7.1 we provide the tool to pull off the valuation component of the coefficients of the characteristic polynomial. Lemmas 7.2 and 7.3 then give us the ability to prove that what remains has unit determinant.

Fix s a positive integer, and let $d \in \mathcal{O}_{\mathbb{C}_3}$ be nonzero. Let $N = (n_{i,j})_{0 \leq i,j \leq 3s-1}$ be a $3s$ by $3s$ matrix with the property that $n_{i,j} \in d^j \mathcal{O}_{\mathbb{C}_3}$ for all $0 \leq i, j \leq 3s-1$. Assume that $n_{i,j} = 0$ when j is not a multiple of 3. Let $P(T) = \det(1 - TN) = 1 + \dots = \sum_{\alpha \geq 0} a_\alpha T^\alpha \in \mathcal{O}_{\mathbb{C}_3}$ denote the ‘‘characteristic power series’’ of N (though it is of course actually a polynomial). For $0 \leq \beta \leq s$, let T_β denote the β by β matrix whose (i, j) th entry is $n_{3i,3j}/d^{3j} \in \mathcal{O}_{\mathbb{C}_3}$.

Lemma 7.1. *We have that $a_\alpha/d^{3\alpha(\alpha-1)/2} \in \mathcal{O}_{\mathbb{C}_3}$, and furthermore, for $\alpha \leq s$ we have that $a_\alpha/d^{3\alpha(\alpha-1)/2} \in \mathcal{O}_{\mathbb{C}_3}^\times$ iff $\det(T_\alpha) \in \mathcal{O}_{\mathbb{C}_3}^\times$.*

Proof. For S a subset of $\{0, 1, 2, \dots, 3s-1\}$ of size α , set

$$d_S = \sum_{\sigma: S \rightarrow S} \text{sgn}(\sigma) \prod_{s \in S} n_{s, \sigma(s)}.$$

By the definition of the determinant, we have that $(-1)^\alpha a_\alpha$ is the sum of the d_S as S ranges over the size α subsets of $\{0, 1, 2, \dots, 3s-1\}$. Note that $d_S = 0$ unless S consists entirely of multiples of 3. In this case, $d^{\sum_{s \in S} s}$ divides d_S , and $\sum_{s \in S} s \geq \frac{3}{2}\alpha(\alpha-1)$, with equality iff $S = S_0 := \{0, 3, 6, \dots, 3\alpha-3\}$. Thus a_α is a sum of multiples of $d^{3\alpha(\alpha-1)/2}$, all but one of which are multiples of $d^{3\alpha(\alpha-1)/2+1}$. Therefore $a_\alpha/d^{3\alpha(\alpha-1)/2} \in \mathcal{O}_{\mathbb{C}_3}$ and in fact, $a_\alpha/d^{3\alpha(\alpha-1)/2} \in \mathcal{O}_{\mathbb{C}_3}^\times$ iff $d_{S_0}/d^{3\alpha(\alpha-1)/2} \in \mathcal{O}_{\mathbb{C}_3}^\times$. But $d_{S_0}/d^{3\alpha(\alpha-1)/2} = \det(T_\alpha)$ and we are done.

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□

We will use this lemma with N as truncations of the matrix of U . The following lemma allows us to find the coefficients of the matrix T_α in this case. Recall that $r(X) = \sum_{m \geq 0} X^{3m}$.

Lemma 7.2. Define $s_{i,j} \in \mathbb{F}_3$ for $0 \leq i, j < \infty$ by

$$\sum_{0 \leq i, j} s_{i,j} X^i Y^j = \frac{1 - X^{-1}r(X^3) - X^{-2}(r(X^3) - r(X^3)^2)}{1 - XY^3},$$

with the equality taking place in $\mathbb{F}_3[[X, Y]]$. Define $t_{i,j} = s_{3i,3j}$ for $0 \leq i, j$. Then

$$\sum_{0 \leq i, j} t_{i,j} X^i Y^j = \frac{1 - r(X)Y + (r(X)^2 - r(X))Y^2}{1 - XY^3}.$$

Proof. Define power series $A(X, Y)$ and $B(X, Y)$ in $\mathbb{F}[[X, Y]]$ by

$$A(X, Y) = \frac{1 - X^{-1}r(X^3)X^{-2}(r(X^3)^2 - r(X^3))}{1 - XY^3},$$

and

$$B(X, Y) = \frac{1 - r(X)Y + (r(X)^2 - r(X))Y^2}{1 - XY^3}.$$

Our desired result is equivalent to the statement that

$$X^3(A(X, Y) - B(X^3, Y^3)) \cap \mathbb{F}[[X^3, Y^3]] = 0.$$

This follows from explicit computation:

$$\begin{aligned} X^3(A(X, Y) - B(X^3, Y^3)) &= \\ &= \frac{(X + X^2Y^3)r(X^3)^2 - (X + X^2 + X^2Y^3 + X^4Y^6)r(X^3) + X^4Y^3 + X^5Y^6}{1 - X^3Y^9}. \quad \square \end{aligned}$$

Finally, we provide another lemma that allows us to conclude that certain matrices have unit determinant.

Lemma 7.3. Fix an integer $\alpha \geq 0$, and let \bar{T}_α be the α by α matrix $(\bar{t}_{i,j})_{0 \leq i, j \leq \alpha}$ with entries in \mathbb{F} defined via the following identity:

$$\sum_{i, j \geq 0} \bar{t}_{i,j} X^i Y^j = \frac{1 - r(X)Y + (r(X)^2 - r(X))Y^2}{1 - XY^3},$$

the equality taking place in $\mathbb{F}[X, Y]/(X^\alpha, Y^\alpha)$. Then $\det(\bar{T}_\alpha) \neq 0$.

Proof. Write $\sum_{i, j} \bar{t}_{i,j} X^i Y^j = \sum_j f_j(X) Y^j$, with $f_j(X) \in V := \mathbb{F}[X]/(X^\alpha)$. It suffices to prove that the $f_j(X)$, $0 \leq j < \alpha$, span V as an \mathbb{F} -vector space. Consider $r = r(X)$ as an element of V . We have

$$\sum_j f_j(X) Y^j = (1 - rY + (r^2 - r)Y^2)(1 + XY^3 + X^2Y^6 + X^3Y^9 + \dots)$$

and by comparing powers of Y we obtain the identities $f_{3t}(X) = X^t$ and $f_{3t+1} = -rX^t$ and $f_{3t+2} = (r^2 - r)X^t$. Using the identity $r - r^3 = X$, we have that $f_{3t} = (r - r^3)^t$ and $f_{3t+1} = -r(r - r^3)^t$ and $f_{3t+2} = (r^2 - r)(r - r^3)^t$ and hence as polynomials in r we have that $\deg(f_n) = n$. Therefore the span of the f_j contains the image of $\mathbb{F}[r]$ in V . This is enough because $r = X + \dots$, so this image is all of $\mathbb{F}[X]/(X^\alpha)$. \square

We now prove a proposition that gives the valuations of the coefficients of the characteristic power series of U . As usual let κ be a weight such that the corresponding w_0 satisfies $1/3 < |w_0| < 1$, and let $(m_{i,j})$ be the matrix representing U in weight κ .

Proposition 7.4. *If $P_\kappa(T) = \sum_{\alpha \geq 0} b_\alpha T^\alpha$ denotes the characteristic power series of U in weight κ , then $|b_\alpha| = |w_0|^{\alpha(\alpha-1)/2}$.*

Proof. If $\beta \geq 0$ and M_β denotes the truncated matrix $(m_{i,j})_{0 \leq i, j < \beta}$, and if we write $P_\beta(T)$ for the characteristic power series $\det(1 - TM_\beta)$ of M_β , then the $P_\beta(T)$ tend to $P_\kappa(T)$ in the sense that if $P_\beta(T) = \sum_\alpha b_{\alpha,\beta} T^\alpha$ then $\lim_{\beta \rightarrow \infty} b_{\alpha,\beta} = b_\alpha$. Therefore it suffices to prove that $|b_{\alpha,\beta}| = |w_0|^{\alpha(\alpha-1)/2}$ for $\beta > 3\alpha$, and we may further assume that β is a multiple of 3. Let N_β be the matrix with elements $(n_{i,j})_{0 \leq i, j < \beta}$ where $n_{i,j} = m_{i,j}(c/w_0)^{i-j}$. Then N_β is easily checked to be a conjugate of M_β , so $P_\beta(T) = \det(1 - TN_\beta)$. Furthermore, one easily checks that Lemma 6.1 implies (substituting X for w_0/cX and Y for c/w_0Y)

$$F(X, Y) := \sum_{0 \leq i, j < \beta} n_{i,j} X^i Y^j = \frac{g_\kappa(X)(1 + 6/w_0X)^3}{(1 + 6/w_0X)^3 - Y^3(w_0^2X + 3w_0X^2 + 9X^3)},$$

as an element of $\mathcal{O}_{\mathbb{C}_3}[X, Y]/(X^\beta, Y^\beta)$. Choose $d \in \mathcal{O}_{\mathbb{C}_3}$ with $d^3 = w_0^2$. The fact that $G(X, Y) := F(X, Y/d)$ satisfies

$$G(X, Y) = \frac{g_\kappa(X)(1 + 6/w_0X)^3}{(1 + 6/w_0X)^3 - Y^3(X + 3/w_0X^2 + 9/w_0^2X^3)}$$

shows that $n_{i,j}/d^j \in \mathcal{O}_{\mathbb{C}_3}$ for all i, j , and the fact that $F(X, Y)$ is a function of X and Y^3 implies that $n_{i,j} = 0$ if j is not a multiple of 3. We are therefore in position to apply Lemma 7.1 to deduce that $|b_{\alpha,\beta}| \leq d^{3\alpha(\alpha-1)/2} = |w_0|^{\alpha(\alpha-1)/2}$, with equality iff the matrix $(n_{3i,3j}/d^{3j})_{0 \leq i, j < \beta}$ has unit determinant. Let T_α denote this matrix, and let \bar{T}_α denote its reduction modulo the maximal ideal of $\mathcal{O}_{\mathbb{C}_3}$. Reducing $G(X, Y)$ modulo the maximal ideal of $\mathcal{O}_{\mathbb{C}_3}$, it becomes

$$\bar{G}(X, Y) = \frac{\bar{g}_\kappa(X)}{1 - XY^3} \in \mathbb{F}[X, Y]/(X^\beta Y^\beta)$$

and by Lemma 5.1 and Lemma 7.2 we deduce that $\bar{T}_\alpha = (\bar{t}_{i,j})_{0 \leq i, j < \alpha}$ with

$$\sum_{0 \leq i, j < \alpha} \bar{t}_{i,j} X^i Y^j = \frac{1 - r(X)Y + (r(X)^2 - r(X))Y^2}{1 - XY^3},$$

the equality taking place in $\mathbb{F}[X, Y]/(X^\alpha Y^\alpha)$. Lemma 7.3 implies that $\det(\bar{T}_\alpha)$ is nonzero, and hence that $\det(T_\alpha) \in \mathcal{O}_{\mathbb{C}_3}^\times$. The second part of Lemma 7.1 now implies the desired equality. \square

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This proposition allows us to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 7.4, the Newton polygon of the characteristic power series of U has vertices $(\alpha, \frac{1}{2}\alpha(\alpha - 1)v)$ and slopes $0, v, 2v, 3v, 4v, \dots$ \square

As in the $p = 2$ case, we see that the eigencurve is geometrically the disjoint union of countably many annuli over the boundary of weight space.

8. Other Work

Daniel Jacobs' thesis [9] uses a different approach to compute the slopes of U_3 on spaces of overconvergent modular forms. He begins with a specific definite quaternion algebra, ramified at 2 and infinity, and then uses the Jacquet-Langlands correspondence to derive results about U_3 . As a consequence of this different methodology, he only obtains a subset of the slopes listed in Theorem 1.1. In addition, the fact that his quaternion algebra is ramified at 2 introduces level structure at 2 beyond just $\Gamma_0(3)$. However, his methods are not subject to the restriction on weight that Theorem 1.1 is: he can find slopes at weight $x \mapsto x^3$ for example.

Smithline's thesis [14] gives partial information on the slopes of U for $p = 3$ and $p = 5$ in weight 0. Loeffler [12] refined this work to compute the slopes of the U operator for $p = 3$, again only for weight 0.

For $p = 5$, Kilford [10] finds that the slopes of U arise in the union of two arithmetic progressions. His work applies to arbitrary weight, but is restricted to conductor 25. Kilford and McMurdy [11] extend these results to $p = 7$ and conductor 49, where again the slopes occur in two arithmetic progressions.

In a related setting, Davis, Wan and Xiao [6] have recently proven that the eigencurve associated to the Artin-Schreier-Witt tower breaks up as a disjoint union of pieces over the boundary of weight space, with slopes in arithmetic progression.

Finally, Diao and Liu [8] have shown that the eigencurve is proper for general p : a geometric motivation for computing the slopes of U .

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Second, my debt to the paper of Buzzard and Kilford [4] will be obvious to anyone who has read it. To a large extent, I follow their structure, their notation and many of their proofs.

References

- [1] Christian Batut, Karim Belabas, Dominique Benardi, Henri Cohen, and Michel Olivier, *User's guide to PARI-GP* (1985-2013), <http://pari.math.u-bordeaux.fr>.

- [2] Kevin Buzzard, *Harvard course math 254z*, 2006. <http://www.math.harvard.edu/ev/documents.html>.
- [3] Kevin Buzzard and Frank Calegari, *Slopes of overconvergent 2-adic modular forms*, *Compositio Math.* **141** (2005), no. 3, 591–604.
- [4] Kevin Buzzard and Lloyd Kilford, *The 2-adic eigencurve at the boundary of weight space*, *Compositio Math.* **141** (2005), no. 3, 605–619.
- [5] Robert Coleman and Barry Mazur, *The eigencurve*, *Galois representations in algebraic geometry*, 1998, pp. 1–113.
- [6] Christopher Davis, Daqing Wan, and Liang Xiao, *Newton slopes for Artin-Schreier-Witt towers*, 2013. arXiv 1310.5311.
- [7] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Springer, New York, 2005.
- [8] Hansheng Diao and Ruochuan Liu, *The eigencurve is proper*, 2014. arXiv:1401.4871.
- [9] Daniel Jacobs, *Slopes of compact Hecke operators*. Ph.D. Thesis, Imperial College, 2003.
- [10] Lloyd Kilford, *On the slopes of the U_5 operator acting on overconvergent modular forms*, *J. Théorie des Nombres des Bordeaux* **20** (2008), no. 1, 165–182.
- [11] Lloyd Kilford and Ken McMurdy, *Slopes of the U_7 operator acting on a space of overconvergent modular forms*, *LMS Journal of Computation and Mathematics* **15** (2012), 113–139.
- [12] David Loeffler, *Slopes of 3-adic overconvergent modular forms*, 2008. Preprint at homepages.warwick.ac.uk/~masiao/math3adicU.pdf.
- [13] Jean-Pierre Serre, *Completely continuous endomorphisms of p -adic Banach spaces*, *Publ. Math. I.H.E.S.* **12** (1962), 69–85.
- [14] Lawren Smithline, *Slopes of p -adic modular forms*. Ph.D. Thesis, University of California at Berkeley, 2000.
- [15] William Stein et al, *Sage Mathematics Software* (2005-2013), <http://www.sagemath.org>.