Homework 17 Solutions

Problems

1. Do the following computations in the given modulus.
   
   (a) \( 6 - 4 \pmod{7} \).
   (b) \( 80 + 21 \pmod{101} \).
   (c) \( 3 - 12 \pmod{15} \).
   (d) \( 456 \times 450 \pmod{457} \).

\[ 2 \cdot 0 \cdot 6 \cdot -1 \times -7 \equiv 7 \]

2. Do the following computations in the given modulus.

   (a) \( 457 \times 458 \pmod{459} \).
   (b) \( 13 \times 44 \pmod{5} \).
   (c) \( 13 \times 44 \pmod{15} \).

   In all three cases, we have \( (-2) \times (-1) \equiv 2 \) in the given modulus.

3. This problem concerns a divisibility rule for 4, i.e., a way of telling if a number is divisible by 4 easily.

   (a) Show why the following divisibility rule for 4 works:
   Add the last digit to twice the second to last digit. If the sum is a multiple of four, then the number is a multiple of 4.

   For example, 16 is a multiple of 4 since \( 6 + 2 \cdot 1 = 8 \) is a multiple of 4.
   
   [Hint: We can write 538 as \( 5 \cdot 10^2 + 3 \cdot 10 + 8 \). What does considering this expression modulo 4 tell us?]

   (b) Is 2736 divisible by 4? Why or why not?

   (c) Is 293847102938470192834701928374 divisible by 4? Why or why not?

   We write out a general number \( a_n a_{n-1} \cdots a_2 a_1 a_0 \) as

   \[ a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_2 \cdot 100 + a_1 \cdot 10 + a_0. \]

   Now note that 100 is divisible by 4, and so is 1000, etc. Rephrasing this using modular arithmetic, we have \( 100 \equiv 0 \pmod{4} \), \( 1000 \equiv 0 \pmod{4} \), etc. Moreover, \( 10 \equiv 2 \pmod{4} \). Thus

   \[ a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_2 \cdot 100 + a_1 \cdot 10 + a_0 \equiv a_n \cdot 0 + a_{n-1} \cdot 0 + \cdots + a_2 \cdot 0 + a_1 \cdot 10 + a_0 \]
   \[ \equiv a_1 \cdot 10 + a_0 \]
   \[ \equiv a_1 \cdot 2 + a_0 \]

   But if two numbers are congruent modulo 4, then one is divisible by 4 if and only if the other one is. So our original number is a multiple of 4 if and only if the sum of its last digit plus twice the second to last digit is a multiple of 4.

   We now apply the rule.

   (a) 2736 is a multiple of 4, since \( 3 \cdot 2 + 6 = 12 \) is.

   (b) 293847102938470192834701928374 is not a multiple of 4 since \( 7 \cdot 2 + 4 = 18 \) is not.