Math 430 – Group Theory Extra Credit Project

Due April 25, 2016

If you choose to work on this project, I will e-mail you an integer \( N \). Your goal is to classify all groups of order \( N \) up to isomorphism. Your main tools will be

- Sylow’s Theorems (15.4, 15.7 and 15.8),
- Semidirect Products (see the introduction below),
- the following proposition, which you should prove.

**Proposition.**

1. Let \( H \) and \( K \) be subgroups of \( G \). Then \( HK \) is a subgroup of \( G \) if and only if \( HK = KH \).
2. If \( K \) is a normal subgroup of \( G \) and \( H \) is any subgroup of \( G \), then \( HK \) is a subgroup of \( G \).

The book *Abstract Algebra* by Dummit and Foote may be useful (especially chapters 4 and 5). E-mail me if you’re not able to get a copy from the library.

**Hint.** As a warmup, you may want to try to classify groups of order \( M \) for some divisors \( M \) of \( N \). Also, try to list as many groups of order \( N \) as you can before attempting to prove that you have a complete list.

**Introduction to Semidirect Products**

Given two groups \( H \) and \( K \), together with a left action of \( K \) on \( H \) (which we’ll denote by \( k \cdot h \)), one can form a new group called the *semidirect product* of \( H \) and \( K \). It will be denoted \( H \rtimes K \).

As a set, its elements are just ordered pairs \((h,k)\), like the direct product \( H \times K \). The operation is defined by

\[
(h_1,k_1) \times (h_2,k_2) = (h_1(k_1 \cdot h_2),k_1k_2).
\]

There are natural injective homomorphisms \( H \rightarrow H \rtimes K \) and \( K \rightarrow H \rtimes K \) given by \( h \mapsto (h,1) \) and \( k \mapsto (1,k) \). However, the image of \( H \) is a normal subgroup,
while the image of $K$ is not (unless the action of $K$ on $H$ is trivial, in which case the semidirect product is the same as the direct product).

Just as with direct products, there is a notion of internal semidirect product. Given a pair of subgroups $H$ and $K$ in $G$ with $H$ normal, multiplication induces an isomorphism between $H \rtimes K$ (with the action defined by $k \cdot h = khk^{-1}$) and $G$ if the following two conditions are satisfied:

1. $H \cap K = \{1\}$,
2. $G = HK$ (or that $\#G = \#H \#K$ if all are finite).

Finally, there is a relationship between semidirect products and quotient groups. Given a normal subgroup $H$ of $G$, let $K_q = G/H$. Then $G \cong H \rtimes K_q$ if and only if there is a subgroup $K \subseteq G$ that is mapped isomorphically onto $K_q$ by the natural quotient map $G \to G/H$.

**Example.** If $G = D_n$ and $H$ is the subgroup of rotations, then letting $K$ be any subgroup of order 2 generated by a reflection we get $G \cong H \rtimes K \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$.

**Example.** If $G = S_n$ and $H = A_n$, letting $K$ be any subgroup of order 2 generated by a transposition we get $G \cong H \rtimes K \cong A_n \rtimes \mathbb{Z}_2$.

**Example.** There is no way to express $Q_8$ as a semidirect product of smaller groups, since every subgroup of order 4 contains the unique subgroup of order 2.

**Example.** The subgroup of order 12 that we didn’t define in class is a semidirect product $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$. The action of $\mathbb{Z}_4$ on $\mathbb{Z}_3$ is: $a$ acts trivially if it is even and acts by $x \mapsto -x$ if it is odd.