Math 430 – Practice Midterm Solutions

1. (a) A group is a set G with an operation $\cdot\ : G \times G \to G$ so that
   i. There is an identity element $1 \in G$ so that $1 \cdot g = g \cdot 1 = g$ for all $g \in G$,
   ii. for each $g \in G$ there is an inverse $g^{-1} \in G$ so that $g \cdot g^{-1} = g^{-1} \cdot g = 1$,
   iii. for all $x, y, z \in G$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

   (b) The cancellation law states that if $gh = gh'$ in a group then $h = h'$.

$$\begin{aligned} gh &= gh' \text{ by assumption} \\ g^{-1}(gh) &= g^{-1}(gh') \text{ multiplying by the same element} \\ (g^{-1}g)h &= (g^{-1}g)h' \text{ by associativity} \\ 1 \cdot h &= 1 \cdot h' \text{ by the definition of inverses} \\ h &= h' \text{ by the definition of the identity} \end{aligned}$$

2. (a) Not a group, since the inverse of 1 is -1, which is not a non-negative integer.

   (b) This is a subgroup: if σ is a product of $2m$ transpositions and τ a product of $2n$ transpositions then τ^{-1} will be the product of those $2n$ transpositions in the reverse order and thus $\sigma\tau^{-1}$ is a product of $2(m+n)$ transpositions and is thus even.

   (c) This is not a subgroup, since the identity permutation is not odd.

   (d) This is a subgroup, since the inverse of a positive rational number is again rational, the product of two rational numbers is rational, and there exist positive rational numbers.

   (e) This is not a subgroup since 1 is not irrational and thus this subset does not contain the identity.

3. (a) This is true. The maps $\exp : \mathbb{R} \to \mathbb{R}^+$ and $\log : \mathbb{R}^+ \to \mathbb{R}$ are inverses of each other and thus bijective. Moreover, $\exp(x + y) = \exp(x)\exp(y)$ and $\log(xy) = \log(x) + \log(y)$, so they are isomorphisms.

   (b) This is false, since $\mathbb{Z}$ is countable while $\mathbb{R}^+$ is uncountable.

   (c) This is false, since the circle group U contains an element $\text{cis}(2\pi/3)$ of order 3, while $\mathbb{R}^*$ contains no elements of order 3.

4. (a) The transposition (12) has order 2, so the subgroup generated by it is just $\{(), (12)\}$.

   (b) The left cosets are $\{(), (12)\}$ and $(13)\{(), (12)\} = \{(13), (123)\}$ and $(23)\{(), (12)\} = \{(23), (132)\}$.

   (c) The subgroup generated by (12) and (23) must contain the products $(12)(12) = ()$ and $(12)(23) = (123)$ and $(23)(12)$ and $(123)(12) = (13)$. It is thus all of S_3.

5. (a) The inverse is $\begin{pmatrix} 3 & 6 & 4 & 1 & 5 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 1 & 3 & 5 & 2 \end{pmatrix}$.

   (b) $\sigma = (134)(26)$.

   (c) $\sigma = (14)(13)(26)$

(d) σ is odd since it is the product of three transpositions.

(e) the order of σ is 6, the least common multiple of 2 and 3 (the lengths of the disjoint cycles in σ).

6. If gH is a left coset of H, consider $\lambda_g : h \mapsto gh$ and $\lambda_{g^{-1}} : a \mapsto g^{-1}a$. By definition, λ_g maps H to gH. Moreover, if $a = gh \in gH$ then $\lambda_{g^{-1}}(a) = g^{-1}gh \in H$ so $\lambda_{g^{-1}}$ maps gH to H. Finally, the two maps are inverses of each other (since multiplication by gg^{-1} or $g^{-1}g$ is the identity), and thus bijections.

7. (a) By the definition of an isomorphism, $\varphi(r/2)^2 = \varphi(2(r/2)) = \varphi(r)$.

   (b) Since φ must be surjective, there is some r with $\varphi(r) = 2$. But then $\varphi(r/2)$ is a positive rational whose square is 2, which is impossible.