Instructions: Attempt all problems. Show your work! No books or notes may be used.

1. (5 points each)
   
   (a) Define the term *group*.

   (b) State and prove the Cancellation Law for groups. Justify all steps using the definition you gave above.
2. (5 points each) In each item below, you are given a group and a subset. In each item, determine whether the given subset is a subgroup. No credit will be given for answers without justification.

(a) The non-negative integers \( \{0, 1, 2, \ldots \} \) as a subset of the group \((\mathbb{Z}, +)\).

(b) The set of all even permutations in \(S_n\).

(c) The set of all odd permutations in \(S_n\).

(d) The set of all positive rational numbers as a subset of \((\mathbb{R}^+, \cdot)\).

(e) The set of all positive irrational numbers as a subset of \((\mathbb{R}^+, \cdot)\).
3. (5 points each) Prove or disprove:

(a) $(\mathbb{R}, +)$ is isomorphic to $(\mathbb{R}^+, \cdot)$.

(b) $(\mathbb{Z}, +)$ is isomorphic to $(\mathbb{R}^+, \cdot)$.

(c) The circle group $(U, \cdot)$ is isomorphic to $(\mathbb{R}^*, \cdot)$. 
4. (5 points each)

(a) Find the subgroup of $S_3$ generated by the transposition $(1, 2)$.

(b) List all left cosets of the subgroup you found in 5a.

(c) Find the subgroup of $S_3$ generated by $\{(1, 2), (2, 3)\}$. 
5. (5 points each) Consider the following permutation in $S_6$:

$$
\sigma = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 6 & 4 & 1 & 5 & 2 
\end{pmatrix}
$$

(a) Find $\sigma^{-1}$.

(b) Express $\sigma$ as a product of disjoint cycles.

(c) Express $\sigma$ as a product of transpositions.

(d) Is $\sigma$ even or odd? Explain.

(e) Find the order of $\sigma$. 
6. (10 points) Let $G$ be a group and let $H$ be a subgroup of $G$. Prove that every left coset of $H$ has the same cardinality as $H$ by exhibiting a bijection between the two sets.
7. (Bonus!) Prove that the groups \((\mathbb{Q}, +)\) and \((\mathbb{Q}^+, \cdot)\) are not isomorphic by completing the following.

(a) Suppose \(\varphi : \mathbb{Q} \to \mathbb{Q}^+\) is an isomorphism. Show that for every rational number \(r\) we have \(\varphi(r/2)^2 = \varphi(r)\).

(b) Deduce from 8a that if \((\mathbb{Q}, +)\) and \((\mathbb{Q}^+, \cdot)\) are isomorphic, then every positive rational number has a rational square root, and conclude that such an isomorphism is impossible.