Math 0430 Midterm Exam Name: ____________________

February 23, 2015

Instructions: Attempt all problems. Show your work! No books or notes may be used.

1. (5 points each)

   (a) Define the term *group*.

   (b) State and prove the Cancellation Law for groups. Justify all steps using the definition you gave above.

2. (5 points each) In each item below, you are given a group and a subset. In each item, determine whether the given subset is a subgroup. No credit will be given for answers without justification.

   (a) The non-negative integers $\{0, 1, 2, \ldots\}$ as a subset of the group $(\mathbb{Z}, +)$.

   (b) The set of all even permutations in S_n.

   (c) The set of all odd permutations in S_n.

   (d) The set of all positive rational numbers as a subset of $(\mathbb{R}^+, \cdot)$.

   (e) The set of all positive irrational numbers as a subset of $(\mathbb{R}^+, \cdot)$.

3. (5 points each) Prove or disprove:

   (a) $(\mathbb{R}, +)$ is isomorphic to $(\mathbb{R}^+, \cdot)$.

   (b) $(\mathbb{Z}, +)$ is isomorphic to $(\mathbb{R}^+, \cdot)$.

   (c) The circle group $(U, \cdot)$ is isomorphic to $(\mathbb{R}^*, \cdot)$.

4. (5 points each)

(a) Find the subgroup of S_3 generated by the transposition $(1, 2)$.

(b) List all left cosets of the subgroup you found in 5a.

(c) Find the subgroup of S_3 generated by $\{(1, 2), (2, 3)\}$.

5. (5 points each) Consider the following permutation in S_6:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 5 & 2 \end{pmatrix}$$

(a) Find σ^{-1}.

(b) Express σ as a product of disjoint cycles.

(c) Express σ as a product of transpositions.

(d) Is σ even or odd? Explain.

(e) Find the order of σ.

6. (10 points) Let G be a group and let H be a subgroup of G. Prove that every left coset of H has the same cardinality as H by exhibiting a bijection between the two sets.

7. (Bonus!) Prove that the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}^+, \cdot)$ are not isomorphic by completing the following.

   (a) Suppose $\varphi : \mathbb{Q} \to \mathbb{Q}^+$ is an isomorphism. Show that for every rational number r we have $\varphi(r/2)^2 = \varphi(r)$.

   (b) Deduce from 8a that if $(\mathbb{Q}, +)$ and $(\mathbb{Q}^+, \cdot)$ are isomorphic, then every positive rational number has a rational square root, and conclude that such an isomorphism is impossible.