Math 430 – Problem Set 5 Solutions

Due March 18, 2016

10.4. Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$; that is, matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$ and $ac \neq 0$. Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

where $x \in \mathbb{R}$.

(a) Show that U is a subgroup of T.

Solution. Taking $x = 0$, we see that the identity matrix is in U. The inverse of $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 1 & -x \\ 0 & 1 \end{smallmatrix}\right)$, which is also in U. Finally,

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix},$$

which is in U.

(b) Prove that U is abelian.

Solution. This follows from the formula for multiplication of elements of U given above, together with the commutativity of addition in $\mathbb{R}$.

(c) Prove that U is normal in T.

Solution.

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} &= \begin{pmatrix} a & ax+b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1/a & -b/(ac) \\ 0 & 1/c \end{pmatrix} \\ &= \begin{pmatrix} 1 & ax/c \\ 0 & 1 \end{pmatrix} \end{aligned}$$

(d) Show that T/U is abelian.

Solution. Note that

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix},$$

so every coset in T/U has a representative that is a diagonal matrices. Since diagonal matrices commute with each other, T/U is commutative.

Alternatively, note that

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}^{-1} &= \begin{pmatrix} aa' & ab'+bc' \\ 0 & cc' \end{pmatrix} \begin{pmatrix} 1/(aa') & -(b'c+bc')/(aca'c') \\ 0 & 1/(cc') \end{pmatrix} \\ &= \begin{pmatrix} 1 & (ab'-b'c)/(cc') \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Since U contains the commutator subgroup of T, T/U is abelian by 10.14.

(e) Is T normal in $\mathrm{GL}_2(\mathbb{R})$?

Solution. No. For example,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

10.7. Prove or disprove: If H is a normal subgroup of G such that H and G/H are abelian, then G is abelian.

Solution. $U \triangleleft T$ from the previous problem provides a counterexample, as does $A_3 \triangleleft S_3$.

10.9. Prove or disprove: If H and G/H are cyclic, then G is cyclic.

Solution. $A_3 \triangleleft S_3$ provides a counterexample, as does $\mathbb{Z}_2 \triangleleft \mathbb{Z}_2 \times \mathbb{Z}_2$.

10.14. Let G be a group and let $G' = \langle aba^{-1}b^{-1} \rangle$; that is, G' is the subgroup of all finite products of elements in G of the form $aba^{-1}b^{-1}$. The subgroup G' is called the commutator subgroup of G.

(a) Show that G' is a normal subgroup of G.

Solution. Suppose $\gamma = aba^{-1}b^{-1}$ is a generator of G'. Since $g\gamma g^{-1} = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1}$, we have that $g\gamma g^{-1} \in G'$. Since conjugation by g is a homomorphism, every product of such elements will also be an element of G'. Thus G' is normal.

Alternatively, note that $g\gamma g^{-1} = g\gamma g^{-1}\gamma^{-1}\gamma \in G'$ since $\gamma \in G'$ and $g\gamma g^{-1}\gamma^{-1}$ is a commutator.

(b) Let N be a normal subgroup of G. Prove that G/N is abelian if and only if N contains the commutator subgroup of G.

Solution. Suppose $a, b \in G$. Then

$$\begin{aligned} (aN)(bN) = (bN)(aN) &\Leftrightarrow Nab = Nba \\ &\Leftrightarrow Naba^{-1}b^{-1} = N \\ &\Leftrightarrow aba^{-1}b^{-1} \in N. \end{aligned}$$

So

$$\begin{aligned} G/N \text{ is abelian} &\Leftrightarrow (aN)(bN) = (bN)(aN) \text{ for all } a, b \in G \\ &\Leftrightarrow aba^{-1}b^{-1} \in N \text{ for all } a, b \in G \\ &\Leftrightarrow G' \subseteq N. \end{aligned}$$

11.2. Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel?

(a) $\phi : \mathbb{R}^* \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\phi(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

Solution. This is a homomorphism since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & ab \end{smallmatrix}\right)$. The kernel is $\{1\} \subset \mathbb{R}^*$.

(b) $\phi : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

Solution. This is a homomorphism since $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ b & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ a+b & 1 \end{smallmatrix}\right)$. The kernel is $\{0\} \subset \mathbb{R}$.

(c) $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d$$

Solution. This is not a homomorphism since it maps the identity to 2, which is not the identity in $\mathbb{R}$.

(d) $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^*$ defined by

$$\phi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = ad - bc$$

Solution. This is a homomorphism, since

$$\begin{aligned}
\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) &= \phi\left(\begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}\right) \\
&= (aa'+bc')(cb'+dd') - (ab'+bd')(ca'+dc') \\
&= ada'd' + bcb'c' - adb'c' - bca'd' \\
&= (ad-bc)(a'd'-b'c') \\
&= \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\phi\left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right).
\end{aligned}$$

The kernel is $\mathrm{SL}_2(\mathbb{R})$, the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of matrices of determinant 1.

(e) $\phi : \mathbb{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\phi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = b,$$

where $\mathbb{M}_2(\mathbb{R})$ is the additive group of 2×2 matrices with entries in $\mathbb{R}$.

Solution. This is a homomorphism, since

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = b + b' = \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \phi\left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right).$$

The kernel is the group (under addition) of lower triangular matrices:

$$\left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R}\right\}.$$

11.9. If $\phi : G \to H$ is a group homomorphism and G is abelian, prove that $\phi(G)$ is abelian.

Solution. If $x, y \in \phi(G)$ then there exist $a, b \in G$ with $x = \phi(a)$ and $y = \phi(b)$. Then $xy = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = yx$, so $\phi(G)$ is abelian.