Math 400 - Practice Final Solutions: Part 1

1. Consider the line L with equation $2x + 3y = 12$.

   (a) What is the slope of L?

   Solution. Solving for y, we have

   $$3y = -2x + 12$$
   $$y = -\frac{2}{3}x + 4$$

   So the slope is $-\frac{2}{3}$.

   (b) Find an equation for the line M through $(6, 4)$ that is perpendicular to L.

   Solution. To find the slope, we use the fact that the slope of a perpendicular line is the negative reciprocal of the original. So the slope is $\frac{3}{2}$. Since this line passes through $(6, 4)$, it has equation

   $$y - 4 = \frac{3}{2}(x - 6)$$

   using point-slope form.

   (c) Find the coordinates of the point P where M intersects L.

   Solution. We solve for the intersection of the two lines. Substituting the first equation into the second, we have

   $$\begin{aligned} -\frac{2}{3}x + 4 - 4 &= \frac{3}{2}(x - 6) \\ -\frac{2}{3}x &= \frac{3}{2}x - 9 \\ -\frac{13}{6}x &= -9 \\ x &= \frac{54}{13}. \end{aligned}$$

   Substituting back, we get $y = -\frac{2}{3}\frac{54}{13} + 4 = \frac{16}{13}$.

   (d) Find the distance between P and $(6, 4)$.

   Solution. We use the distance formula. The distance is $\sqrt{(6 - \frac{54}{13})^2 + (4 - \frac{16}{13})^2}$.

2. Consider the system

   $$\begin{aligned} 3y - 3z &= 3 \\ x - y + 2z &= 5 \\ 2x - 3y + 5z &= 9 \end{aligned}$$

   (a) Write down the augmented matrix corresponding to this system.

Solution.

$$\left[\begin{array}{ccc|c} 0 & 3 & -3 & 3 \\ 1 & -1 & 2 & 5 \\ 2 & -3 & 5 & 9 \end{array}\right]$$

(b) Find the row-reduced echelon form of this matrix.

Solution.

$$\xrightarrow[R_3-2R_1]{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 3 & -3 & 3 \\ 0 & -1 & 1 & -1 \end{array}\right] \xrightarrow[R_3+R_2]{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(c) Does this system have a unique solution, infinitely many solutions or no solutions? Why? If there are solutions, write down a general form for the solution.

Solution. It has solutions, since there is no row corresponding to the equation $0 = 1$. It has infinitely many, since there is a column without any leading ones (which will correspond to a parameter).

Since there is one column without any leading ones, we need one parameter; say t. The we set $z = t$, and the first two rows translate to equations

$$\begin{aligned} x + z &= 6 \\ y - z &= 1, \end{aligned}$$

so

$$\begin{aligned} x &= 6 - t \\ y &= 1 + t \\ z &= t. \end{aligned}$$

3. Solve the system

$$\begin{aligned} 3x - 5y &= 2 \\ -x + 3y &= 4 \end{aligned}$$

using matrix inverses.

Solution. We translate to a matrix equation, yielding

$$\begin{bmatrix} 3 & -5 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

We use the formula for the inverse of a 2×2 matrix to find

$$\begin{bmatrix} 3 & -5 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{3 \cdot 3 - (-5) \cdot (-1)} \begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix}.$$

Multiplying both sides of the equation by this inverse gives

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 13/2 \\ 7/2 \end{bmatrix}$$

4. Construct the truth table for the compound proposition $(p \to q) \leftrightarrow (p \wedge q)$.

Solution.

p	q	$p \to q$	$p \wedge q$	$(p \to q) \leftrightarrow (p \wedge q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

5. For each of the following propositions, use *exactly one* of the laws of logic to transform it into an equivalent proposition.

   (a) $p \wedge (q \vee r)$.

   Solution. This is the distributive law for conjunction (and) over disjunction (or). The equivalent proposition is $p \wedge q \vee p \wedge r$.

   (b) $\sim (p \wedge q)$.

   Solution. This is De Morgan's law. The equivalent proposition is $\sim p \vee \sim q$.

   (c) $p \wedge p$.

   Solution. The equivalent proposition is just p.

6. Determine whether the following argument is valid:

   If Al goes to the gym, then he is an athlete.
   If Brenda does not go to the gym, then Cindy lifts weights.
   Cindy does not lift weights
   ___
   Therefore, Al does not go to the gym, and Brenda does not go to the gym.

   Solution. Let p be the proposition "Al goes to the gym," q be the proposition "Al is an athlete," r the proposition "Brenda does not go to the gym" and s the proposition "Cindy lifts weights." Then the argument takes the symbolic form

   $p \to q$
   $r \to s$
   $\sim s$
   ___
   $\sim p \wedge r$

   One can check the validity by making a truth table with sixteen rows. But an easier way is to note that p and q only appear in the first premise and in the conclusion, so it should be possible to make all of the premises true and the conclusion false. In fact, if p is True, q is True, s is False and r is False then all of the premises are True but the conclusion is False. So the argument is invalid.

7. What is the effective rate of return on a savings account that pays 5% nominal interest, compounded monthly? You do not need to simplify your answer.

   Solution. The effective rate is $(1 + \frac{0.05}{12})^{12} - 1$.

8. If you deposit \$100 per month for one year, earning 12% interest compounded monthly, how much money will you have at the end of the year? (You make a total of 12 deposits, and the first earns interest 12 times)

   Solution. You will have $100(1.01 + 1.01^2 + \cdots + 1.01^{12}) = 100 \cdot 1.01 \cdot \frac{1.01^{12}-1}{1.01-1}$ dollars.