Math 400 - Practice Exam 1 Solutions

1. Consider the points with coordinates $P(2, 1)$ and $Q(8, 9)$.

   (a) Find the coordinates of the point R lying halfway in between P and Q.

   Solution. R should have x-coordinate halfway between 2 and 8, and y-coordinate halfway between 1 and 9. So $R = (\frac{2+8}{2}, \frac{1+9}{2}) = (5, 5)$.

   (b) Find an equation for the circle passing through P and Q with center R.

   Solution. The radius of the circle is the distance from R to P, or $\sqrt{(5-2)^2 + (5-1)^2} = 5$ (note that this is also the distance from R to Q since R is the midpoint of $\overline{PQ}$). Therefore the equation of the circle is

   $$(x-5)^2 + (y-5)^2 = 5^2.$$

   (c) Find an equation for the line passing through R that is perpendicular to the line segment $\overline{PQ}$.

   Solution. The slope of the line segment $\overline{PQ}$ is $\frac{9-1}{8-2} = \frac{4}{3}$, so the slope of a perpendicular line is $-\frac{3}{4}$. Therefore the equation of the line passing through R that is perpendicular to $\overline{PQ}$ is

   $$y - 5 = -\frac{3}{4}(x-5).$$

2. A factory manufacturing bicycles has fixed costs of \$10000 per month and a production cost of \$120 per bicycle. They sell the bicycles for \$160 each.

   (a) What is the cost function?

   Solution. Let n be the number of bicycles produced. Then the cost function is

   $$C(n) = 10000 + 120n.$$

   (b) What is the revenue function?

   Solution. The revenue function is

   $$R(n) = 160n.$$

   (c) What is the profit function?

   Solution. The profit is the difference, or

   $$P(n) = 160n - (10000 + 120n) = 40n - 10000.$$

   (d) What is their break-even point?

   Solution. The break-even point can be found either by setting cost equal to revenue, or profit equal to zero. Solving for n gives

   $$\begin{aligned} 40n - 10000 &= 0 \\ n &= 250. \end{aligned}$$

(e) What is their profit if they sell 500 bicylces?

Solution. We compute

$$P(500) = 40 \cdot 500 - 10000 = 10000.$$

So the profit is \$10000.

3. Consider the system of equations

$$\begin{aligned} x + y + 2z &= 4 \\ 2x + 2y + 6z &= 10. \end{aligned}$$

(a) Write down the augmented matrix corresponding to this system.

Solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & 2 & 6 & 10 \end{array}\right]$$

(b) Use row operations to find the row-reduced echelon form of the augmented matrix of part (a).

Solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & 2 & 6 & 10 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(c) Does this system have a unique solution, infinitely many solutions, or no solutions? If there are solutions, write down a general form for the solution.

Solution. This system has infinitely many solutions, since it is consistent (there is no row corresponding to an equation $0 = 1$) and there is a column without a leading one (the second column). Writing $y = t$, we get the parametric solution

$$\begin{aligned} x &= 2 - t \\ y &= t \\ z &= 1 \end{aligned}$$

(d) Write down the matrix equation corresponding to this system.

Solution.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

(e) Why is it impossible to solve this system using matrix inverses?

Solution. The matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$

can't possibly have an inverse since it is not square.

4. Consider the system of equations

$$\begin{aligned} x + 2y \phantom{{}-4z} &= 2 \\ y - 4z &= 4 \\ 2z &= 8 \end{aligned}$$

(a) Express the system as a matrix equation $A\mathbf{v} = \mathbf{b}$.

Solution.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

(b) Find A^{-1}.

Solution. We augment A with a 3×3 identity matrix and row reduce.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{R_2 \to R_2 + 4R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -4 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

(c) Solve the system using the matrix inverse.

Solution.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -38 \\ 20 \\ 4 \end{bmatrix}$$

5. For what value(s) of k does the matrix

$$\begin{pmatrix} 2 & 3 \\ 4 & k \end{pmatrix}$$

have no inverse?

Solution. A matrix has no inverse precisely when its determinant is zero. For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $ad - bc$, which is $2k - 12$ in this case. So this matrix will have no inverse when

$$2k - 12 = 0,$$

or $k = 6$.

6. Construct the truth table for the compound proposition $(\sim p \vee q) \wedge (p \vee q)$.

Solution.

p	q	$\sim p$	$\sim p \vee q$	$p \vee q$	$(\sim p \vee q) \wedge (p \vee q)$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

7. Which of these steps is not justified by one of the laws of logic?

$$\begin{aligned} (\sim p \vee q) \wedge (p \vee q) &\Leftrightarrow (\sim p \wedge p) \vee (q \wedge q) \\ &\Leftrightarrow c \vee (q \wedge q) \\ &\Leftrightarrow c \vee q \\ &\Leftrightarrow q \end{aligned}$$

Solution. The first logical equivalence is the only one that's not justified. If we used the distributive law properly, we would instead have

$$\begin{aligned}(\sim p \vee q) \wedge (p \vee q) &\Leftrightarrow ((\sim p \vee q) \wedge p) \vee ((\sim p \vee q) \wedge q) \\ &\Leftrightarrow (\sim p \wedge p) \vee (q \wedge p) \vee (\sim p \wedge q) \vee (q \wedge q)\end{aligned}$$

8. Consider the argument

   If you prepare for the exam, then you will pass.
   You will pass the exam.

   Therefore, you prepared for the exam.

   (a) What are the premises and what is the conclusion?

   Solution. The premises are the two propositions

   i. "If you prepare for the exam, then you will pass."
   ii. "You will pass the exam."

   The conclusion is "You prepared for the exam."

   (b) Write the argument symbolically, and determine whether it is valid.

   Solution. If p is the proposition that you prepared for the exam and q is the proposition that you will pass the exam, then the argument takes the form

   $$\begin{array}{l} p \to q \\ \underline{q \qquad} \\ p \end{array}$$

   This is not a valid argument, since when p is false and q is true then both premises are true but the conclusion is false.