Math 400 - Exam 2 Solutions

Name: ______________________________

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Problem 1 (8 points)	
Problem 2 (8 points)	
Problem 3 (12 points)	
Problem 4 (26 points)	
Problem 5 (14 points)	
Problem 6 (6 points)	
Problem 7 (12 points)	
Problem 8 (14 points)	
Total (100 points)	

1. (8 points) Suppose you invest \$1000 in a savings account, and you leave it there for 45 years (until you retire). If the account earns a nominal rate of 4% per year and is compounded twice a year, give an expression for the amount of money in the account when you retire?

   Solution. $A = P(1 + \frac{r}{m})^{mt} = 1000(1 + \frac{0.04}{2})^{2 \cdot 45}$.

2. (8 points) How many ways are there to choose 3 apples from a pile of 6 apples, ignoring order? Give you answer as.a number (rather than leaving it unsimplified).

   Solution. This is the binomial coefficient $\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$.

3. Find the following sums (show your work; adding terms one by one will not receive credit)

(a) (6 points) $5 + 9 + 13 + 17 + \cdots + 85$

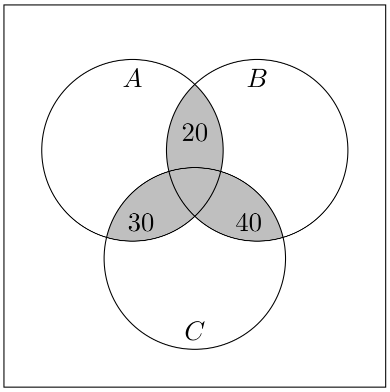
Solution. Here $a = 5$, $d = 4$ and $a_n = a + (n-1)d = 85$, so $n = 21$. Thus the sum is $(85 + 5) \cdot \frac{21}{2} = 945$.

(b) (6 points) $2 + 6 + 18 + 54 + \cdots + 2 \cdot 3^7$

Solution. Here $a = 2$, $r = 3$ and $n - 1 = 7$ so $n = 8$. Thus the sum is

$$2 \cdot \frac{3^8 - 1}{3 - 1} = 3^8 - 1.$$

4. Consider the follow Venn diagram of the sets A, B and C, where the numbers indicate the number of elements in that region.



(a) (6 points) Give an expression in terms of A, B and C, intersections, unions and complements that describes the shaded region (Hint: there are multiple correct answers. One possibility gives it as the union of three sets).

Solution. $(A \cap B) \cup (B \cap C) \cup (C \cap A)$.

(b) (10 points) Suppose that A, B and C all contain 100 elements, and $A \cup B \cup C$ contains 190 elements. How many elements are contained in the shaded region?

Solution. Let x be the number of elements in the small, central region. Then the top left region has $50 - x$ elements (since A has 100), the top right has $40 - x$ (since B has 100), and the bottom region has $30 - x$ (since C has 100). Thus the overall count is

$$x + 20 + 30 + 40 + (50 - x) + (40 - x) + (30 - x) = 210 - 2x.$$

Since this is supposed to equal 190, we get that $x = 10$ and thus the shaded region has 100 elements.

(c) (5 points) If an element of $A \cup B \cup C$ is selected uniformly at random, what is the probability that it lies in the shaded region?

Solution. There are 100 elements in the shaded region and 190 in $A \cup B \cup C$, so the probability is $\frac{100}{190} = \frac{10}{19}$.

(d) (5 points) Given that an element selected lies in the shaded region, what is the probability that it lies in A?

Solution. There are 60 elements in the intersection of the shaded region with A, and 100 in the shaded region overall. So the probability is $\frac{60}{100} = \frac{3}{5}$.

5. You line up five books on a shelf at random. Three of the books (titled A, B and C) are blue and two (titled D and E) are yellow.

(a) (6 points) How many different arrangements of the five books are possible?

Solution. There are $5! = 120$ arrangements of the books if you consider them all to be distinct. I also gave full credit for $\binom{5}{2} = \binom{5}{3} = 10$, which is the number of ways to arrange the colors (ignoring the titles).

(b) (8 points) What is the probability that all of the blue books end up to the left of the yellow books?

Solution. The probability is $\frac{1}{10}$. One way to see this is by the multiplicative principle: there are 3 choices for the first blue book, 2 for the second, 1 for the third, 2 for the first yellow book and 1 for the second. Dividing by 120 gives $\frac{1}{10}$.

If your answer was 10 to part (a), then another natural way to think about it is to say that, among those 10 colorings, exactly 1 is the specific coloring BBBYY.

6. (6 points) Pennsylvania license plates have three letters and then four digits (e.g. ABA-1233). How many license PA license plates are possible?

Solution. There are 26 choices for each letter and 10 for each digit. By the multiplication principle, the total number of license plates is therefore $26^3 \cdot 10^4$.

7. Suppose that events A and B are independent with $P(A) = 0.5$ and $P(B) = 0.4$.

   (a) (6 points) What is the probability that both A and B occur?

   Solution. $P(A \cap B) = P(A) \cdot P(B) = 0.5 \cdot 0.4 = 0.2.$

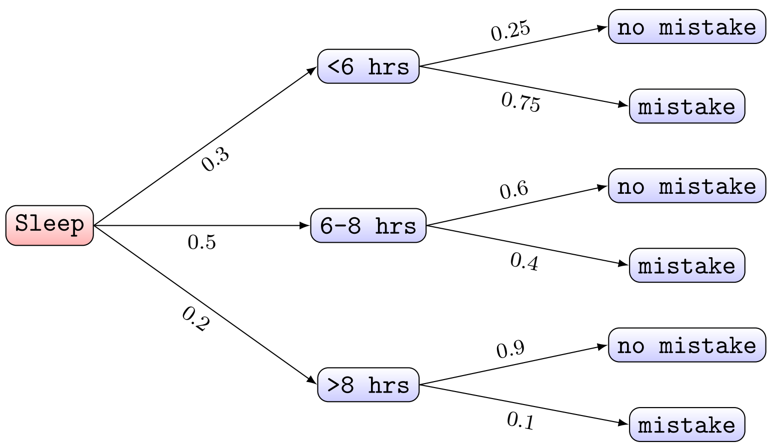
   (b) (6 points) What is the probability that either A or B occurs?

   Solution. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.4 - 0.2 = 0.7.$

8. Suppose that the probability you make a mistake on your homework depends on how much sleep you got the previous night. If you got at least 8 hours of sleep, there is a 10% chance of a mistake. If you got between 6 and 8 hours of sleep, there is a 40% chance of a mistake. And if you got less than 6 hours of sleep, there is a 75% chance of a mistake. The likelihood that you get at least 8 hours of sleep is 20% and that you get between 6 and 8 hours of sleep is 50%.

   (a) (6 points) Draw a tree diagram corresponding to this setup.

   Solution.



   (b) (8 points) What is the probability that you make a mistake on your homework?

   Solution. There are three endpoints leading two a mistake, one for each of the different amounts of sleep. The probability of each is the product of the probabilities along the branches leading to that result, and the total probability is the sum of the three (since these events are mutually exclusive). Thus the overall probability is

$$(0.3)(0.75) + (0.5)(0.4) + (0.2)(0.1) = 0.445.$$