Math 400 - Exam 1 Solutions

1. Find the points on the x-axis that are a distance of 5 from the point $(1, 4)$.

   Solution. The circle of radius 5 around $(1, 4)$ has equation

   $$(x-1)^2 + (y-4)^2 = 25.$$

   Points along the x-axis have $y = 0$, so we need to solve

   $$(x-1)^2 + 16 = 25,$$

   yielding $x = 4$ or $x = -2$. So the two points are $(-2, 0)$ and $(4, 0)$.

2. Let L be the line passing through the points $P(1, 1)$ and $Q(5, 7)$ and L' be the line passing through the points $P'(-3, 0)$ and $Q'(0, 2)$.

   (a) Find equations for L and L'.

   Solution. The slope of L is $\frac{7-1}{5-1} = \frac{3}{2}$ and the slope of L' is $\frac{2-0}{0-(-3)} = \frac{2}{3}$. So the equations (in point slope form) are

   $$\begin{aligned} y - 1 &= \frac{3}{2}(x-1) \\ y - 2&\frac{2}{3}x. \end{aligned}$$

   (b) Is L perpendicular to L'? Why or why not?

   Solution. No: to be perpendicular the slopes would need to be negative reciprocals of each other, but both slopes are positive.

   (c) Find the intersection of L and L'.

   Solution. We substitute and multiply by 6, giving

   $$\begin{aligned} \frac{2}{3}x + 2 - 1 &= \frac{3}{2}(x-1) \\ 4x + 6 &= 9x - 9 \\ 5x &= 15 \\ x &= 3 \end{aligned}$$

   and thus $y = 4$. So the intersection is the point $(3, 4)$.

3. A factory is making three kinds of items (puzzles, rattles, and abacuses) using three kinds of components (screws, beads, and dowels). It takes one screw, two beads and one dowel to make a puzzle; it takes one bead and two dowels to make a rattle; and it takes three screws, four beads and one dowel to make an abacus. At the end of the day, the manager determines that she has eleven screws, seventeen beads and seven dowels remaining. She wants to determine how many puzzles, rattles and abacuses should she make in order to use all of these materials.

(a) Write a system of equations whose solution will give the number of puzzles, rattles and abacuses to make.

Solution. Each equation corresponds to a component. If the number of puzzles, rattles and abacuses made are p, r and a, then we have

$$\begin{aligned} p \phantom{{}+r} + 3a &= 11 \\ 2p + r + 4a &= 17 \\ p + 2r + a &= 7 \end{aligned}$$

(b) Use row operations to solve the system.

Solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 2 & 1 & 4 & 17 \\ 1 & 2 & 1 & 7 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & 1 & -2 & -5 \\ 0 & 2 & -2 & -4 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_1-3R_3]{R_2+2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Thus the manager should make two puzzles, one rattle and three abacuses.

4. Consider the system of equations

$$\begin{aligned} x + 3y &= 1 \\ 2x - y &= 2 \\ 3x + 2y &= 3 \end{aligned}$$

(a) Write down the augmented matrix corresponding to this system.

Solution.

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & -1 & 2 \\ 3 & 2 & 3 \end{array}\right]$$

(b) Use row operations to find the row-reduced echelon form of the augmented matrix of part (a).

Solution.

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & -1 & 2 \\ 3 & 2 & 3 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -4 & 0 \\ 0 & -7 & 0 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & -7 & 0 \end{array}\right] \xrightarrow{R_3+7R_2} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(c) Does this system have a unique solution, infinitely many solutions, or no solutions? If there are solutions, write down a general form for the solution.

Solution. This system has a unique solution since it is consistent and every column corresponding to a variable has a leading one. The solution is $x = 1$ and $y = 0$, as we can see by translating the reduced row echelon form back to a system of equations.

(d) Write down the matrix equation corresponding to this system.

Solution.

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(e) Why is it impossible to solve this system using matrix inverses?

Solution. $\begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 3 & 2 \end{bmatrix}$ cannot have an inverse since it is not square.

5. Let $A = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$.

(a) Find A^{-1}.

Solution. Using the formula for the inverse of a 2×2 matrix, we have

$$A^{-1} = \frac{1}{(-3)(-5) - (4)(4)} \begin{bmatrix} -5 & -4 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}.$$

(b) Use A^{-1} to solve the equation

$$\begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Solution. Multiplying by A^{-1} we get

$$\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 17 \end{bmatrix}.$$

Adding $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to both sides, we get $x = 23$ and $y = 18$.

(c) Explain why the equation

$$\begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

has no solutions.

Solution. You can't multiply a 2×2 matrix by a 3×1 matrix: the sizes don't match.

6. Construct the truth table for the compound proposition $\sim p \wedge (q \to r)$.

Solution.

p	q	r	$\sim p$	$q \to r$	$\sim p \wedge (q \to r)$
T	T	T	F	T	F
T	T	F	F	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

7. Let p be the proposition "The milk's expiration date has passed" and q the proposition "The milk has gone sour."

(a) Express $p \to q$ in words.

Solution. If the milk's expiration date has passed then the milk has gone sour.

(b) Express the contrapositive of $p \to q$ both symbolically and in words.

Solution. $\sim q \to \sim p$: If the milk has not gone sour then the milk's expiration date has not passed.

(c) Express the converse of $p \to q$ both symbolically and in words.

Solution. $q \to p$: If the milk has gone sour then the milk's expiration date has passed.

(d) Express the inverse of $p \to q$ both symbolically and in words.

Solution. $\sim p \to \sim q$: If the milk's expiration date has not passed then the milk has not gone sour.

(e) Which of these statements are logically equivalent?

Solution. The original is equivalent to the contrapositive and the converse is equivalent to the inverse.

8. Consider the argument $\begin{array}{l} p \wedge \sim q \\ p \to q \\ \hline p \end{array}$

(a) What are the premises and what is the conclusion?

Solution. The premises are $p \wedge \sim q$ and $p \to q$.

(b) Determine whether this argument is valid, justifying your reasoning either with a truth table or the laws of logic.

Solution. This argument is valid. We need to check that $(p \wedge \sim q \wedge (p \to q)) \to p$ is a tautology. But $p \to q$ is the same as $q \vee \sim p$, so we get $(p \wedge \sim q \wedge (q \vee \sim p)) \to p$, or $(p \wedge \sim q \wedge q \vee p \wedge \sim q \wedge \sim p) \to p$. But $\sim q \wedge q$ and $p \wedge \sim p$ are both contradictions, so this is $(c \vee c) \to p$ or $c \to p$. This is a tautology.

Alternatively, you can construct the truth table.