Math 240 Practice Problems

Note that a few of these questions are somewhat harder than questions on the final will be, but they will all help you practice the material from this semester.

1. Consider the three points $A = (3, 1, 4)$, $B = (6, 4, 4)$, $C = (3, 4, 1)$.

   (a) Find the area of the triangle formed by A, B, and C.

   Solution. We use cross products. The area of the triangle is

$$\begin{aligned}
\frac{1}{2}|\vec{AB} \times \vec{AC}| &= \frac{1}{2}|\langle 3, 3, 0\rangle \times \langle 0, 3, -3\rangle| \\
&= \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 0 \\ 0 & 3 & -3 \end{vmatrix} \\
&= \frac{1}{2}|\langle -9, 9, 9\rangle| \\
&= \frac{9}{2}\sqrt{3}.
\end{aligned}$$

   (b) Find the angles of this triangle.

   Solution.

$$\begin{aligned}
\angle BAC &= \cos^{-1}\Big(\frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}|\,|\vec{AC}|} \\
&= \cos^{-1}\left(\frac{\langle 3, 3, 0\rangle \cdot \langle 0, 3, -3\rangle}{|\langle 3, 3, 0\rangle|\,|\langle 0, 3, -3\rangle|}\right) \\
&= \cos^{-1}\left(\frac{9}{18}\right) \\
&= \frac{\pi}{3}
\end{aligned}$$

   A similar calculation shows that $\angle ABC = \frac{\pi}{3}$, and then the last angle is as well by subtraction from π. So this triangle is equilateral.

   (c) Find an equation for the plane containing A, B and C.

   Solution. We found the cross product in part (a), and we can divide by 9 since any choice of normal works. This gives an equation

$$\begin{aligned}
-x + y + z &= -3 + 1 + 4 \\
-x + y + z &= 2
\end{aligned}$$

2. Let $\mathbf{v} = \langle 7, 6, 5\rangle$ and $\mathbf{w} = \langle 3, 2, -1\rangle$. Express $\mathbf{v}$ as the sum of two perpendicular vectors, one of which points in the direction of $\mathbf{w}$.

   Solution. We project $\mathbf{v}$ onto $\mathbf{w}$, getting

$$\begin{aligned}
\text{proj}_{\mathbf{w}}\,\mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} \\
&= \frac{21 + 12 - 5}{9 + 4 + 1}\langle 3, 2, -1\rangle \\
&= \langle 6, 4, -2\rangle.
\end{aligned}$$

   Thus we can express $\langle 7, 6, 5\rangle$ as a sum of $\langle 6, 4, -2\rangle$ and $\langle 1, 2, 7\rangle = \langle 7, 6, 5\rangle - \langle 6, 4, -2\rangle$, which are perpendicular to each other.

3. The eight vertices of a cube centered at $(0,0,0)$ of side length 2 are at $(\pm 1, \pm 1, \pm 1)$.

   (a) Find the four vertices of the cube, including $(1,1,1)$, that form a regular tetrahedron.

   Solution. The four vertices are $A = (1,1,1)$, $B = (1,-1,-1)$, $C = (-1,1,-1)$ and $D = (-1,-1,1)$.

   (b) A methane molecule consists of a hydrogen atom at each of the vertices of a regular tetrahedron and a carbon atom at the center. Find the "bond angle," i.e. the angle made by the vectors from the carbon atom to two hydrogen atoms.

   Solution. The carbon atom is at the center, so the bond angle is just given by the angle between the position vectors for A and for B. This is

$$\cos^{-1}\left(\frac{1-1-1}{\sqrt{3}\sqrt{3}}\right) \approx 1.911 \approx 109.5^\circ$$

   (c) Find the angle between two adjacent edges of the tetrahedron, and the angle between two opposite edges.

   Solution. We can either use dot products to find the angle between adjacent sides,

$$\begin{aligned}
\angle BAC &= \cos^{-1}\left(\frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\,|\vec{AC}|}\right) \\
&= \cos^{-1}\left(\frac{\langle 0,-2,-2\rangle \cdot \langle -2,0,-2\rangle}{\sqrt{8}\sqrt{8}}\right) \\
&= \cos^{-1}(1/2) \\
&= \frac{\pi}{3}
\end{aligned}$$

   (d) Find the area of a face of the tetrahedron using vectors.

   Solution. The area is half the length of the cross product of the vectors along the edges, or $\frac{1}{2}|\langle 0,-2,-2\rangle \times \langle -2,0,-2\rangle = 2\sqrt{3}$.

   (e) Find the volume of the tetrahedron using vectors.

   Solution. The scalar triple product gives the volume of the parallelogram spanned by the vectors, and the tetrahedron has volume one sixth as large – you can see an example of this by examining the triple integral

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dV = \frac{1}{6}.$$

   So the volume of the regular tetrahedron is

$$\frac{1}{6}\begin{vmatrix} 0 & -2 & -2 \\ -2 & 0 & -2 \\ -2 & -2 & 0 \end{vmatrix} = \frac{8}{3}.$$

4. Describe the set of vectors $\mathbf{v}$ such that $\frac{\mathbf{v}}{|\mathbf{v}|} \cdot \langle 1,1,1\rangle = 1$.

   Solution. We can rescale to get the equation

$$\frac{\mathbf{v}}{|\mathbf{v}|} \cdot \langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\rangle = 1/\sqrt{3}.$$

   Since both $\frac{\mathbf{v}}{|\mathbf{v}|}$ and $\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\rangle$ are unit vectors, this is equivalent to the condition that $\frac{\mathbf{v}}{|\mathbf{v}|}$ and $\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\rangle$ are at an angle of $\cos^{-1}(1/\sqrt{3})$, which is equivalent to the condition that $\mathbf{v}$ and $\langle 1,1,1\rangle$ are at an angle of $\cos^{-1}(1/\sqrt{3})$. Since $\langle 1,1,1\rangle$ is fixed, the set of such $\mathbf{v}$ forms a cone.

5. Consider the helix given by the parametric equation $\mathbf{r}(t) = (\cos t, \sin t, t)$.

   (a) Find an equation for the line tangent to the helix at $\mathbf{r}(t)$ (use u for the parameter of the line).

   Solution. The tangent vector is $\langle -\sin t, \cos t, 1\rangle$ so the parametric equation for the tangent line is

$$L(u) = \langle \cos t, \sin t, t\rangle + u\langle -\sin t, \cos t, 1\rangle = (\cos t - u\sin t, \sin t + u\cos t, t + u).$$

   (b) Find a parameterization for the curve traced out by the intersection of the xy-plane and the tangent lines described above.

Solution. The line will intersect the xy-plane when $z = 0$, ie when $t + u = 0$ or $u = -t$. We get a parameterization for the curve by substituting into the x and y coordinates of $L(u)$, giving $(\cos t + t\sin t, \sin t - t\cos t)$.

6. Consider the curve defined parametrically by $\mathbf{r}(t) = (4e^{3t}, e^{2t}, 2e^{2t})$.

   (a) At what value t_0 is the tangent vector to $\mathbf{r}$ at $\mathbf{r}(t_0)$ parallel to the plane $\mathbf{P}$ given by the equation $x+2y-7z = 17$?

   Solution. The tangent at $\mathbf{r}(t)$ is $\langle 12e^{3t}, 2e^{2t}, 4e^{2t}\rangle$, which will be parallel to $\mathbf{P}$ when $\langle 12e^{3t}, 2e^{2t}, 4e^{2t}\rangle \cdot \langle 1, 2, -7\rangle = 0$, ie when $12e^{3t} - 24e^{2t} = 0$, or $t_0 = \ln(2)$.

   (b) Determine the parametric and symmetric equations of the line $\mathbf{L}$ tangent to $\mathbf{r}$ at $\mathbf{r}(t_0)$.

   Solution. Plugging in t_0, we have $\mathbf{r}(t_0) = (4(2)^3, (2)^2, 2(2)^2) = (32, 4, 8)$ and $\mathbf{r}'(t_0) = \langle 96, 8, 16\rangle$. Rescaling the direction vector for simplicity, we can instead use $\langle 12, 1, 2\rangle$. So the parametric equations are

$$\begin{aligned} x(t) &= 32 + 12t \\ y(t) &= 4 + t \\ z(t) &= 8 + 2t \end{aligned}$$

   and the symmetric equations are

$$\frac{x-32}{12} = y - 4 = \frac{z-8}{2}.$$

   Note that there are other correct solutions.

   (c) Determine the parametric equation of the line passing through $\mathbf{r}(t_0)$, parallel to the plane $\mathbf{P}$ and perpendicular to the line $\mathbf{L}$.

   Solution. We can use the cross product to find the direction vector: $\langle 12, 1, 2\rangle \times \langle 1, 2, -7\rangle = \langle -11, 86, 23\rangle$. We get parametric equations

$$\begin{aligned} x(t) &= 32 - 11t \\ y(t) &= 4 + 86t \\ z(t) &= 8 + 23t. \end{aligned}$$

7. Let Q be the point $(1, 4, 3)$ and $\mathcal{P}$ be the plane given by the equation $2x + y - z = 4$.

   (a) Give parametric and symmetric equations of the line passing through Q and perpendicular to $\mathcal{P}$.

   Solution. The direction vector for this line is just $\langle 2, 1, -1\rangle$, so we get parametric equations

$$\begin{aligned} x(t) &= 1 + 2t \\ y(t) &= 4 + t \\ z(t) &= 3 - t \end{aligned}$$

   and symmetric equations

$$\frac{x-1}{2} = y - 4 = 3 - z.$$

   (b) Find the distance from Q to $\mathcal{P}$.

   Solution. We use the formula from the book, yielding a distance of

$$\frac{|2+4-3-4|}{\sqrt{4+1+1}} = \frac{\sqrt{6}}{6}.$$

8. Let $\mathcal{P}$ be the tangent plane to $f(x, y) = 4x^3 - 3y^2$ at the point $(2, 4)$. Find the intersection of $\mathcal{P}$ with the xy-plane.

   Solution. The z-coordinate at $(2, 4)$ is $32 - 48 = -16$ and the partials of f are $f_x(2, 4) = 12(2)^2 = 48$ and $f_y(2, 4) = -6(4) = -24$. So a tangent vector is $\langle 48, -24, -1\rangle$ and an equation for the tangent plane is

$$48(x-2) - 24(y-4) - (z+16) = 0.$$

   Intersecting with the plane $z = 0$ gives a line with equation

$$48x - 24y - 16 = 0.$$

9. Approximate the function $f(x, y, z) = x^2y - 2xz + 1$ near the point $(1, 1, 0)$ by a linear function.

   Solution. The value of f at $(1, 1, 0)$ is $(1)^2(1) - 2(1)(0) + 1 = 2$, and its partial derivatives are

$$
\begin{aligned}
f_x(x, y, z) &= 2(1)(1) - 2(0) = 2 \\
f_y(x, y, z) &= (1)^2 = 1 \\
f_z(x, y, z) &= -2(1) = -2.
\end{aligned}
$$

   The linear approximation should have the same value and partial derivatives, so is given by

$$L(x, y, z) = 2(x - 1) + (y - 1) - 2z + 2.$$

10. Suppose $f(x, y, z) = xyz^2 + 2x - yz$. Compute the directional derivative of f at the point $(1, 1, 1)$ in the direction of $\langle 1, 2, -1\rangle$.

    Solution. We compute the partials

$$
\begin{aligned}
f_x(1, 1, 1) &= (1)(1)^2 + 2 = 3 \\
f_y(1, 1, 1) &= (1)(1)^2 - (1) = 0 \\
f_z(1, 1, 1) &= 2(1)(1)(1) - (1) = 1.
\end{aligned}
$$

    Rescaling $\langle 1, 2, -1\rangle$ to be a unit vector, we get that the directional derivative is

$$\frac{1}{\sqrt{6}}((3)(1) + (0)(2) + (1)(-1)) = \frac{\sqrt{6}}{3}.$$

11. Consider the surface $z = f(x, y) = (2x^2 + 3y^2)e^{-x^2-y^2}$.

    (a) Find the first and second order partials of f.

    Solution.

$$
\begin{aligned}
f_x &= (4x - 2x(2x^2 + 3y^2))e^{-x^2-y^2} = (4x - 4x^3 - 6xy^2)e^{-x^2-y^2} \\
f_y &= (6y - 2y(2x^2 + 3y^2))e^{-x^2-y^2} = (6y - 4x^2y - 6y^3)e^{-x^2-y^2} \\
f_{xx} &= (4 - 12x^2 - 6y^2 - 2x(4x - 4x^3 - 6xy^2))e^{-x^2-y^2} = (4 - 20x^2 - 6y^2 + 8x^4 + 12x^2y^2)e^{-x^2-y^2} \\
f_{xy} &= (-12xy - 2y(4x - 4x^3 - 6xy^2))e^{-x^2-y^2} = (-20xy + 8x^3y + 12xy^3)e^{-x^2-y^2} \\
f_{yy} &= (6 - 4x^2 - 18y^2 - 2y(6y - 4x^2y - 6y^3))e^{-x^2-y^2} = (6 - 4x^2 - 30y^2 + 8x^2y^2 + 12y^4)e^{-x^2-y^2}
\end{aligned}
$$

    (b) Now imagine pouring water onto this surface at the point $(\frac{1}{2}, \frac{1}{3})$. What is the tangent plane to the surface at this point?

    Solution. We have $f(\frac{1}{2}, \frac{1}{3}) = \frac{5}{6}e^{-13/36}$ and $f_x(\frac{1}{2}, \frac{1}{3}) = \frac{7}{6}e^{-13/36}$ and $f_y(\frac{1}{2}, \frac{1}{3}) = \frac{13}{9}e^{-13/36}$. So an equation for the tangent plane is

$$\frac{7}{6}e^{-13/36}\left(x - \frac{1}{2}\right) + \frac{13}{9}e^{-13/36}\left(y - \frac{1}{3}\right) - \left(z - \frac{5}{6}e^{-13/36}\right) = 0.$$

    (c) In what direction will the water flow?

    Solution. Water flows downhill. So it will flow in the direction opposite the gradient, or $\left\langle -\frac{7}{6}, -\frac{13}{9}\right\rangle$.

    (d) As the water level rises, it will gradually fill up a bounded xy region until spilling over and flowing out to infinity. At what depth will this occur and at what point (or points) will the water spill over?

    Solution. This will occur at the saddle points, which can be found by classifying the critical points of f. Setting f_x and f_y equal to 0 and noting that $e^{-x^2-y^2}$ is never zero, we get the system

$$
\begin{aligned}
4x - 4x^3 - 6xy^2 &= 0, \\
6y - 4x^2y - 6y^3 &= 0.
\end{aligned}
$$

The first equation holds if $x = 0$ or $4x^2 + 6y^2 = 4$, and the second holds if $y = 0$ or $4x^2 + 6y^2 = 6$. Since $4x^2 + 6y^2$ can't simultaneously equal 4 and 6, the possibilities are $(0, 0)$, $(0, \pm 1)$ and $(\pm\sqrt{3/2}, 0)$. Substituting these into $D = f_{xx}f_{yy} - f_{xy}^2$ gives 24, $\frac{24}{e^2}$ and 0 respectively, and $f_{xx} = 4$ and $\frac{-2}{e}$ and $\frac{-8}{e^{3/2}}$ respectively. Thus $(0, 0)$ is a minimum, $(0, \pm 1)$ are maxima, and the second derivative test is inconclusive at $(\pm\sqrt{3/2}, 0)$.

To determine what's happening at $(\pm\sqrt{3/2}, 0)$, we can restrict to $y = 0$ and to the ellipse $2x^2 + 3y^2 = 3$. When $y = 0$, we have $f(x, 0) = 2x^2e^{-x^2}$, which has maxima at $x = \pm\sqrt{3/2}$ (using the computation of f_x and f_xx above). Along the ellipse on the other hand, $f(x, y) = 3e^{-x^2-y^2} = 3e^{(y^2-3)/2}$, which has a minimum at $y = 0$. Since one of these restrictions is a maximum and the other a minimum, $f(x, y)$ has a saddle point at $(\pm\sqrt{3/2}, 0)$.

The water will spill over at the points $(\pm\sqrt{3/2}, 0)$ at a depth of $f(\pm\sqrt{3/2}, 0) = 3e^{-3/2}$.

12. Let $f(x, y) = 4xy - x^3y - xy^3$.

(a) Compute all first and second order partial derivatives of f.

Solution.

$$\begin{aligned}
f_x &= 4y - 3x^2y - y^3 \\
f_y &= 4x - x^3 - 3xy^2 \\
f_{xx} &= -6xy \\
f_{xy} &= 4 - 3x^2 - 3y^2 \\
f_{yy} &= -6xy.
\end{aligned}$$

(b) Find and classify (as a local minimum, maximum or saddle point) all critical points of f.

Solution. Setting the first derivatives to 0 gives the system

$$\begin{aligned}
y(4 - 3x^2 - y^2) &= 0 \\
x(4 - x^2 - 3y^2) &= 0
\end{aligned}$$

so either $x = y = 0$ or $y = 0$ and $4 - x^2 = 0$ or $x = 0$ and $4 - y^2 = 0$ or $3x^2 + y^2 = x^2 + 3y^2 = 4$. The last option gives $2(x + y)(x - y) = 0$ by subtraction, and then $x = \pm y = \pm 1$. So the critical points are $(0, 0), (\pm 2, 0), (0, \pm 2)$ and $(\pm 1, \pm 1)$.

Computing $D = f_{xx}f_{yy} - fxy^2$ we get -16, -64, -64 and 32 respectively. Moreover, $f_{xx} = 6$ at $(1, -1)$ and $(-1, 1)$ and $f_{xx} = -6$ at $(1, 1)$ and $(-1, -1)$. So we have saddle points at $(0, 0), (\pm 2, 0)$ and $(0, \pm 2)$, local minima at $(1, -1)$ and $(-1, 1)$ and local maxima at $(1, 1)$ and $(-1, -1)$.

(c) Find the global minimum and global maximum value of f on the disk of radius 2 around the origin.

Solution. Note that $f(x, y) = xy(4 - x^2 - y^2)$, so $f(x, y) = 0$ on the circle of radius 2. Thus the minimum value is -2 occurring at $(1, -1)$ and $(-1, 1)$ and the maximum value is 2 occurring at $(1, 1)$ and $(-1, -1)$.

If you try to use Lagrange multipliers, you'll find that every point on the boundary is a critical point (since the function is identically zero along that boundary).

(d) Find the gradient of f at the point $(2, 2)$.

Solution. The gradient at $(2, 2)$ is $\langle f_x, f_y\rangle = \langle -24, -24\rangle$.

(e) Find the tangent plane to f at the point $(2, 2)$.

Solution. Noting that $f(2, 2) = -16$, an equation for the tangent plane is given by

$$24(x - 2) + 24(y - 2) + (z + 16) = 0.$$

(f) Find a direction $\mathbf{u}$ such that $D_{\mathbf{u}}(f) = \frac{-24}{5}$.

Solution. If $\mathbf{u} = \langle a, b\rangle$ then we need

$$\begin{aligned}
a^2 + b^2 &= 1 \\
-24a - 24b &= -24/5.
\end{aligned}$$

There are two solutions to this system: $\langle 4/5, -3/5\rangle$ and $\langle -3/5, 4/5\rangle$.

13. Let $f(x, y) = 3xy - x^3 - y^3$. Find and classify (as a local minimum, maximum or saddle point) all critical points of f.

Solution. We have

$$
\begin{aligned}
f_x &= 3y - 3x^2 \\
f_y &= 3x - 3y^2 \\
f_{xx} &= -6x \\
f_{xy} &= 3 \\
f_{yy} &= -6y.
\end{aligned}
$$

Setting the first derivatives equal to 0 gives $y = x^2 = y^4$, which has two solutions: $(0, 0)$ and $(1, 1)$. Using the second derivative test, we find that the first is a saddle and the second is a maximum.

14. Let $f(x, y) = xy + x^5 - y^3$. Find and classify (as a local minimum, maximum or saddle point) all critical points of f.

Solution. We have

$$
\begin{aligned}
f_x &= y + 5x^4 \\
f_y &= x - 3y^2 \\
f_{xx} &= 20x^3 \\
f_{xy} &= 1 \\
f_{yy} &= -6y.
\end{aligned}
$$

Setting the first derivatives equal to 0 gives $y = -5x^4 = -405y^8$. So $x = y = 0$ or $y = \frac{-1}{\sqrt[7]{405}}$ and $x = \frac{3}{\sqrt[7]{164025}}$. Using the second derivative test, we find that the first is a saddle and the second is a minimum.

15. Find the minimum and maximum values of the function $f(x, y, z) = xy^3 - z$ when restricted to the surface $xy + yz = -3$.

Solution. We try Lagrange multipliers.

$$
\begin{aligned}
y^3 &= \lambda y \\
3xy^2 &= \lambda(x + z) \\
-1 &= \lambda y.
\end{aligned}
$$

The first and third equations imply that $y = -1$ and $\lambda = 1$. Thus $\lambda(x+z) = (x+z) = 3$ and the second equation becomes $xy^2 = 1$ or $x = 1$. Now $xy + yz = -3$ implies $z = 2$. There is thus only one critical point along this constraint, at $(1, -1, 2)$ where $f(1, -1, 2) = -3$.

The fact that there is only one critical point should give us pause: we need to determine what behavior $f(x, y, z)$ has "at infinity." To do so, we use the constraint $xy + yz = -3$ to write $xy = -3 - yz$ and then $f(x, y, z) = (-3 - yz)y^2 - z = -3y^2 - (y^3 + 1)z$. We can make this arbitrarily negative by setting $z = 0$ and allowing y to increase to ∞, and arbitrarily positive by setting $y = 0$ and allowing z to decrease to $-\infty$. Thus $f(x, y, z)$ has **no** minimum or maximum value on this surface.

This was kind of a trick question: I won't do this on the exam.

16. Find the minimum and maximum values of the function $f(x, y, z) = x^2y - yz$ when restricted to the surface $x^2 + yz + \frac{y^2}{2} = 4$.

Solution. With the previous problem as a model, we don't use Lagrange multipliers but instead solve the constraint for yz getting $yz = 4 - x^2 - \frac{y^2}{2}$ and then note that $f(x, y, z) = x^2y + x^2 + \frac{y^2}{2} - 4$ along this surface. Note that we can allow almost any values of x and y by setting $z = \frac{4-x^2-y^2/2}{y}$, with the exception that if $y = 0$ we can only allow $x = \pm 2$.

Note that we may make the expression for $f(x, y, z)$ as large as we like by setting $x = 0$ and allowing y to increase. And we may make it as small as we like by setting $y = -2$ and letting x increase. So this function has no minimum or maximum value on the surface.

Again, this was a trick question that I won't duplicate on the exam.

17. Suppose that the variables x, y, z satisfy an equation $g(x,y,z)=0$. Assume the point $P(1,1,1)$ lies on this level surface of g and that $\nabla g(1,1,1)=\langle -1,1,2\rangle$. Let $f(x,y,z)$ be another function, and assume that $\nabla f(1,1,1)=\langle 1,2,1\rangle$. Find the gradient of the function $w=f(x,y,z(x,y))$ of the two independent variables x and y at the point $x=1$, $y=1$.

    Solution. We use the chain rule to compute w_x and w_y. We see that

$$\begin{aligned} w_x &= f_x + f_z z_x \\ w_y &= f_y + f_z z_y. \end{aligned}$$

    Since $z(x,y)$ is defined by the condition that $g(x,y,z(x,y))=0$, we have that

$$\begin{aligned} 0 &= g_x + g_z z_x \\ 0 &= g_y + g_z z_y. \end{aligned}$$

    Thus

$$\begin{aligned} w_x &= f_x - f_z \frac{g_x}{g_z} = 1 - 1\cdot\frac{-1}{2} = \frac{3}{2}. \\ w_y &= f_y - f_z \frac{g_y}{g_z} = 2 - 1\cdot\frac{1}{2} = \frac{3}{2}. \end{aligned}$$

    So $\nabla w = \langle \frac{3}{2}, \frac{3}{2}\rangle$.

18. Suppose f, g and h are differentiable functions of two variables, and u and v are differentiable functions of one variable. Suppose that you know that $u(\pi)=2$, that $v(\pi)=4$, that $g(2,4)=-1$ and $h(2,4)=0$. In addition, I tell you that $u'(\pi)=1$, that $v'(\pi)=-1$, that the gradient of g at $(1,-1)$ is $\langle 8,0\rangle$, that the gradient of g at $(2,4)$ is $\langle 5,3\rangle$, that $h(x,y)=x^3-xy$, and that the gradient of f at $(-1,0)$ is $\langle 2,3\rangle$. Let $\alpha(t)=f(g(u(t),v(t)),h(u(t),v(t)))$. Find $a'(\pi)$.

    Solution. We use the chain rule again. If we consider f as a function of g and h, and g and h as functions of u and v, we have that

$$\begin{aligned} \alpha'(t) &= f_g g_t + f_h h_t \\ &= f_g(g_u u_t + g_v v_t) + f_h(h_u u_t + h_v v_t). \end{aligned}$$

    We need to evaluate f_g and f_h at $(g(u(\pi),v(\pi)),h(u(\pi),v(\pi)))=(g(2,4),h(2,4))=(-1,0)$, so $f_g=2$ and $f_h=3$. We need to evaluate g_u and g_v at $(u(\pi),v(\pi))=(2,4)$, so $g_u=5$ and $g_v=3$. Similarly, we need to evaluate h_u and h_v at $(u(\pi),v(\pi))=(2,4)$. Using the formula for h, we have that $h_u=3u^2-v=8$ and $h_v=-u=-2$. Finally, we are given that $u'(\pi)=1$ and $v'(\pi)=-1$. Putting it all together, we have that

$$\alpha'(\pi) = 2(5\cdot 1 + 3\cdot(-1)) + 3(8\cdot 1 + (-2)\cdot(-1)) = 34.$$

19. Consider the region bounded by the two cylinders $y^2+z^2=1$ and $x^2+z^2=1$. Find an expression that gives the average distance of a point in this region to $(0,0,1)$ (you do not have to evaluate any integrals).

    Solution. If z is the inner variable, we will need to have cases depending on where in the xy-plane we are, so let's make z the outer variable. The other two variables are symmetric, so let's choose the order $dx\,dy\,dz$. Since x does not appear in the equation of the first cylinder, we have lower an upper limits for the inner integral of $\pm\sqrt{1-z^2}$. In the yz-plane, our region is just a disc bounded by $y^2+z^2=1$. The distance from a point (x,y,z) to $(0,0,1)$ is $\sqrt{x^2+y^2+(z-1)^2}$. Since we're computing an average over the region, we need to divide by the volume of the region. Overall, we get

$$\frac{\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \sqrt{x^2+y^2+(z-1)^2}\,dx\,dy\,dz}{\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dx\,dy\,dz}$$

20. Integrate the function x^2+z^2 over the region enclosed by the planes $z=0$ and $z=1$ and the cone $z^2=x^2+y^2$.

Solution. We use cylindrical coordinates, getting

$$\begin{aligned}\int_0^{2\pi}\int_0^1\int_r^1 (r^2\cos^2(\theta)+z^2)\, r dz\, dr\, d\theta &= \int_0^{2\pi}\cos^2(\theta)\, d\theta\int_0^1 r^3(1-r)\, dr + \int_0^{2\pi} d\theta\int_0^1 \frac{1}{3}(1-r^3) r\, dr\\ &= \pi\left(\frac{1}{4}-\frac{1}{5}\right)+2\pi\left(\frac{1}{6}-\frac{1}{15}\right)\\ &= \frac{\pi}{4}\end{aligned}$$

21. Find the area of the ellipse $(4x-y)^2+(x-3y)^2<1$ using an appropriate change of coordinates.

    Solution. We use $u=4x-y$ and $v=x-3y$. Solving for x and y we get

$$\begin{aligned}x &= \frac{3u}{11}-\frac{v}{11}\\ y &= \frac{u}{11}-\frac{4v}{11}.\end{aligned}$$

    The Jacobian is then the determinant of $\begin{pmatrix} 3/11 & -1/11\\ 1/11 & -4/11\end{pmatrix}$, or $-\frac{1}{11}$. The corresponding region in the uv-plane is the disk $u^2+v^2<1$, so the area is

$$\int_0^{2\pi}\int_0^1 \frac{1}{11} r\, dr\, d\theta = \frac{\pi}{11}.$$

22. Consider the region bounded by the surface

$$z=(3x+5y)^2+(2x+4y)^2$$

    and the plane

$$z=10x+18y+2.$$

    (a) Set up an integral to find the volume of this region.
    Hint: $10x+18y=2(3x+5y)+2(2x+4y)$

    Solution. If we set $u=3x+5y$ and $v=2x+4y$, then the surface has equation $z=u^2+v^2$ and the plane equation $z=2u+2v+2$. They intersect where $u^2+v^2=2u+2v+2$, or $(u-1)^2+(v-1)^2=4$. This is the circle of radius 2 around $(1,1)$, suggesting that we then want to make the change to polar coordinates with $u-1=r\cos(\theta)$ and $v-1=r\sin(\theta)$.
    The Jacobian is the inverse of $\det\left(\begin{smallmatrix}3 & 5\\ 2 & 4\end{smallmatrix}\right)$, or $\frac{1}{2}$. Switching to polar introduces a factor of r, so we get the integral

$$\frac{1}{2}\int_0^{2\pi}\int_0^2\int_{(r\cos(\theta)+1)^2+(r\sin(\theta)+1)^2}^{2r\cos(\theta)+2r\sin(\theta)+4} r\, dz\, dr\, d\theta.$$

    (b) Evaluate the integral you found in the first part.

    Solution.

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi}\int_0^2\int_{(r\cos(\theta)+1)^2+(r\sin(\theta)+1)^2}^{2r\cos(\theta)+2r\sin(\theta)+6} r\, dz\, dr\, d\theta &= \frac{1}{2}\int_0^{2\pi}\int_0^2 4r-r^3\cos^2(\theta)-r^3\sin^2(\theta)\, dr\, d\theta\\ &= \int_0^{2\pi} 4-2\cos^2(\theta)-2\sin^2(\theta)\, d\theta\\ &= 4\pi.\end{aligned}$$

23. Give an expression for the arclength of the Archemedian spiral $r=\theta$ between the origin and the point $(2\pi,0)$.

    Solution. We have parametric equations

$$\begin{aligned}x(\theta) &= \theta\cos(\theta)\\ y(\theta) &= \theta\sin(\theta),\end{aligned}$$

    so an integral giving the arclength is

$$\int_0^{2\pi}\sqrt{(\cos(\theta)-\theta\sin(\theta))^2+(\sin(\theta)+\theta\cos(\theta))^2}\, d\theta = \int_0^{2\pi}\sqrt{1+\theta^2}\, d\theta.$$

24. Let C be the spiral given in polar coordinates by the equation $r = \theta$, $0 \le \theta \le 2\pi$, traced out starting from the origin. Let $\mathbf{F}(x, y) = (x^2, y)$. Reduce the following line integral to a single variable integral, but do not evaluate the resulting single variable integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Solution. We use the same parameterization as above, giving

$$\int_0^{2\pi} \theta^2 \cos^2(\theta)(\cos(\theta) - \theta\sin(\theta)) + \theta\sin(\theta)(\sin(\theta) + \theta\cos(\theta))\, d\theta$$

25. Let C be the twisted cubic (t, t^2, t^3) with $1 \le t \le 2$, and let $\mathbf{F}(x, y, z) = (y^z x^{y^z}, \ln(x) z y^{z-1} x^{y^z}, \ln(x)\ln(y) y^z x^{y^z})$. Evaluate $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}$.

Solution. We find a potential function for $\mathbf{F}$. Integrating the x component with respect to y gives x^{y^z}, and then differentiating with respect to y and z gives exactly the y and z components. So if we set $f(x, y, z) = x^{y^z}$ then $\mathbf{F} = \nabla f$. Thus

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = f(2, 4, 8) - f(1, 1, 1) = 2^{4^8} - 1^{1^1} = 2^{65536} - 1.$$

26. Consider the vector field $\mathbf{F}(x, y, z) = (2xy + z, x^2 + 3y^2 z, y^3 + x - 4z^3)$.

(a) Is $\mathbf{F}$ conservative? Why or why not?

Solution. We compute the curl:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z & x^2 + 3y^2 z & y^3 + x - 4z^3 \end{vmatrix} \\ &= \left\langle 3y^2 - 3y^2, 1 - 1, 2x - 2x \right\rangle \\ &= \langle 0, 0, 0 \rangle.\end{aligned}$$

So $\mathbf{F}$ is conservative.

(b) If $\mathbf{F}$ is conservative, find a potential function. If not, find a closed path C such that $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$.

Solution. Suppose $\nabla f = \mathbf{F}$. Integrating the first component, we get that $f(x, y, z) = x^2 y + xz + g(y, z)$. Differentiating with respect to y, we see that

$$x^2 + \frac{\partial g}{\partial y} = x^2 + 3y^2 z,$$

so $g(y, z) = y^3 z + h(z)$. Finally, differentiating with respect to z we have

$$x + y^3 + h'(z) = x + y^3 - 4z^3,$$

so $h(z) = z^4 + C$. Overall, we may take $f(x, y, z) = x^2 y + xz + y^3 z + z^4$ as a potential function.

27. For each of the following vector fields, determine whether it is conservative. If it is, find a potential function.

(a) $\mathbf{F}_1(x, y, z) = \langle 4x^3 y + 3z, x^4 - z + 2, 3x + y \rangle$.

Solution.

$$\begin{aligned}\nabla \times \mathbf{F}_1 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x^3 y + 3z & x^4 - z + 2 & 3x + y \end{vmatrix} \\ &= \left\langle 1 + 1, 3 - 3, 4x^3 - 4x^3 \right\rangle \\ &= \langle 2, 0, 0 \rangle.\end{aligned}$$

Thus $\mathbf{F}_1$ is not conservative.

(b) $\mathbf{F}_2(x, y, z) = \langle 2x + y\cos(xy), x\cos(xy) - ze^{yz}, 2 - ye^{yz} \rangle$.

Solution.

$$\nabla \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + y\cos(xy) & x\cos(xy) - ze^{yz} & 2 - ye^{yz} \end{vmatrix}$$
$$= \langle -(e^{yz} + yze^{yz}) + e^{yz} + yze^{yz}, 0 - 0, \cos(xy) - xy\sin(xy) - (\cos(xy) - xy\sin(xy)) \rangle$$
$$= \langle 0, 0, 0 \rangle.$$

So $\mathbf{F}_2$ is conservative, and we find that we can take $f_2(x, y, z) = x^2 + \sin(xy) - e^{yz} + 2z$ as a potential function.

28. Evaluate the following line integrals.

(a) $\int_{C_1} \mathbf{F}_1(x, y, z) \cdot d\mathbf{r}$ where $\mathbf{F}_1(x, y, z) = \langle 4x^3y + 3z, x^4 - z + 2, 3x + y \rangle$ as in the previous problem and C_1 is the portion of the parabola $x = y = z^2$ from $(1, 1, -1)$ to $(1, 1, 1)$.

Solution. Since $\mathbf{F}_1$ is not conservative, we parameterize the curve and compute the line integral explicitly. We can take $\mathbf{r}(t) = \langle t^2, t^2, t \rangle$, giving $\mathbf{r}'(t) = \langle 2t, 2t, 1 \rangle$ and

$$\int_{C_1} \mathbf{F}_1(x, y, z) \cdot d\mathbf{r} = \int_{-1}^{1} (4t^8 + 3t)(2t) + (t^8 - t + 2)(2t) + (4t^2)(1)\, dt$$
$$= \int_{-1}^{1} 10t^9 + 8t^2 + 4t\, dt$$
$$= \frac{16}{3}.$$

(b) $\int_{C_2} \mathbf{F}_2(x, y, z) \cdot d\mathbf{r}$ where $\mathbf{F}_2(x, y, z) = \langle 2x + y\cos(xy), x\cos(xy) - ze^{yz}, 2 - ye^{yz} \rangle$ as in the previous problem and C_2 is the twisted cubic given by $\mathbf{r}(t) = (t, t^2, t^3), 0 \le t \le 1$.

Solution. Now we can use the potential function, giving

$$\int_{C_2} \mathbf{F}_2(x, y, z) \cdot d\mathbf{r} = f_2(1, 1, 1) - f_2(0, 0, 0)$$
$$= 4 + \sin(1) - e$$

29. Let $\mathbf{F}(x, y, z) = \left(ax^2y + z^2, x^3 + 4y^3z, bxz + y^4\right)$.

(a) For what values of a and b will $\mathbf{F}$ be conservative?

Solution.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2y + z^2 & x^3 + 4y^3z & bxz + y^4 \end{vmatrix}$$
$$= \left\langle 4y^3 - 4y^3, 2z - bz, 3x^2 - ax^2 \right\rangle.$$

In order for $\mathbf{F}$ to be conservative, the curl must be 0, so $a = 3$ and $b = 2$.

(b) Using these values of a and b, find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

Solution. Integrating the first component, we find that $f(x, y, z) = x^3y + xz^2 + g(y, z)$, and differentiating with respect to the other two components gives $f(x, y, z) = x^3y + xz^2 + y^4z$.

(c) Again using these values of a and b, give a defining equation for a surface S with the property that

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = 0$$

for any two points P and Q lying on S and any path between them.

Solution. Any surface whose equation is of the form $f(x, y, z) = k$ will have this property since the value of the potential function will be the same at P and at Q. For example, we can take the surface

$$x^3y + xz^2 + y^4z = 1.$$

30. Let C be the curve formed by intersecting the paraboloid $z = (x+y)^2+(x+2y+1)^2$ and the plane $x-y-z+2 = 0$, oriented counterclockwise when viewed from above. Let $f(x, y, z)$ be an arbitrary smooth function, and set $\mathbf{F}(x, y, z) = \nabla f + xy\mathbf{i} - xy\mathbf{j} - xy\mathbf{k}$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution. We use Stokes' theorem and take advantage of the fact that $\nabla \times \nabla f = \mathbf{0}$.

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -xy & -xy \end{vmatrix} \\ &= \langle -x, y, -y-x \rangle .\end{aligned}$$

Stokes' theorem allows us to use either surface, so we'll use the plane since it's simpler. Then the normal vector is $\mathbf{n} = \langle -1, 1, 1\rangle$ and $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = (-x)(-1) + (y)(1) + (-y - x)(1) = 0$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

31. Parameterize each of the following curves or surfaces. *Hint:* for many of these, you can use vector addition to break up the problem into easier pieces.

(a) Consider a circle rolling along a line without slipping. Parametrize the path a point on this circle traces out.

Solution. The vector from the origin to the moving point on the edge of the circle can be described as a sum of the vector from the origin to the center of the circle plus the vector from the center of the circle to the point on the edge. Suppose the circle has radius r, is traveling in the positive x direction with velocity v, begins above the origin (so that the center is at $(0, r)$) and the point of interest starts out at the origin. The vector from the origin to the center of the circle moves in a line, parameterized by $\langle 0, r\rangle + t\langle v, 0\rangle$.. The circle will make one full revolution in time $\frac{2\pi r}{v}$, so the vector to the point on the edge is given by $\langle -r \sin(\frac{vt}{2\pi r}), -r \cos(\frac{vt}{2\pi r})\rangle$. Thus a parameterization for the curve is given by

$$\langle vt - r \sin(\frac{vt}{2\pi r}), r - r \cos(\frac{vt}{2\pi r})\rangle.$$

(b) The cone with vertex half-angle α and axis pointing along the positive x-axis.

Solution. We need two parameters for this surface. Because the angle is fixed, there is a relationship between the distance out along the x-axis and the radius, namely that $\tan(\alpha) = \frac{r}{x}$. Taking θ as the angle made with the positive y-axis (counterclockwise in the yz-plane), we can take x and θ for our parameters and get the equation

$$\langle x, x \tan(\alpha) \cos(\theta), x \tan(\alpha) \sin(\theta)\rangle.$$

Here $0 \le \theta < 2\pi$ and $0 \le x < \infty$.

(c) A Möbius strip (a Möbius strip can be formed by taking a rectangular strip of paper, making a half twist and then taping the ends together). You may choose any constants necessary, or leave them as variables.

Solution. A Möbius strip has a central circle that the band twists around (with a half twist for a full revolution of the circle). Say the radius of this central circle is a and the band has width $2b$. Let θ be the angle around the circle and u a parameter across the width of the band, ranging from $-b$ to b. In the rz-plane, the band is centered at $(r, z) = (a, 0)$, and makes angle $\theta/2$ with the r-axis (this factor of 2 is key: it makes the band twist halfway around as θ ranges from 0 to 2π). So we can take $r = a + u \cos(\theta/2)$ and $z = a + u \sin(\theta/2)$. Converting back to x, y, z we have

$$\begin{aligned} x(u, \theta) &= (a + u \cos(\theta/2)) \cos(\theta) \\ y(u, \theta) &= (a + u \cos(\theta/2)) \sin(\theta) \\ z(u, \theta) &= u \sin(\theta/2), \end{aligned}$$

with $0 \le \theta < 2\pi$ and $-b \le u \le b$.

(d) A spiraling seashell: a tubular surface centered on the helix $(\cos t, \sin t, t)$, with radius away from the helix proportional to t.

Solution. Let k be the ratio of the radius at $(\cos t, \sin t, t)$ to t, so that we need to draw a circle of radius kt. We use the normal and binormal vectors, which span a plane perpendicular to the helix. As we did in the review session, we can find

$$\begin{aligned}
\mathbf{T} &= \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle \\
\mathbf{N} &= \langle -\cos t, -\sin t, 0\rangle \\
\mathbf{B} &= \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle.
\end{aligned}$$

Letting θ be a parameter for the angle tracing out the circle around $(\cos t, \sin t, t)$, we get

$$\mathbf{r}(t,\theta) = \langle \cos t, \sin t, t\rangle + \langle -\cos t, -\sin t, 0\rangle kt\cos\theta + \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle kt\sin\theta,$$

or

$$\begin{aligned}
x(t,\theta) &= \cos(t)(1 - kt\cos(\theta)) + \frac{kt\sin(t)\sin(\theta)}{\sqrt{2}} \\
y(t,\theta) &= \sin(t)(1 - kt\cos(\theta)) - \frac{kt\cos(t)\sin(\theta)}{\sqrt{2}} \\
z(t,\theta) &= t + \frac{kt\sin(\theta)}{\sqrt{2}},
\end{aligned}$$

with $0 \le t < \infty$ and $0 \le \theta < 2\pi$.

(e) Consider two helices around the z-axis of radius 1, separated by a 180° rotation (a double helix). Parametrize the surface formed by connecting corresponding points in these helices with line segments.

Solution. We can take the first helix as $\langle \cos t, \sin t, t\rangle$ and the second as $\langle -\cos t, -\sin t, t\rangle$, using the same parameter for both. The line segment between them is just given by $\langle u\cos t, u\sin t, t\rangle$ for $-1 \le u \le 1$. So we get

$$\begin{aligned}
x(t,u) &= u\cos t \\
y(t,u) &= u\sin t \\
z(t,u) &= t,
\end{aligned}$$

with $\infty < t < \infty$ and $-1 \le u \le 1$.

32. Let S be a sphere of radius 1, and let the density be equal to the distance *along the sphere* from the north pole of the sphere. Find the mass.

Solution. In spherical coordinates, the density is just ϕ. Using the parameterization

$$\mathbf{r}(\phi,\theta) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$$

we have

$$\begin{aligned}
|\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & 0 \end{vmatrix} \\
&= \sqrt{\cos^2(\theta)\sin^4(\phi) + \sin^2(\theta)\sin^4(\phi) + \cos^2(\phi)\sin^2(\phi)} \\
&= \sin(\phi).
\end{aligned}$$

So the mass is

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi} \phi\sin(\phi)\,d\phi\,d\theta &= 2\pi[\sin(\phi) - \phi\cos(\phi)]_0^{\pi} \\
&= 2\pi^2.
\end{aligned}$$

33. Let R be the region bounded above by the surface $z = 16 - x^4 - 2x^2y^2 - y^4$ and below by the lower half of the sphere of radius 2 centered at the origin. If the density of the region is given by $\delta(x,y,z) = x^2 + y^2$, find the center of mass of R.

Solution. The mass is

$$\begin{aligned}
m &= \int_0^{2\pi} \int_0^2 \int_{-\sqrt{4-r^2}}^{16-r^4} r^3 \, dz \, dr \, d\theta \\
&= 2\pi \int_0^2 (16 - r^4 + \sqrt{4-r^2}) r^3 \, dr \\
&= 2\pi \left[4r^4 - \frac{r^8}{8} - \frac{1}{15}(3r^2+8)(4-r^2)^{3/2} \right]_0^2 \\
&= \frac{1088\pi}{15}
\end{aligned}$$

using the substitution $u = 4 - r^2$. Since both the region and the density is symmetric about the z-axis, the center of mass has $\bar{x} = \bar{y} = 0$.

$$\begin{aligned}
\bar{z} &= \frac{1}{m} \int_0^{2\pi} \int_0^2 \int_{-\sqrt{4-r^2}}^{16-r^4} zr^3 \, dz \, dr \, d\theta \\
&= \frac{15}{1088} \int_0^2 ((16-r^4)^2 - (4-r^2))r^3 \, dr \\
&= \frac{15}{1088} \int_0^2 252r^3 + r^5 - 32r^7 + r^11 \, dr \\
&= \frac{315}{68}
\end{aligned}$$

So the center of mass is $\left(0, 0, \frac{315}{68}\right)$.

34. Consider the ellipsoid E given by the equation $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{36} = 1$. Let $f(x,y,z) = \sqrt{2x^2 + 2y^2 + 1}$. Calculate the value of

$$\iint_E f(x,y,z) \, dS$$

Hint: parameterize E similarly to a sphere.

Solution. We parameterize E by $\mathbf{r}(\phi, \theta) = (2\cos(\theta)\sin(\phi), 2\sin(\theta)\sin(\phi), 6\cos(\phi))$. Then

$$\begin{aligned}
|\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos(\theta)\cos(\phi) & 2\sin(\theta)\cos(\phi) & -6\sin(\phi) \\ -2\sin(\theta)\sin(\phi) & 2\cos(\theta)\sin(\phi) & 0 \end{vmatrix} \\
&= \sqrt{144\cos^2(\theta)\sin^4(\phi) + 144\sin^2(\theta)\sin^4(\phi) + 16\cos^2(\phi)\sin^2(\phi)} \\
&= 4\sin(\phi)\sqrt{9\sin^2(\phi) + \cos^2(\phi)} \\
&= 4\sin(\phi)\sqrt{8\sin^2(\phi) + 1}.
\end{aligned}$$

We then substitute $(2\cos(\theta)\sin(\phi), 2\sin(\theta)\sin(\phi), 6\cos(\phi))$ into the formula for $f(x,y,z)$, giving

$$\begin{aligned}
f(\mathbf{r}(\phi,\theta)) &= \sqrt{2(2\cos(\theta)\sin(\phi))^2 + 2(2\sin(\theta)\sin(\phi))^2 + 1} \\
&= \sqrt{8\sin^2(\phi) + 1}.
\end{aligned}$$

So the desired surface integral is given by

$$\begin{aligned}
\iint_S f(x,y,z) \, dS &= \int_0^{2\pi} \int_0^{\pi} 4\sin(\phi)(8\sin^2(\phi) + 1) \, d\phi \, d\theta \\
&= 8\pi \int_0^{\pi} \sin(\phi)(9 - 8\cos^2(\phi)) \, d\phi \\
&= 8\pi \left[-9\cos(\phi) + \frac{8}{3}\cos^3(\phi) \right]_0^{\pi} \\
&= \frac{304\pi}{3}
\end{aligned}$$

35. Consider the sphere S of radius 1 centered at the point $(0, 0, 1)$

(a) Parameterize S. *Hint:* how would the parameterizations of a sphere centered at the origin and a sphere centered at $(0, 0, 1)$ differ?

Solution. The easiest way to parameterize this shifted sphere is to just add 1 to the z-component in the normal parameterization of a sphere around the origin. We get

$$\mathbf{r}(\phi, \theta) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) + 1)$$

(b) Let $f(x, y, z) = x^2z + y^2z - x^2 - y^2$. Compute

$$\iint_S f(x, y, z)\, dS.$$

Solution. Note that adding 1 to the z-component has no effect on the partial derivatives $\mathbf{r}_\phi$ and $\mathbf{r}_\theta$, so we still have

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin(\phi)$$

as in problem 32. Moreover, $f(x, y, z) = (x^2 + y^2)(z - 1) = \sin^2(\phi)\cos(\phi)$. So

$$\begin{aligned}\iint_S f(x, y, z)\, dS &= \int_0^{2\pi}\int_0^{\pi} \sin^2(\phi)\cos(\phi)\sin(\phi)\, d\phi\, d\theta \\ &= 2\pi\left[\frac{1}{4}\sin^4(\phi)\right]_0^{\pi} \\ &= 0.\end{aligned}$$

In retrospect, we can realize that $f(x, y, z)$ is negated when (x, y, z) is reflected across the plane $z = 1$ that passes through the center of the sphere. So the negative contribution of the bottom half of the sphere will exactly balance the positive contribution of the top half, leaving 0 for the integral.

(c) Let $\mathbf{F}(x, y, z) = \langle x + 3y, 2y - z, 4z + x\rangle$. Compute

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

Solution. We use the divergence theorem, since S is a closed surface. The divergence of $\mathbf{F}$ is $1+2+4 = 7$, so the result is just 7 times the volume of the sphere, or $\frac{28\pi}{3}$.

(d) Now consider just the upper half of S (above the $z = 1$ plane). Call this surface S_1 and equip it with the upward pointing normal. Let $\mathbf{F} = \langle z - y - 1, x + z^2 + z - 2, xy^2 + xz - 4\rangle$. Compute

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

Solution. We use Stokes' theorem, parameterizing the circle of radius 1 in the $z = 1$ plane as $\mathbf{r}(t) = \langle\cos(t), \sin(t), 1\rangle$. Then

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= \langle -\sin(t), \cos(t), \cos(t)\sin^2(t) + \cos(t) - 4\rangle \cdot \langle -\sin(t), \cos(t), 0\rangle\, dt \\ &= dt.\end{aligned}$$

So

$$\begin{aligned}\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS &= \int_0^{2\pi} dt \\ &= 2\pi.\end{aligned}$$

36. For each of the following integrals, use either Green's Theorem, the Divergence Theorem or Stokes' Theorem to write the given integral as an iterated integral with a different number of integral signs (by iterated integral I mean a triple integral, a double integral or a single integral: ie, you need to transform surface integrals and line integrals to double and single integrals and put appropriate limits on your integrals).

(a) Let C be the curve given parametrically by

$$\mathbf{r}(t) = \left(\left(t^3 - \frac{\pi^2 t}{9}\right)^2 \cos t, \left(t^3 - \frac{\pi^2 t}{9}\right)^2 \sin t, 0\right),$$

where t runs from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$. Let

$$\mathbf{F}(x, y, z) = (xy, x^2 + yz, x + y + 3z).$$

Modify the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Solution. Since $z = 0$, we can use Green's theorem and ignore the z-component of $\mathbf{F}$ (equivalently, the z-component of the curl will disappear when dotting with $\mathbf{k}$). This gives

$$\iint_D 2x - x\, dA,$$

where D is the region enclosed by C. But C is just the graph of the curve given in polar coordinates by $r = (\theta^3 - \frac{\pi^2\theta}{9})^2$. So we get the double integral

$$\int_{-\pi/3}^{\pi/3} \int_0^{(\theta^3 - \pi^2\theta/9)^2} r^2 \cos(\theta)\, dr\, d\theta.$$

(b) Let S be the half of the unit sphere that lies above the xy-plane, with upward pointing unit normal. Let $\mathbf{F}(x, y, z) = (2y, 0, 2x)$. Modify the integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

Solution. We would like to use Stokes' theorem. Since this vector field has divergence 0, we should be able to express it as the curl of something, and since it is so simple we can carry that out in practice. Suppose that $\mathbf{F} = \nabla \times \langle a(x, y, z), b(x, y, z), c(x, z, z)\rangle$. Computing the curl, we need

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a(x, y, z) & b(x, y, z) & c(x, y, z) \end{vmatrix} \\
&= \left\langle \frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}, \frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}, \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right\rangle \\
&= \langle 2y, 0, 2x\rangle.
\end{aligned}$$

One can take $c = y^2$ and $b = x^2$ and $a = 0$. The unit circle C is then parameterized by $\langle \cos t, \sin t, 0\rangle$, so we get a line integral

$$\int_C \langle 0, x^2, y^2\rangle \cdot d\mathbf{r} = \int_0^{2\pi} \cos^2(t)\cos(t)\, dt.$$

(c) Let S be the surface obtained by rotating the circle

$$(x - 1)^2 + z^2 = 1$$

about the z-axis, oriented outward.
Let $\mathbf{F}(x, y, z) = (x^3 - x, y - y^2 z, yz^2)$. Modify the integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

Solution. The outward normal and flux integral suggest the divergence theorem, we just need to find $\nabla \cdot \mathbf{F}$ and bounds on the region.

$$\nabla \cdot \mathbf{F} = (3x^2 - 1) + (1 - 2yz) + (2yz) = 3x^2.$$

The rotation of the circle around the z-axis means that the actual equation for the surface is $(r-1)^2+z^2=1$, so we can use either cylindrical coordinates with $z=\pm\sqrt{2r-r^2}$ or spherical with $\rho=2\sin(\phi)$. These give

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^{2\pi}\int_0^2\int_{-\sqrt{2r-r^2}}^{\sqrt{2r-r^2}} 3r^2\cos^2(\theta) r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{2\sin(\phi)} 3\rho^2\cos^2(\theta)\sin^2(\phi)\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta\end{aligned}$$

respectively.

37. Let S be the unit sphere centered at the origin with outward pointing normal and let $\mathbf{F}(x,y,z)=(x,0,z)$.

(a) Find the flux of $\mathbf{F}$ through S directly.

Solution. We use the parameterization of problem 32:

$$\mathbf{r}(\phi,\theta)=(\cos(\theta)\sin(\phi),\sin(\theta)\sin(\phi),\cos(\phi)).$$

The we have

$$\begin{aligned}\mathbf{r}_\phi\times\mathbf{r}_\theta &= \det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi)\\ -\sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & 0\end{pmatrix}\\ &= \langle\cos(\theta)\sin^2(\phi),\sin(\theta)\sin^2(\phi),\cos(\phi)\sin(\phi)\rangle.\end{aligned}$$

Thus

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^{2\pi}\int_0^{\pi}(\cos(\theta)\sin(\phi))\cos(\theta)\sin^2(\phi)+(\cos(\phi))\cos(\phi)\sin(\phi)\,d\phi\,d\theta\\ &= \int_0^{2\pi}\int_0^{\pi}\cos^2(\theta)\sin(\phi)(1-\cos^2(\phi))+\cos^2(\phi)\sin(\phi)\,d\phi\,d\theta\\ &= \int_0^{2\pi}\cos^2(\theta)\,d\theta\int_0^{\pi}\sin(\phi)\,d\phi+\int_0^{2\pi}(1-\cos^2(\theta))\,d\theta\int_0^{\pi}\sin(\phi)\cos^2(\phi)\,d\phi\\ &= \pi\cdot 2+\pi\cdot\frac{2}{3}\\ &= \frac{8\pi}{3}.\end{aligned}$$

(b) Use the divergence theorem to check your answer

Solution. The divergence of $\mathbf{F}$ is 2, so the flux integral should be twice the volume of the ball: $2\cdot\frac{4\pi}{3}=\frac{8\pi}{3}$.

38. Let S be the part of the paraboloid $z=1-x^2-y^2$ above the xy-plane, and let $\mathbf{F}(x,y,z)=(x^2+yz^4-\sin(z^2),x-4y+z,4z+1)$. Compute

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

by using the divergence theorem to reduce to a simpler surface integral.

Solution. Let S' be the unit disk in the xy-plane, oriented upward, and let E be the region between S and S'. Then the divergence theorem implies that

$$\iiint_E(\nabla\cdot\mathbf{F})\,dV=\iint_S\mathbf{F}\cdot\mathbf{n}\,dS-\iint_{S'}\mathbf{F}\cdot\mathbf{n}\,dS.$$

Since $\nabla\cdot\mathbf{F}=2x-4+4=2x$ is odd with respect to x and E is symmetric about the plane $x=0$, $\iiint_E(\nabla\cdot\mathbf{F})\,dV=0$. On S' we have $z=0$ and $\mathbf{n}=\mathbf{k}$, so

$$\iint_{S'}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{S'}1dS,$$

which is just the area of the disc, or π. So

$$\iint_S\mathbf{F}\cdot\mathbf{n}\,dS=\pi.$$

39. Consider the part of the surface $z = -r^2+3r-2$ that lies above the xy-plane (r is the r of cylindrical coordinates). Call this surface S. Let

$$\mathbf{F}(x, y, z) = \langle x + 3y - \sin^4(z^5), x^2 - y^2, -z - 4\rangle.$$

Evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

Hint: use the divergence theorem so that the surface integral that you actually compute is easier.

Solution. The given surface intersects the xy-plane when $-r^2 + 3r - 2 = 0$, ie when $r = 1$ or $r = 2$. This suggests we relate the given flux integral to the annulus in the xy-plane described by $1 \le r \le 2$; call this annulus (with upward pointing normal) S'. Let E be the region in between.

Computing the divergence of $\mathbf{F}$ we get $1 - 2y - 1 = -2y$. On S', we have $z = 0$ and $\mathbf{n} = \mathbf{k}$, so $\mathbf{F} \cdot \mathbf{n} = -4$ and $\iint_{S'} \mathbf{F} \cdot \mathbf{n}\, dS = -4(4\pi - \pi) = -12\pi$. Moreover, since the region E is symmetric about the plane $y = 0$ and the divergence of $\mathbf{F}$ is odd with respect to y, $\iiint_E \nabla \cdot \mathbf{F}\, dV = 0$. Thus

$$\begin{aligned}
\iiint_E \nabla \cdot \mathbf{F}\, dV &= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS - \iint_{S'} \mathbf{F} \cdot \mathbf{n}\, dS \\
0 &= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS - (-12\pi) \\
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= -12\pi.
\end{aligned}$$

40. Consider a donut: a cylindrical region wrapped into a circular ring. Suppose that the radius from the center to the inner edge of the donut is 3 units, and to the outer edge is 5 units.

(a) Find limits of integration for the interior of the donut in a coordinate system of your choice.

Solution. The donut intersects the rz-plane in a circle of radius 1 around $(r, z) = (4, 0)$, so it has equation $(r - 4)^2 + z^2 = 1$. So the interior of the donut is given in cylindrical coordinates by

$$\int_0^{2\pi} \int_3^5 \int_{-\sqrt{8r-r^2-15}}^{\sqrt{8r-r^2-15}} r\, dz\, dr\, d\theta.$$

(b) Find a parameterization for the surface of the donut.

Solution. Let θ be the angle around the z-axis (so, the standard cylindrical θ) and t be the angle around the smaller circle of radius 1. Then, in the rz-plane the donut slice is parameterized by $(r, z) = (4+\cos(t), \sin(t))$. Using cylindrical coordinates we get

$$\begin{aligned}
x(\theta, t) &= (4 + \cos(t)) \cos(\theta) \\
y(\theta, t) &= (4 + \cos(t)) \sin(\theta) \\
z(\theta, t) &= \sin(t).
\end{aligned}$$

41. Consider the torus of inner radius 3, outer radius 5, central axis the y-axis and with central plane the xz-plane (this is related to the donut in the previous problem). Let S be the part of this torus above the xy-plane, with outward pointing normal. Define

$$\mathbf{F}(x, y, z) = \langle xy + \cos(\frac{\pi}{2} e^{xyz}) \sin(xy), x + \ln(x^2 + y^2 + z^2) z^{x^2+y^2}, x^2 + xy^4 - 2x \cos(y) - e^{5y-3\cos(x)}\rangle.$$

Compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

Solution. This torus intersects the xy-plane in a pair of circles, each of radius 1 and centered around the points $(4, 0)$ and $(-4, 0)$. The outward orientation on the torus induces a positive orientation on both circles. We clearly need to use Stokes' theorem since we are asked to compute the flux of the curl of $\mathbf{F}$. Substituting $z = 0$ and using the fact that that the z-component will be perpendicular to $d\mathbf{r}$, we can simplify $\mathbf{F}$ to

$$\mathbf{F}(x, y) = \langle xy, x\rangle.$$

On the circle C_+ around $(4, 0)$, we compute

$$\begin{aligned}
\int_{C_+} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (4 + \cos(\theta))(\sin(\theta))(-\sin(\theta)) + (4 + \cos(\theta))(\cos(\theta))\, d\theta \\
&= \int_0^{2\pi} -4\sin^2(\theta) - \sin^2(\theta)\cos(\theta) + 4\cos(\theta) + \cos^2(\theta)\, d\theta \\
&= -3\pi
\end{aligned}$$

Similarly, on the circle C_- around $(-4, 0)$, we compute

$$\begin{aligned}
\int_{C_-} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-4 + \cos(\theta))(\sin(\theta))(-\sin(\theta)) + (-4 + \cos(\theta))(\cos(\theta))\, d\theta \\
&= \int_0^{2\pi} 4\sin^2(\theta) - \sin^2(\theta)\cos(\theta) - 4\cos(\theta) + \cos^2(\theta)\, d\theta \\
&= 5\pi
\end{aligned}$$

So we get

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS &= \int_{C_+} \mathbf{F} \cdot d\mathbf{r} + \int_{C_-} \mathbf{F} \cdot d\mathbf{r} \\
&= 2\pi.
\end{aligned}$$