Math 240 Practice Problems

Note that a few of these questions are somewhat harder than questions on the final will be, but they will all help you practice the material from this semester.

1. Consider the three points $A = (3, 1, 4)$, $B = (6, 4, 4)$, $C = (3, 4, 1)$.
   (a) Find the area of the triangle formed by A, B, and C.
   (b) Find the angles of this triangle.
   (c) Find an equation for the plane containing A, B and C.
2. Let $\mathbf{v} = \langle 7, 6, 5\rangle$ and $\mathbf{w} = \langle 3, 2, -1\rangle$. Express $\mathbf{v}$ as the sum of two perpendicular vectors, one of which points in the direction of $\mathbf{w}$.
3. The eight vertices of a cube centered at $(0, 0, 0)$ of side length 2 are at $(\pm 1, \pm 1, \pm 1)$.
   (a) Find the four vertices of the cube, including $(1, 1, 1)$, that form a regular tetrahedron.
   (b) A methane molecule consists of a hydrogen atom at each of the vertices of a regular tetrahedron and a carbon atom at the center. Find the "bond angle," i.e. the angle made by the vectors from the carbon atom to two hydrogen atoms.
   (c) Find the angle between two adjacent edges of the tetrahedron, and the angle between two opposite edges.
   (d) Find the area of a face of the tetrahedron using vectors.
   (e) Find the volume of the tetrahedron using vectors.
4. Describe the set of vectors $\mathbf{v}$ such that $\frac{\mathbf{v}}{|\mathbf{v}|} \cdot \langle 1, 1, 1\rangle = 1$.
5. Consider the helix given by the parametric equation $\mathbf{r}(t) = (\cos t, \sin t, t)$.
   (a) Find an equation for the line tangent to the helix at $\mathbf{r}(t)$ (use u for the parameter of the line).
   (b) Find a parameterization for the curve traced out by the intersection of the xy-plane and the tangent lines described above.
6. Consider the curve defined parametrically by $\mathbf{r}(t) = (4e^{3t}, e^{2t}, 2e^{2t})$.
   (a) At what value t_0 is the tangent vector to $\mathbf{r}$ at $\mathbf{r}(t_0)$ parallel to the plane $\mathbf{P}$ given by the equation $x+2y-7z = 17$?
   (b) Determine the parametric and symmetric equations of the line $\mathbf{L}$ tangent to $\mathbf{r}$ at $\mathbf{r}(t_0)$.
   (c) Determine the parametric equation of the line passing through $\mathbf{r}(t_0)$, parallel to the plane $\mathbf{P}$ and perpendicular to the line $\mathbf{L}$.
7. Let Q be the point $(1, 4, 3)$ and $\mathcal{P}$ be the plane given by the equation $2x + y - z = 4$.
   (a) Give parametric and symmetric equations of the line passing through Q and perpendicular to $\mathcal{P}$.
   (b) Find the distance from Q to $\mathcal{P}$.
8. Let $\mathcal{P}$ be the tangent plane to $f(x, y) = 4x^3 - 3y^2$ at the point $(2, 4)$. Find the intersection of $\mathcal{P}$ with the xy-plane.
9. Approximate the function $f(x, y, z) = x^2y - 2xz + 1$ near the point $(1, 1, 0)$ by a linear function.
10. Suppose $f(x, y, z) = xyz^2 + 2x - yz$. Compute the directional derivative of f at the point $(1, 1, 1)$ in the direction of $\langle 1, 2, -1\rangle$.
11. Consider the surface $z = f(x, y) = (2x^2 + 3y^2)e^{-x^2-y^2}$.
    (a) Find the first and second order partials of f.

(b) Now imagine pouring water onto this surface at the point $(\frac{1}{2}, \frac{1}{3})$. What is the tangent plane to the surface at this point?

(c) In what direction will the water flow?

(d) As the water level rises, it will gradually fill up a bounded xy region until spilling over and flowing out to infinity. At what depth will this occur and at what point (or points) will the water spill over?

12. Let $f(x, y) = 4xy - x^3y - xy^3$.

    (a) Compute all first and second order partial derivatives of f.

    (b) Find and classify (as a local minimum, maximum or saddle point) all critical points of f.

    (c) Find the global minimum and global maximum value of f on the disk of radius 2 around the origin.

    (d) Find the gradient of f at the point $(2, 2)$.

    (e) Find the tangent plane to f at the point $(2, 2)$.

    (f) Find a direction $\mathbf{u}$ such that $D_{\mathbf{u}}(f) = \frac{-24}{5}$.

13. Let $f(x, y) = 3xy - x^3 - y^3$. Find and classify (as a local minimum, maximum or saddle point) all critical points of f.

14. Let $f(x, y) = xy + x^5 - y^3$. Find and classify (as a local minimum, maximum or saddle point) all critical points of f.

15. Find the minimum and maximum values of the function $f(x, y, z) = xy^3 - z$ when restricted to the surface $xy + yz = -3$.

16. Find the minimum and maximum values of the function $f(x, y, z) = x^2y - yz$ when restricted to the surface $x^2 + yz + \frac{y^2}{2} = 4$.

17. Suppose that the variables x, y, z satisfy an equation $g(x, y, z) = 0$. Assume the point $P(1, 1, 1)$ lies on this level surface of g and that $\nabla g(1, 1, 1) = \langle -1, 1, 2\rangle$. Let $f(x, y, z)$ be another function, and assume that $\nabla f(1, 1, 1) = \langle 1, 2, 1\rangle$. Find the gradient of the function $w = f(x, y, z(x, y))$ of the two independent variables x and y at the point $x = 1$, $y = 1$.

18. Suppose f, g and h are differentiable functions of two variables, and u and v are differentiable functions of one variable. Suppose that you know that $u(\pi) = 2$, that $v(\pi) = 4$, that $g(2, 4) = -1$ and $h(2, 4) = 0$. In addition, I tell you that $u'(\pi) = 1$, that $v'(\pi) = -1$, that the gradient of g at $(1, -1)$ is $\langle 8, 0\rangle$, that the gradient of g at $(2, 4)$ is $\langle 5, 3\rangle$, that $h(x, y) = x^3 - xy$, and that the gradient of f at $(-1, 0)$ is $\langle 2, 3\rangle$. Let $\alpha(t) = f(g(u(t), v(t)), h(u(t), v(t)))$. Find $a'(\pi)$.

19. Consider the region bounded by the two cylinders $y^2 + z^2 = 1$ and $x^2 + z^2 = 1$. Find an expression that gives the average distance of a point in this region to $(0, 0, 1)$ (you do not have to evaluate any integrals).

20. Integrate the function $x^2 + z^2$ over the region enclosed by the planes $z = 0$ and $z = 1$ and the cone $z^2 = x^2 + y^2$.

21. Find the area of the ellipse $(4x - y)^2 + (x - 3y)^2 < 1$ using an appropriate change of coordinates.

22. Consider the region bounded by the surface

$$z = (3x + 5y)^2 + (2x + 4y)^2$$

and the plane

$$z = 10x + 18y + 2.$$

    (a) Set up an integral to find the volume of this region.
    Hint: $10x + 18y = 2(3x + 5y) + 2(2x + 4y)$

    (b) Evaluate the integral you found in the first part.

23. Give an expression for the arclength of the Archemedian spiral $r = \theta$ between the origin and the point $(2\pi, 0)$.

24. Let C be the spiral given in polar coordinates by the equation $r = \theta$, $0 \leq \theta \leq 2\pi$, traced out starting from the origin. Let $\mathbf{F}(x, y) = (x^2, y)$. Reduce the following line integral to a single variable integral, but do not evaluate the resulting single variable integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

25. Let C be the twisted cubic (t, t^2, t^3) with $1 \le t \le 2$, and let $\mathbf{F}(x, y, z) = (y^z x^{y^z}, \ln(x) z y^{z-1} x^{y^z}, \ln(x)\ln(y) y^z x^{y^z})$. Evaluate $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}$.

26. Consider the vector field $\mathbf{F}(x, y, z) = (2xy + z, x^2 + 3y^2 z, y^3 + x - 4z^3)$.

    (a) Is $\mathbf{F}$ conservative? Why or why not?

    (b) If $\mathbf{F}$ is conservative, find a potential function. If not, find a closed path C such that $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$.

27. For each of the following vector fields, determine whether it is conservative. If it is, find a potential function.

    (a) $\mathbf{F}_1(x, y, z) = \langle 4x^3 y + 3z, x^4 - z + 2, 3x + y \rangle$.

    (b) $\mathbf{F}_2(x, y, z) = \langle 2x + y\cos(xy), x\cos(xy) - ze^{yz}, 2 - ye^{yz} \rangle$.

28. Evaluate the following line integrals.

    (a) $\int_{C_1} \mathbf{F}_1(x, y, z) \cdot d\mathbf{r}$ where $\mathbf{F}_1(x, y, z) = \langle 4x^3 y + 3z, x^4 - z + 2, 3x + y \rangle$ as in the previous problem and C_1 is the portion of the parabola $x = y = z^2$ from $(1, 1, -1)$ to $(1, 1, 1)$.

    (b) $\int_{C_2} \mathbf{F}_2(x, y, z) \cdot d\mathbf{r}$ where $\mathbf{F}_2(x, y, z) = \langle 2x + y\cos(xy), x\cos(xy) - ze^{yz}, 2 - ye^{yz} \rangle$ as in the previous problem and C_2 is the twisted cubic given by $\mathbf{r}(t) = (t, t^2, t^3), 0 \le t \le 1$.

29. Let $\mathbf{F}(x, y, z) = \left(ax^2 y + z^2, x^3 + 4y^3 z, bxz + y^4\right)$.

    (a) For what values of a and b will $\mathbf{F}$ be conservative?

    (b) Using these values of a and b, find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

    (c) Again using these values of a and b, give a defining equation for a surface S with the property that

    $$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = 0$$

    for any two points P and Q lying on S and any path between them.

30. Let C be the curve formed by intersecting the paraboloid $z = (x+y)^2 + (x+2y+1)^2$ and the plane $x - y - z + 2 = 0$, oriented counterclockwise when viewed from above. Let $f(x, y, z)$ be an arbitrary smooth function, and set $\mathbf{F}(x, y, z) = \nabla f + xy\mathbf{i} - xy\mathbf{j} - xy\mathbf{k}$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

31. Parameterize each of the following curves or surfaces. *Hint:* for many of these, you can use vector addition to break up the problem into easier pieces.

    (a) Consider a circle rolling along a line without slipping. Parametrize the path a point on this circle traces out.

    (b) The cone with vertex half-angle α and axis pointing along the positive x-axis.

    (c) A Möbius strip (a Möbius strip can be formed by taking a rectangular strip of paper, making a half twist and then taping the ends together). You may choose any constants necessary, or leave them as variables.

    (d) A spiraling seashell: a tubular surface centered on the helix $(\cos t, \sin t, t)$, with radius away from the helix proportional to t.

    (e) Consider two helices around the z-axis of radius 1, separated by a $180°$ rotation (a double helix). Parametrize the surface formed by connecting corresponding points in these helices with line segments.

32. Let S be a sphere of radius 1, and let the density be equal to the distance *along the sphere* from the north pole of the sphere. Find the mass.

33. Let R be the region bounded above by the surface $z = 16 - x^4 - 2x^2y^2 - y^4$ and below by the lower half of the sphere of radius 2 centered at the origin. If the density of the region is given by $\delta(x, y, z) = x^2 + y^2$, find the center of mass of R.

34. Consider the ellipsoid E given by the equation $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{36} = 1$. Let $f(x, y, z) = \sqrt{2x^2 + 2y^2 + 1}$. Calculate the value of

    $$\iint_E f(x, y, z) dS$$

    Hint: parameterize E similarly to a sphere.

35. Consider the sphere S of radius 1 centered at the point $(0, 0, 1)$

(a) Parameterize S. *Hint:* how would the parameterizations of a sphere centered at the origin and a sphere centered at $(0, 0, 1)$ differ?

(b) Let $f(x, y, z) = x^2z + y^2z - x^2 - y^2$. Compute

$$\iint_S f(x, y, z)\, dS.$$

(c) Let $\mathbf{F}(x, y, z) = \langle x + 3y, 2y - z, 4z + x\rangle$. Compute

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

(d) Now consider just the upper half of S (above the $z = 1$ plane). Call this surface S_1 and equip it with the upward pointing normal. Let $\mathbf{F} = \langle z - y - 1, x + z^2 + z - 2, xy^2 + xz - 4\rangle$. Compute

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

36. For each of the following integrals, use either Green's Theorem, the Divergence Theorem or Stokes' Theorem to write the given integral as an iterated integral with a different number of integral signs (by iterated integral I mean a triple integral, a double integral or a single integral: ie, you need to transform surface integrals and line integrals to double and single integrals and put appropriate limits on your integrals).

(a) Let C be the curve given parametrically by

$$\mathbf{r}(t) = \left(\left(t^3 - \frac{\pi^2 t}{9}\right)^2 \cos t, \left(t^3 - \frac{\pi^2 t}{9}\right)^2 \sin t, 0\right),$$

where t runs from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$. Let

$$\mathbf{F}(x, y, z) = (xy, x^2 + yz, x + y + 3z).$$

Modify the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(b) Let S be the half of the unit sphere that lies above the xy-plane, with upward pointing unit normal. Let $\mathbf{F}(x, y, z) = (2y, 0, 2x)$. Modify the integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

(c) Let S be the surface obtained by rotating the circle

$$(x - 1)^2 + z^2 = 1$$

about the z-axis, oriented outward.
Let $\mathbf{F}(x, y, z) = (x^3 - x, y - y^2z, yz^2)$. Modify the integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

37. Let S be the unit sphere centered at the origin with outward pointing normal and let $\mathbf{F}(x, y, z) = (x, 0, z)$.

(a) Find the flux of $\mathbf{F}$ through S directly.

(b) Use the divergence theorem to check your answer

38. Let S be the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane, and let $\mathbf{F}(x, y, z) = (x^2 + yz^4 - \sin(z^2), x - 4y + z, 4z + 1)$. Compute

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

by using the divergence theorem to reduce to a simpler surface integral.

39. Consider the part of the surface $z = -r^2 + 3r - 2$ that lies above the xy-plane (r is the r of cylindrical coordinates). Call this surface S. Let

$$\mathbf{F}(x, y, z) = \langle x + 3y - \sin^4(z^5), x^2 - y^2, -z - 4 \rangle.$$

Evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

Hint: use the divergence theorem so that the surface integral that you actually compute is easier.

40. Consider a donut: a cylindrical region wrapped into a circular ring. Suppose that the radius from the center to the inner edge of the donut is 3 units, and to the outer edge is 5 units.

    (a) Find limits of integration for the interior of the donut in a coordinate system of your choice.

    (b) Find a parameterization for the surface of the donut.

41. Consider the torus of inner radius 3, outer radius 5, central axis the y-axis and with central plane the xz-plane (this is related to the donut in the previous problem). Let S be the part of this torus above the xy-plane, with outward pointing normal. Define

$$\mathbf{F}(x, y, z) = \langle xy + \cos(\frac{\pi}{2} e^{xyz}) \sin(xy), x + \ln(x^2 + y^2 + z^2) z^{x^2+y^2}, x^2 + xy^4 - 2x \cos(y) - e^{5y - 3\cos(x)} \rangle.$$

Compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.$$