# Math 0240 - Analytic Geometry & Calculus 3
# Final Exam, Fall 2016

YOUR NAME (PLEASE PRINT):______________________________

PEOPLESOFT NUMBER:______________________________

**Please circle the name of your instructor:**

Constantine    Hajlasz    Sparling    Trofimov    Wang    Xu

**Please circle your lecture time:**

9 AM    11 AM    12 PM    1 PM    2 PM

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**INSTRUCTIONS:**

- **No calculators, electronic devices, notes, books or memory cards are allowed.**
- Please write legibly and logically, and show all work. **Incomplete explanations may receive little or no credit.**
- **Warning:** You are expected to work independently.

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**There are ten problems for a total of 100 points.**

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|---|---|---|---|---|
| **1**______(10) | **2**______(10) | **3**______(10) | **4**______(10) | **5**______(10) |
| **6**______(10) | **7**______(10) | **8**______(10) | **9**______(10) | **10**______(10) |

TOTAL__________ (100)

1. (10pts) Let $C$ be the curve given by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j} + 3e^{t-1}\mathbf{k}$. At the point $P(1, 1, 3)$, find the curvature of $C$ and parametric or symmetric equations of the line tangent to $C$.

2. (10pts) Let $f(x, y, z) = \dfrac{x-y}{z} + 4\sqrt{x+3z}$ and $P$ be the point $P(1, 1, 1)$. The following three parts are relevant. You do not need to repeat any calculation.

   (a) (4pts) What is the direction in which the maximum rate of change of $f$ occurs at the point $P$?

   (b) (3pts) Compute the directional derivative of $f(x, y, z)$ at the point $P$ in the direction of the vector $v = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

   (c) (3pts) At the point $P(1, 1, 1)$, the equation $\dfrac{x-y}{z} + 4\sqrt{x+3z} = 8$ holds. Use the Implicit Function Theorem to find $z_x(1, 1)$.

3. (10pts) Let $S$ be the ellipsoid given by the equation $x^2 + y^2 - xz + z^2 = 2$. That is, $S$ is a level surface of the function $F(x, y, z) = x^2 + y^2 - xz + z^2$. Find all points on $S$ where the tangent plane is parallel to the plane $x + 2y + z = 10$. (Hint: Use the fact that the coordinates of such a point satisfy the equation of $S$.)

4. (10pts) Find all critical points of the function $f(x, y) = 2x^2 + y^2 - x^2y$. For each critical point determine if it is a local maximum, a local minimum, or a saddle point.

5. (10pts) Suppose that the volume of a solid $E$ can be represented by the triple integral

$$\iiint_E dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz dy dx.$$

Find the mass of the solid $E$, if the density function is given by $\rho(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$.

6. (10pts) Given the vector field $\mathbf{F}(x, y) = (2x \ln y - y)\ \mathbf{i} + \left(x^2 y^{-1} - x\right)\ \mathbf{j}$ defined on $\{(x, y) \mid y > 0\}$.

   (a) (6pts) Show that $\mathbf{F}$ is conservative and find a potential function $f$.

   (b) (4pts) A particle, under the influence of the vector field $\mathbf{F}$, moves along the curve $C$ given by $\mathbf{r}(t) = (3t)\mathbf{i} + (2t^2 + 1)\mathbf{j}$ from $t = 0$ to $t = 1$. Use the Fundamental Theorem of line integrals to find the work done.

7. (10pts)

(a) (4pts) Use Green's Theorem to show that a region $R$ enclosed by a simple closed curve $C$, oriented clockwise, has area $\int_C y\,dx$.

(b) (6pts) Use part a) to compute the area of the region $D$ enclosed by the arch of the cycloid $C_1 : x = t - \sin t,\ y = 1 - \cos t$ from $(0,0)$ to $(2\pi, 0)$ and the line segment $C_2 : x = t,\ y = 0$ from $(2\pi, 0)$ to $(0,0)$. See the sketch below.

8. (10pts) Find the area of the surface $S$ that is the part of the cylinder $x^2 + y^2 = 1$, below the plane $z = 3 - x - y$ and above the plane $z = 0$.

9. (10pts) Use Stoke's Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2 y\, \mathbf{i} + \frac{1}{3}x^3\, \mathbf{j} + xy\, \mathbf{k}$, and where $C$ is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the hyperbolic paraboloid $z = y^2 - x^2$, oriented counterclockwise when viewed from above.

10. (10pts) Let $S$ be the boundary surface of the solid $E$ enclosed by the paraboloids $z = 1 + x^2 + y^2$ and $z = 2(x^2 + y^2)$, with the normal pointing outward. Compute the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \left(e^{\sin z} - x^2\right) \mathbf{i} + 2xy\mathbf{j} + (z^2 - \cos y)\mathbf{k}$.

1. $r(t) = \langle t, t^2, 3e^{t-1} \rangle$ Find Curvature @ $P(1,1,3)$
Tangent line

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$$r(t) = \langle t, t^2, 3e^{t-1} \rangle$$

$$r'(t) = \langle 1, 2t, 3e^{t-1} \rangle$$

(2) $$r''(t) = \langle 0, 2, 3e^{t-1} \rangle$$

$i \quad j \quad k$

$P(1,1,3) \Rightarrow t=1 \Rightarrow r'(1) = \langle 1, 2, 3 \rangle$

(2) $r''(1) = \langle 0, 2, 3 \rangle$

(2) $r'(1) \times r''(1) = \langle 0, -3, 2 \rangle$

$$\kappa(1) = \frac{|r'(1) \times r''(1)|}{|r'(1)|^3} = \frac{\sqrt{9+4}}{\left(\sqrt{1+4+9}\right)^3} = \frac{\sqrt{13}}{14^{3/2}}.$$
(2)

(2) $$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-3}{3}$$

2. $f(x,y,z) = \frac{(x-y)}{z} + 4\sqrt{x+3z}$

a) $\nabla f = \langle f_x, f_y, f_z \rangle$

$= \langle \frac{1}{z} + 4 \cdot \frac{1}{2\sqrt{x+3z}}, -\frac{1}{z}, (x-y)\frac{-1}{z^2} + 4 \cdot \frac{1}{2\sqrt{x+3z}} \cdot 3 \rangle$

$P(1,1,1) \Rightarrow \nabla f(1,1,1) = \langle 2, -1, 3 \rangle$

b) $D_u f(1,1,1) = \nabla f \cdot \frac{v}{|v|} = \langle 2, -1, 3 \rangle \cdot \frac{\langle 2, 3, 1 \rangle}{\sqrt{14}}$

$= \frac{4}{\sqrt{14}}$.

c) $z_x(1,1) = -\frac{f_x(1,1,1)}{f_z(1,1,1)} = -\frac{2}{3}$

3.

Tangent plane.

Let $F(x,y,z) = x^2 - xz + z^2 + y^2$

$F_x = 2x - z$

3 $F_y = 2y$

$F_z = -x + 2z$

$n$ of $x + 2y + z = 10$ is $\langle 1, 2, 1 \rangle$

$$3 \quad \nabla F \parallel n \Rightarrow \begin{cases} 2x - z = \lambda \\ 2y = 2\lambda \\ -x + 2z = \lambda \end{cases}$$

$z = 2x - \lambda$

$\Rightarrow y = \lambda$

$-x + 2(2x - \lambda) = \lambda$

$-x + 4x - 2\lambda = \lambda$

$3x = 3\lambda$

3 $\Rightarrow x = \lambda$

$\Rightarrow z = \lambda$

$\Rightarrow \lambda^2 - \lambda^2 + \lambda^2 + \lambda^2 = 2$

$\Rightarrow \lambda = \pm 1$

1 $\Rightarrow (1, 1, 1)$ and $(-1, -1, -1)$.

4

$f(x,y) = 2x^2 + y^2 - x^2 y$

critical pts.

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2 $f_x = 4x + 0 - 2xy = 2x(2-y) = 0 \Rightarrow x=0 \quad y=2$

$f_y = 0 + 2y - x^2$

3 $x=0,\ y=0, \qquad y=2 \Rightarrow 4 - x^2 = 0 \Rightarrow x = \pm 2$

$(0,0) \qquad (2,2) \qquad (-2,2)$

$D = 2(4-2y) - 4x^2$

2 $f_{xx} = 4 - 2y \qquad f_{xy} = -2x$

$f_{yx} = -2x \qquad f_{yy} = 2$

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1 $D(0,0) = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} > 0, \quad 4 > 0 \quad$ local min

1 $D(2,2) = \begin{vmatrix} 0 & -4 \\ -4 & 2 \end{vmatrix} < 0 \quad$ saddle

1 $D(-2,2) = \begin{vmatrix} 0 & 4 \\ 4 & 2 \end{vmatrix} < 0 \quad$ saddle

6.

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$F = \langle \overset{P}{2x \ln y - y}, \ \overset{Q}{x^2 y^{-1} - x} \rangle$

a) $Q_x - P_y = (2xy^{-1} - 1) - (2x \cdot \frac{1}{y} - 1) = 0$ (2) yes.

$f = \int f_x \, dx = \int 2x \ln y - y \, dx = x^2 \ln y - xy + h(y)$ (2)

(2) $\begin{cases} f_y = x^2 \cdot y^{-1} - x + h'(y) \\ Q = x^2 y^{-1} - x \end{cases} \Rightarrow h'(y) = 0 \Rightarrow f = x^2 \ln y - xy.$

$h(y) = \text{constant}$

b). (2) $\begin{cases} r(t) = \langle 3t, \ 2t^2 + 1 \rangle \\ t = 0 \Rightarrow (0, 1) \\ t = 1 \Rightarrow (3, 3) \end{cases}$

(2) $\begin{cases} W = \int_C F \cdot dr = f(3,3) - f(0,1) = (3^2 \ln 3 - 3 \cdot 3) - (0 - 0) \\ = 9 \ln 3 - 9. \end{cases}$

5 Volume → Density.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} \, dz\,dy\,dx$$

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$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^1 e^{\rho^3} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \frac{\pi}{2} \cdot \int_0^{\frac{\pi}{4}} \sin\varphi \, d\varphi \cdot \frac{1}{3} e^{\rho^3} \Big|_0^1$$

$$= \frac{\pi}{2} \cdot \cos\varphi \Big|_{\frac{\pi}{4}}^{0} \cdot \frac{1}{3}(e-1)$$

$$= \frac{\pi}{2} \cdot \left(1-\frac{\sqrt{2}}{2}\right) \cdot \frac{1}{3}(e-1)$$

6.

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$F = \langle 2x\ln y - y,\ x^2y^{-1} - x \rangle$

a) $Q_x - P_y = (2xy^{-1} - 1) - (2x \cdot \frac{1}{y} - 1) = 0$ yes.

$f = \int f_x\, dx = \int 2x\ln y - y\, dx = x^2\ln y - xy + h(y)$

$\left.\begin{array}{l} f_y = x^2 \cdot y^{-1} - x + h'(y) \\ Q = x^2y^{-1} - x \end{array}\right\} \Rightarrow h'(y) = 0 \Rightarrow f = x^2\ln y - xy.$

$h(y) = \text{constant}$

b). $r(t) = \langle 3t,\ 2t^2 + 1 \rangle$

$t = 0 \Rightarrow (0, 1)$

$t = 1 \Rightarrow (3, 3)$

$W = \int_C F \cdot dr = f(3,3) - f(0,1) = (3^2 \cdot \ln 3 - 3 \cdot 3) - (0 - 0)$

$= 9\ln 3 - 9.$

Green's thm.

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④ $\int_C y\,dx \underset{Green}{=} -\iint_D Q_x - P_y \, dA = -\iint_D 0-(-1)\,dA = \iint_D 1\,dA$

$= A(D).$

$P = y \quad Q = 0.$

⑥ $A(D) = \int_C y\,dx = \int_0^{2\pi} (1-\cos t)\cdot(1-\cos t)\,dt + \int_{2\pi}^{0} 0\,dx$

$= \int_0^{2\pi} 1 - 2\cos t + \cos^2 t \,dt \quad + 0$

$= \int_0^{2\pi} 1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t \,dt$

$= \left(t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi}$

$= 2\pi - 0 + \pi + 0$

$= 3\pi.$

$C_1 \ \langle t - \sin t,\ 1-\cos t\rangle \quad (0, 2\pi)$

$\Rightarrow C_2 \ \langle x, 0\rangle$

8.

$\frac{40}{25}$
15

$S:\ x=\cos\theta,\quad y=\sin\theta\quad z=z$

$0\le\theta\le 2\pi$

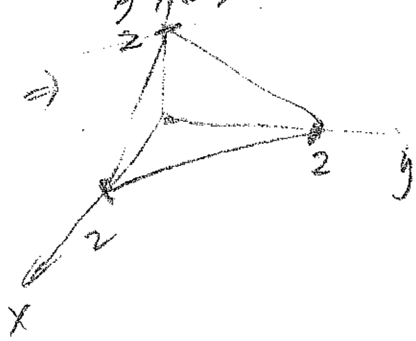
3 $\quad 0\le z\le 3-x-y=3-\cos\theta-\sin\theta.$

$z=3-x-y$

$\Rightarrow x+y+z=3$

$\Rightarrow \frac{x}{3}+\frac{y}{3}+\frac{z}{3}=1$

$\Rightarrow$




$\Rightarrow\quad 0\le\theta\le 2\pi$

$n=r_\theta\times r_z=\begin{vmatrix} i & j & k\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}$

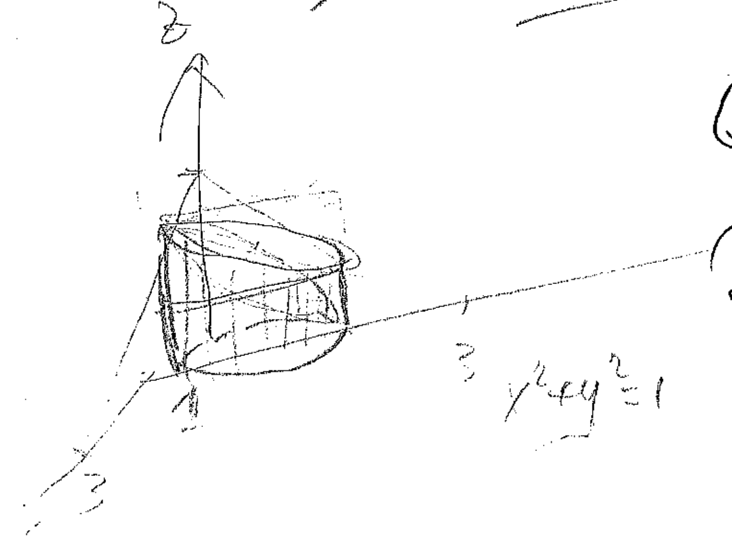
2 $\quad =\langle\cos\theta,\ \sin\theta,\ 0\rangle$

$|r_\theta\times r_z|=\sqrt{\phantom{xx}}=1$

3 $A(S)=\int_0^{2\pi}\int_0^{3-\cos\theta-\sin\theta} 1\cdot 1\, dz\, d\theta$

2 $\quad =\int_0^{2\pi} 3-\cos\theta-\sin\theta\, d\theta$

$=3\cdot 2\pi=6\pi$




lateral surface

3 $\ell:\ x=\cos t\quad y=\sin t\qquad 0\le t\le 2\pi$

$f(x,y)=3-x-y.$

3 $A(S)=\int_\ell f(x,y)\,ds=\int_0^{2\pi} 3-\cos t-\sin t\,\cdot$

2 $\sqrt{(-\sin t)^2+(\cos t)^2}\cdot dt$

2 $=\int_0^{2\pi} 3-\cos t-\sin t\, dt$

$=3\cdot 2\pi=6\pi.$

9

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$S:\quad x = x,\ \ y = y \quad z = y^2 - x^2$

$D:\quad x^2 + y^2 = 1$

$n: = \langle -2x, -2y, 1\rangle = \langle -2x, 2y, 1\rangle$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & \frac{x^3}{3} & xy \end{vmatrix} = \langle x, -y, 0\rangle$$

$$\oint_C F\,dr \underset{\text{Stokes}}{=} \iint_S \operatorname{curl} F \cdot d\vec{S} = \iint_D \langle x, -y, 0\rangle \cdot \langle -2x, 2y, 1\rangle\, dA$$

$$= -\iint_D 2x^2 + 2y^2\, dA = -2\int_0^{2\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta = -2\cdot 2\pi \cdot \frac{r^4}{4}\Big|_0^1$$

$$= -\pi.$$

10. $\vec{F} = \langle e^{\sin z} - x^2,\ 2xy,\ z^2 - \cos y \rangle$

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$\operatorname{Div} F = -2x + 2x + 2z = 2z$

$$\iint_S \vec{F} \cdot d\vec{S} \overset{+}{\underset{\text{div.}}{=}} \iiint_E 2z\, dv = \int_0^{2\pi}\int_0^1\int_{2r^2}^{1+r^2} 2z \ r\, dz\, dr\, d\theta$$

$$= 2\pi \cdot \int_0^1 z^2\Big|_{2r^2}^{1+r^2} \cdot r\, dr = 2\pi\int_0^1 r\left[(1+r^2)^2 - (2r^2)^2\right] dr$$

$$= 2\pi \int_0^1 r\left(1 + 2r^2 + r^4 - 4r^4\right) dr$$

$$= 2\pi \int_0^1 r + 2r^3 - 3r^5\, dr$$

$$= 2\pi \left(\frac{r^2}{2} + \frac{2}{4}r^4 - \frac{3}{6}r^6\right)\Big|_0^1$$

$$= 2\pi\left(\frac{1}{2} + \frac{1}{2} - \frac{1}{2}\right) = \pi.$$