1. Evaluate \[ \int_0^1 \int_{x^2}^1 x^3 \sin(y^3) \, dy \, dx. \]

**Solution.** We switch the order of integration. The region described is that bounded by \( y = x^2, \ y = 1 \) and \( x = 0 \). Switching to a type II integral, we have

\[
\int_0^1 \int_{x^2}^1 x^3 \sin(y^3) \, dy \, dx = \frac{1}{4} \int_0^1 (\sqrt{y})^4 \sin(y^3) \, dy
\]

\[
= \frac{1}{12} \int_0^1 \sin(u) \, du
\]

\[
= \frac{1 - \cos(1)}{12},
\]

using the substitution \( u = y^3 \).

2. Consider a laminate of radius 1 whose density at any point is proportional to the distance from the center. Find the moment of inertia about its center.

**Solution.** We use polar coordinates, first finding the mass. The density can be written as \( Kr \) where \( K \) is a constant, and then the mass is given by

\[
m = \int_0^{2\pi} \int_0^1 Kr \cdot r \, dr \, d\theta
\]

\[
= \frac{2K\pi}{3}.
\]

The moment of inertia about its center is then given by

\[
I = \frac{1}{m} \int \rho(x, y)(x^2 + y^2) \, dA
\]

\[
= \frac{3}{2K\pi} \int_0^{2\pi} \int_0^1 Kr \cdot r^2 \cdot r \, dr \, d\theta
\]

\[
= \frac{3}{5}.
\]

3. Find the limits of integration when rewriting the following integral with a different order of integration.

\[
\int_{-1}^1 \int_{\sqrt{1-x^2}}^{1} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2}} f(x, y, z) \, dz \, dy \, dx = \int_{?}^? \int_{?}^? \int_{?}^? f(x, y, z) \, dx \, dy \, dz.
\]
4. Evaluate the following integrals:

Solution. The region of integration is that above the cone \( z = \sqrt{x^2 + y^2} \) and below \( z = 1 \). So the outer limits are \( z = 0 \) to \( z = 1 \). The projection of the cone onto the \( yz \)-plane is the triangle bounded by \( z = 1 \) and the lines \( y = z \) and \( y = -z \), so the middle limits are \( y = -z \) to \( y = z \). Finally, we solve \( z = \sqrt{x^2 + y^2} \) for \( x \), giving \( x = \pm \sqrt{2^2 - y^2} \). Thus the new limits are

\[
\int_0^1 \int_z^{-z} \int_{\sqrt{x^2-y^2}}^{\sqrt{z^2-y^2}} f(x, y, z) \, dx \, dy \, dz.
\]

4. Evaluate the following integrals:

(a)

\[
\int_0^2 \int_{\sqrt{4-x^2}}^{\sqrt{2x^2-y^2}} \int_{\sqrt{2^2-x^2-y^2}}^{\sqrt{2^2+y^2+z^2}} xyz \sqrt{x^2+y^2+z^2} \, dz \, dy \, dx.
\]

Solution. We use spherical coordinates, as hinted by the \( \sqrt{x^2 + y^2 + z^2} \) in the integrand and the limit \(-\sqrt{8-x^2-y^2}\), which is the bottom half of a sphere of radius \( \sqrt{8} \). The projection to the \( xy \)-plane is a semicircle of radius 2 in the first and fourth quadrants, corresponding to a range on \( \theta \) from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\). The intersection between the cone \( z = -\sqrt{x^2+y^2} \) and the hemisphere \( z = -\sqrt{8-x^2-y^2} \) also occurs along the circle \( x^2 + y^2 = 4 \), so the region described is the one below the cone \( z = -\sqrt{x^2+y^2} \), above the sphere \( x^2 + y^2 + z^2 = 8 \) and on the positive side of the plane \( x = 0 \). In spherical coordinates, the corresponding limits are

\[
\int_{-\pi/2}^{\pi/2} \int_{\pi/4}^{\pi} \int_0^{\sqrt{8}} \rho \cos(\theta) \sin(\phi) \rho \sin(\phi) d\rho \, d\phi \, d\theta.
\]

Translating the integrand to spherical coordinates and remembering the volume element \( \rho^2 \sin(\phi) \), we get

\[
\int_{-\pi/2}^{\pi/2} \int_{\pi/4}^{\pi} \int_0^{\sqrt{8}} \rho \cos(\theta) \sin(\phi) \rho \cos(\phi) \cdot \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} \int_{\pi/4}^{\pi} \int_0^{\sqrt{8}} \rho^3 \sin^2(\phi) \, d\rho \, d\phi \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} \int_{\pi/4}^{\pi} \left[ \frac{1}{3} \sin^3(\phi) \right]_0^{\pi} [\rho^6]_0^{\sqrt{8}} \, d\phi \, d\theta
\]

\[
= -128\sqrt{2} \frac{\pi}{9}
\]

(b)

\[
\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{-1}^{1-\sqrt{x^2+y^2}} yz(2 + \sqrt{x^2+y^2}) \, dz \, dy \, dx.
\]

Solution. We use cylindrical coordinates, as hinted by the repeated appearances of \( \sqrt{x^2 + y^2} \). Note that the middle limit \( y = \sqrt{2x-x^2} \) is a semicircle, as we can see by squaring both sides and then completing the square, yielding \( y^2 + (x-1)^2 = 1 \). This translates to the equation \( r = 2 \cos(\theta) \) in polar coordinates, with \( \theta \) ranging from 0 to \( \frac{\pi}{2} \) since the lower limit on \( y \) is 0. The limits on \( z \)
are handled by just changing \( \sqrt{x^2 + y^2} \) to \( r \). We get

\[
\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{-1}^{1-\sqrt{x^2+y^2}} yz(2 + \sqrt{x^2 + y^2}) \, dz \, dy \, dx
\]

\[
= \int_0^{\pi/2} \int_0^{\sin^2(\theta)} \int_{-1}^{1-r} r \sin(\theta)z(2 + r) \, dz \, dr \, d\theta
\]

\[
= \frac{1}{2} \int_0^{\pi/2} \int_0^{\sin^2(\theta)} r^2(2 + r) ((1 - r)^2 - (-1)^2) \, dr \, d\theta
\]

\[
= \frac{1}{2} \int_0^{\pi/2} \int_0^{\sin^2(\theta)} r^2 - 4r^3 \, dr \, d\theta
\]

\[
= \int_0^1 \frac{16}{3}u^6 - 8u^4 \, du
\]

\[
= 16 \cdot \frac{8}{21} - \frac{8}{5}
\]

\[
= -\frac{88}{105}
\]

5. Let \( R \) be the region bounded by the ellipse \( 2x^2 - 2xy + y^2 = 5 \). Using the change of variables

\[
x = 2u + v \\
y = u + 3v,
\]

evaluate

\[
\iint_R x^2 \, dA.
\]

**Solution.** Substituting the change of variables into the equation for the ellipse yields

\[
2(2u + v)^2 - 2(2u + v)(u + 3v) + (u + 3v)^2 = 5
\]

\[
8u^2 + 8uv + 2v^2 - 4u^2 - 14uv - 6u^2 + u^2 + 6uv + 9v^2 = 5
\]

\[
5u^2 + 5v^2 = 5
\]

\[
u^2 + v^2 = 1,
\]

so the corresponding region in the uv-plane is a disc of radius 1. Switching to polar coordinates with \( u = r \cos(\theta) \) and \( v = r \sin(\theta) \) yields

\[
\iint_R x^2 \, dA = \int_0^{2\pi} \int_0^1 (2r \cos(\theta) + r \sin(\theta))^2 \, r \, dr \, d\theta
\]

\[
= \left( \int_0^{2\pi} 4 \cos^2(\theta) + 4 \sin(\theta) \cos(\theta) + \sin^2(\theta) \, d\theta \right) \left( \int_0^1 r^3 \, dr \right)
\]

\[
= \frac{5\pi}{4}
\]

6. Find the maximum and minimum values of \( x^2 + 2y^2 + 2z^2 \) subject to the constraint \( x + y + z = 4 \).

**Solution.** We use Lagrange multipliers with \( f(x, y, z) = x^2 + 2y^2 + 2z^2 \) and \( g(x, y, z) = x + y + z - 4 \). This yields the system of equations

\[
2x = \lambda
\]

\[
4y = \lambda
\]

\[
4z = \lambda
\]

\[
4 = x + y + z.
\]
Solving for $x, y, z$ in the first three equations and substituting into the last gives $\lambda = 4$ and thus $x = 2$ and $y = z = 1$. The value of $f(x, y, z)$ is a minimum at $(2, 1, 1)$ and increases without bound along the plane $x + y + z = 4$ in all directions: there is no maximum.

Of course, the constraint is simple enough that you can also solve for $z$, substitute and then find critical points for the resulting function of $x$ and $y$. 