Math 240 – Practice Exam 2 Solutions

1. Evaluate
$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx.$$

   Solution. *We switch the order of integration. The region described is that bounded by* $y = x^2$*,* $y = 1$ *and* $x = 0$*. Switching to a type II integral, we have*
$$\begin{aligned}\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx &= \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy \\ &= \frac{1}{4}\int_0^1 (\sqrt{y})^4 \sin(y^3)\, dy \\ &= \frac{1}{12}\int_0^1 \sin(u)\, du \\ &= \frac{1-\cos(1)}{12},\end{aligned}$$
   using the substitution $u = y^3$*.*

2. Consider a laminate of radius 1 whose density at any point is proportional to the distance from the center. Find the moment of inertia about its center.

   Solution. *We use polar coordinates, first finding the mass. The density can be written as* Kr *where* K *is a constant, and then the mass is given by*
$$\begin{aligned} m &= \int_0^{2\pi} \int_0^1 Kr \cdot r\, dr\, d\theta \\ &= \frac{2K\pi}{3}.\end{aligned}$$
   The moment of inertia about its center is then given by
$$\begin{aligned} I &= \frac{1}{m} \iint \rho(x,y)(x^2+y^2)\, dA \\ &= \frac{3}{2K\pi} \int_0^{2\pi} \int_0^1 Kr \cdot r^2 \cdot r\, dr\, d\theta \\ &= \frac{3}{5}.\end{aligned}$$

3. Find the limits of integration when rewriting the following integral with a different order of integration.
$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 f(x,y,z)\, dz\, dy\, dx = \int_?^? \int_?^? \int_?^? f(x,y,z)\, dx\, dy\, dz.$$

Solution. *The region of integration is that above the cone* $z = \sqrt{x^2+y^2}$ *and below* $z = 1$. *So the outer limits are* $z = 0$ *to* $z = 1$. *The projection of the cone onto the* yz*-plane is the triangle bounded by* $z = 1$ *and the lines* $y = z$ *and* $y = -z$, *so the middle limits are* $y = -z$ *to* $y = z$. *Finally, we solve* $z = \sqrt{x^2+y^2}$ *for* x, *giving* $x = \pm\sqrt{z^2-y^2}$. *Thus the new limits are*

$$\int_0^1 \int_{-z}^{z} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\, dx\, dy\, dz.$$

4. Evaluate the following integrals:

(a)

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{8-x^2-y^2}}^{-\sqrt{x^2+y^2}} xz\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

Solution. *We use spherical coordinates, as hinted by the* $\sqrt{x^2+y^2+z^2}$ *in the integrand and the limit* $-\sqrt{8-x^2-y^2}$, *which is the bottom half of a sphere of radius* $\sqrt{8}$. *The projection to the* xy*-plane is a semicircle of radius* 2 *in the first and fourth quadrants, corresponding to a range on* θ *from* $-\frac{\pi}{2}$ *to* $\frac{\pi}{2}$. *The intersection between the cone* $z = -\sqrt{x^2+y^2}$ *and the hemisphere* $z = -\sqrt{8-x^2-y^2}$ *also occurs along the circle* $x^2+y^2 = 4$, *so the region described is the one below the cone* $z = -\sqrt{x^2+y^2}$, *above the sphere* $x^2+y^2+z^2 = 8$ *and on the positive side of the plane* $x = 0$. *In spherical coordinates, the corresponding limits are*

$$\int_{-\pi/2}^{\pi/2} \int_{3\pi/4}^{\pi} \int_0^{\sqrt{8}} \cdots\, d\rho\, d\phi\, d\theta.$$

Translating the integrand to spherical coordinates and remembering the volume element $\rho^2\sin(\phi)$, *we get*

$$\begin{aligned}
&\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{8-x^2-y^2}}^{-\sqrt{x^2+y^2}} xz\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx \\
&\qquad = \int_{-\pi/2}^{\pi/2} \int_{3\pi/4}^{\pi} \int_0^{\sqrt{8}} \rho\cos(\theta)\sin(\phi)\cdot\rho\cos(\phi)\cdot\rho\cdot\rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta \\
&\qquad = \Big[\sin(\theta)\Big]_{-\pi/2}^{\pi/2} \left[\frac{1}{3}\sin^3(\phi)\right]_{3\pi/4}^{\pi} \left[\frac{1}{6}\rho^6\right]_0^{\sqrt{8}} \\
&\qquad = -\frac{128\sqrt{2}}{9}
\end{aligned}$$

(b)

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{-1}^{1-\sqrt{x^2+y^2}} yz(2+\sqrt{x^2+y^2})\, dz\, dy\, dx.$$

Solution. *We use cylindrical coordinates, as hinted by the repeated appearances of* $\sqrt{x^2+y^2}$. *Note that the middle limit* $y = \sqrt{2x-x^2}$ *is a semicircle, as we can see by squaring both sides and then completing the square, yielding* $y^2+(x-1)^2 = 1$. *This translates to the equation* $r = 2\cos(\theta)$ *in polar coordinates, with* θ *ranging from* 0 *to* $\frac{\pi}{2}$ *since the lower limit on* y *is* 0. *The limits on* z

are handled by just changing $\sqrt{x^2+y^2}$ *to* r. *We get*

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{-1}^{1-\sqrt{x^2+y^2}} & yz(2+\sqrt{x^2+y^2})\ dz\ dy\ dx \\
&= \int_0^{\pi/2} \int_0^{2\cos(\theta)} \int_{-1}^{1-r} r\sin(\theta) z(2+r)\ r\ dz\ dr\ d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \sin(\theta) \int_0^{2\cos(\theta)} r^2(2+r)\left((1-r)^2-(-1)^2\right)\ dr\ d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \sin(\theta) \int_0^{2\cos(\theta)} r^5 - 4r^3\ dr\ d\theta \\
&= \int_0^1 \frac{16}{3}u^6 - 8u^4\ du \\
&= \frac{16}{21} - \frac{8}{5} \\
&= -\frac{88}{105}
\end{aligned}$$

5. Let R be the region bounded by the ellipse $2x^2 - 2xy + y^2 = 5$. Using the change of variables

$$\begin{aligned}
x &= 2u + v \\
y &= u + 3v,
\end{aligned}$$

evaluate

$$\iint_R x^2\ dA.$$

Solution. *Substituting the change of variables into the equation for the ellipse yields*

$$\begin{aligned}
2(2u+v)^2 - 2(2u+v)(u+3v) + (u+3v)^2 &= 5 \\
8u^2 + 8uv + 2v^2 - 4u^2 - 14uv - 6v^2 + u^2 + 6uv + 9v^2 &= 5 \\
5u^2 + 5v^2 &= 5 \\
u^2 + v^2 &= 1,
\end{aligned}$$

so the corresponding region in the uv-plane is a disc of radius 1. *Switching to polar coordinates with* $u = r\cos(\theta)$ *and* $v = r\sin(\theta)$ *yields*

$$\begin{aligned}
\iint_R x^2\ dA &= \int_0^{2\pi} \int_0^1 (2r\cos(\theta) + r\sin(\theta))^2\ r\ dr\ d\theta \\
&= \left(\int_0^{2\pi} 4\cos^2(\theta) + 4\sin(\theta)\cos(\theta) + \sin^2(\theta)\ d\theta\right)\left(\int_0^1 r^3\ dr\right) \\
&= \frac{5\pi}{4}
\end{aligned}$$

6. Find the maximum and minimum values of $x^2 + 2y^2 + 2z^2$ subject to the constraint $x + y + z = 4$.

Solution. *We use Lagrange multipliers with* $f(x,y,z) = x^2 + 2y^2 + 2z^2$ *and* $g(x,y,z) = x + y + z - 4$. *This yields the system of equations*

$$\begin{aligned}
2x &= \lambda \\
4y &= \lambda \\
4z &= \lambda \\
4 &= x + y + z.
\end{aligned}$$

Solving for x, y, z in the first three equations and substituting into the last gives $\lambda = 4$ and thus $x = 2$ and $y = z = 1$. The value of $f(x, y, z)$ is a minimum at $(2, 1, 1)$ and increases without bound along the plane $x + y + z = 4$ in all directions: there is no maximum.

Of course, the constraint is simple enough that you can also solve for z, substitute and then find critical points for the resulting function of x and y.