Math 240 – Practice Exam 1

1. Find the distance between the lines

$$L_1 : \langle t-1, t+1, 2t\rangle$$
$$L_2 : \langle 2t+1, 0, t-1\rangle$$

Solution. *We project a vector* $\mathbf{a}$ *from a point on* L_1 *to a point on* L_2 *onto the cross product* $\mathbf{b}$ *of their direction vectors, which is perpendicular to both. The absolute value of the component in this perpendicular direction is the distance.*

$$\begin{aligned}\mathbf{b} &= \langle 1,1,2\rangle \times \langle 2,0,1\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \\ &= \langle 1-0, 4-1, 0-2\rangle \\ &= \langle 1,3,-2\rangle.\end{aligned}$$

The vector $\mathbf{a}$ *can be chosen as the difference between the base points on the lines:* $\mathbf{a} = (1,0,-1) - (-1,1,0) = \langle 2,-1,-1\rangle$. *We then have*

$$\begin{aligned}\operatorname{comp}_{\mathbf{b}} \mathbf{a} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} \\ &= \frac{2-3+2}{\sqrt{1^2+3^2+(-2)^2}} \\ &= \frac{1}{\sqrt{14}}.\end{aligned}$$

Thus the distance between the two lines is the absolute value of this quantity, or $\frac{1}{\sqrt{14}}$.

2. Consider the curve parameterized by

$$\mathbf{r}(t) = \left\langle t\sin(t), \frac{2\sqrt{2}}{3}t^{3/2}, t\cos(t)\right\rangle.$$

Find the arclength along this curve between $(0,0,0)$ and $(0, \frac{8}{3}\pi^{3/2}, 2\pi)$.

Solution. *Note that* $(0,0,0)$ *corresponds to* $t = 0$ *and* $(0, \frac{8}{3}\pi^{3/2}, 2\pi)$ *to* $t = 2\pi$. *We compute the derivative, finding*

$$\mathbf{r}'(t) = \langle \sin(t) + t\cos(t), \sqrt{2t}, \cos(t) - t\sin(t) \rangle.$$

The arclength is then

$$\begin{aligned}
\int_0^{2\pi} |\mathbf{r}'(t)|\ \mathrm{d}t &= \int_0^{2\pi} \sqrt{(\sin(t) + t\cos(t))^2 + (\sqrt{2t})^2 + (\cos(t) - t\sin(t))^2}\ \mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{1 + 2t + t^2}\ \mathrm{d}t \\
&= \int_0^{2\pi} t + 1\ \mathrm{d}t \\
&= \left[\frac{t^2}{2} + t\right]_0^{2\pi} \\
&= 2(\pi^2 + \pi).
\end{aligned}$$

3. Consider the curve parameterized by

$$\mathbf{r}(t) = \langle t^2, t^3 - t, t \rangle.$$

Find the curvature as a function of t.

Solution. *We use the formula* $\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 2t, 3t^2 - 1, 1 \rangle \\
\mathbf{r}''(t) &= \langle 2, 6t, 0 \rangle \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 3t^2 - 1 & 1 \\ 2 & 6t & 0 \end{vmatrix} \\
&= \langle -6t, 2, 6t^2 + 2 \rangle \\
\frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} &= \frac{\sqrt{36t^4 + 60t^2 + 8}}{(9t^4 - 2t^2 + 2)^{3/2}}.
\end{aligned}$$

4. Suppose that S is a surface that contains the two curves given by

$$\begin{aligned}
\mathbf{r}_1(t) &= \langle t\cos(t) - 1, t^2 + t + 1, 2e^t \rangle \\
\mathbf{r}_2(t) &= \langle \ln(t) - t, \sqrt{t}, \frac{2}{t} \rangle
\end{aligned}$$

Find an equation for the tangent plane to S at the point $(-1, 1, 2)$.

Solution. *We need to find the normal vector* $\mathbf{n}$ *to* S *at* $(-1, 1, 2)$. *Both curves pass through this point (*$\mathbf{r}_1(0)$ *and* $\mathbf{r}_2(1)$*), and their derivatives must* *be perpendicular to* $\mathbf{n}$ *since we are told that they lie within* S. *We can thus compute* $\mathbf{n}$ *using a cross product:*

$$\begin{aligned}
\mathbf{r}_1'(t) &= \langle \cos(t) - t\sin(t), 2t+1, 2e^t \rangle \\
\mathbf{r}_1'(0) &= \langle 1, 1, 2 \rangle \\
\mathbf{r}_2'(t) &= \langle \frac{1}{t} - 1, \frac{1}{2\sqrt{t}}, \frac{-2}{t^2} \rangle \\
\mathbf{r}_2'(1) &= \langle 0, \frac{1}{2}, -2 \rangle \\
\mathbf{n} &= \langle 1, 1, 2 \rangle \times \langle 0, \frac{1}{2}, -2 \rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 0 & \frac{1}{2} & -2 \end{vmatrix} \\
&= \langle -3, 2, \frac{1}{2} \rangle.
\end{aligned}$$

We can rescale by 2 *to eliminate the fraction (optional), then write down an equation for the tangent plane*

$$-6(x+1) + 4(y-1) + (z-2) = 0.$$

5. Suppose that $f(x, y)$ has continuous second partial derivatives, and that

$$g(u, v) = f(u^2 + v^2, u^2 - v^2).$$

Suppose that the derivatives of $f(x, y)$ are given in the following table.

	f	f_x	f_y	f_{xx}	f_{xy}	f_{yy}
$(1, 1)$	1	2	0	-1	1	3
$(2, 0)$	0	-1	1	2	0	1

Find the value of $g_{uu}(1, 1)$.

Solution. *We use a mix of notation for partial derivatives in what follows for clarity. Computing the derivative* g_u *uses the chain rule. In differentiating again, we need to the product rule on each summand, and then the* *chain rule when differentiating f_x or f_y with respect to u.*

$$\begin{aligned}
g_u &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\
&= f_x x_u + f_y y_u \\
g_{uu} &= \frac{\partial}{\partial u}(f_x x_u) + \frac{\partial}{\partial u}(f_y y_u) \\
&= \frac{\partial f_x}{\partial u} x_u + f_x \frac{\partial x_u}{\partial u} + \frac{\partial f_y}{\partial u} y_u + f_y \frac{\partial y_u}{\partial u} \\
&= \left(\frac{\partial f_x}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial u}\right) x_u + f_x x_{uu} + \left(\frac{\partial f_y}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial u}\right) y_u + f_y y_{uu} \\
&= (f_{xx}x_u + f_{xy}y_u)x_u + f_x x_{uu} + (f_{yx}x_u + f_{yy}y_u)y_u + f_y y_{uu} \\
&= (2\cdot 2 + 0\cdot 2)\cdot 2 + (-1)\cdot 2 + (0\cdot 2 + 1\cdot 2)\cdot 2 + 1\cdot 2 \\
&= 12.
\end{aligned}$$

Note that we only use the second row of the table, since the derivatives of f should be evaluated at $(x(u,v), y(u,v)) = (2,0)$. On other other hand, the derivatives of x and y all have value 2 at $(1,1)$, since $x_u = y_u = 2u$ and $x_{uu} = y_{uu} = 2$.

6. Let $f(x,y,z) = xe^{xyz}$.

   (a) In what direction is $f(x,y,z)$ increasing the most rapidly near $(1,1,1)$? How quickly is it increasing in this direction?

   Solution. *The direction of most rapid increase is given by the gradient*

$$\begin{aligned}
\nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\
&= \langle (1+yz)e^{xyz}, xze^{xyz}, xye^{xyz} \rangle \\
&= \langle 2e, e, e \rangle.
\end{aligned}$$

   The rate of increase is the length of the gradient, $|\nabla f| = \sqrt{6}e$.

   (b) Find an equation for the tangent plane to $f(x,y,z) = e$ at the point $(1,1,1)$.

   Solution. *The gradient is perpendicular to the level surface. We may rescale and divide by e, giving the tangent plane*

$$2(x-1) + (y-1) + (z-1) = 0.$$

7. Let $f(x,y) = x^2 - 4xy + 2y^4$. Find the critical points of f and determine which are local minima, local maxima or saddle points.

Solution. *We set the partial derivatives of f equal to zero and solve.*

$$\begin{aligned} f_x &= 2x - 4y = 0 \\ f_y &= -4x + 8y^3 = 0 \end{aligned}$$

Thus $x = 2y$ and $-8y + 8y^3 = 0$ so $y = 0$ or $y = 1$ or $y = -1$. Thus the critical points are

$$(-2, -1), (0, 0), (2, 1).$$

To determine which are local mins, local maxes and saddle points, we use the second derivative test.

$$\begin{aligned} f_{xx} &= 2 \\ f_{yy} &= 24y^2 \\ f_{xy} &= -4 \\ D &= f_{xx} f_{yy} - f_{xy}^2 \\ &= 48y^2 - 16. \end{aligned}$$

Since $D < 0$ at $(0, 0)$, this is a saddle point. Since $D > 0$ and $f_{xx} > 0$ at $(-2, -1)$ and $(2, 1)$, these are both local minima.