Math 240 - Exam 2 Solutions

1. (10 points) Write down a double integral that gives the volume of the region that is bounded above by the surface $z = 1 - x^4 - y^4$ and below by the xy-plane. You **do not** need to evaluate the integral.

   Solution.
$$\int_{-1}^{1} \int_{-\sqrt[4]{1-x^4}}^{\sqrt[4]{1-x^4}} 1 - x^4 - y^4 \, dy \, dx.$$

2. (15 points) Find the area of the region contained within the curve $r = 1 - \cos(\theta)$ (this is called a *cardoid*).

   Solution.
$$\int_0^{2\pi} \int_0^{1-\cos(\theta)} r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} (1 - \cos(\theta))^2 \, d\theta = \frac{3\pi}{2}.$$

3. (20 points) A metal hubcap is modeled as the region bounded below by the plane $z = 1$ and above by the sphere $x^2 + y^2 + z^2 = 2$, with density given by the function $\frac{2}{\sqrt{x^2+y^2+z^2}}$. Find its mass.

   Solution. The intersection of these two surfaces occurs where $x^2 + y^2 = 1$ (ie $r = 1$) and $z = 1$, ie where $\phi = \pi/4$ in spherical coordinates. Thus the mass is given by
$$\begin{aligned} \int_0^{2\pi} \int_0^{\pi/4} \int_{1/\cos(\phi)}^{\sqrt{2}} \frac{2}{\rho} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta &= 2\pi \int_0^{\pi/4} \left(2 - \frac{1}{\cos^2(\phi)} \right) \sin(\phi) \, d\phi \\ &= 2\pi \left[-2\cos(\phi) - \frac{1}{\cos(\phi)} \right]_0^{\pi/4} \\ &= 2\pi(3 - 2\sqrt{2}) \end{aligned}$$

4. (15 points) Let R be the region outside the cylinder $x^2 + y^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 2$. Provide limits of integration in cylindrical and spherical coordinates.
$$\iiint_R f(x, y, z) \, dV = \int_?^? \int_?^? \int_?^? ? \quad d\rho \, d\phi \, d\theta = \int_?^? \int_?^? \int_?^? ? \quad dz \, dr \, d\theta$$

   Solution.
$$\begin{aligned} \iiint_R f(x, y, z) \, dV &= \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{1/\sin(\phi)}^{\sqrt{2}} f(\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))\rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} f(r\cos(\theta), r\sin(\theta), z) r \, dz \, dr \, d\theta \end{aligned}$$

5. (20 points) Let R be the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$ and $(1, -1)$. Using an appropriate change of variables, evaluate
$$\iint_R (x^2 - y^2) e^{x^2 - y^2} \, dA.$$

Solution. The square has edges with equations

$$\begin{aligned} x+y&=0\\ x+y&=2\\ x-y&=0\\ x-y&=2, \end{aligned}$$

suggesting that we use $u=x+y$ and $v=x-y$. Solving for x and y we have $x=\frac{u+v}{2}$ and $y=\frac{u-v}{2}$ and the Jacobian is $\left|\det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\right| = \frac{1}{2}$. So

$$\iint_R (x^2-y^2)e^{x^2-y^2}\, dA = \frac{1}{2}\int_0^2\int_0^2 uve^{uv}\, du\, dv.$$

This integral does not have a closed form expression using elementary functions. I apologize; everyone who got to this stage received full credit.

6. (20 points) Find the maximum and minimum values of

$$f(x,y,z)=xyz$$

subject to the constraint that

$$x^2+y^2+2z^2=3.$$

Solution. Using Lagrange multipliers we get the system

$$\begin{aligned} yz&=2\lambda x\\ xz&=2\lambda y\\ xy&=4\lambda z\\ x^2+y^2+2z^2&=3. \end{aligned}$$

Multiplying the first equation by x, the second by y and the third by z yields $x^2=y^2=2z^2$, and by the fourth equation they must all equal 1. Thus the solutions are $(\pm 1, \pm 1, \pm\frac{\sqrt{2}}{2})$. Multiplying these together, we see that the minimum value is $-\frac{\sqrt{2}}{2}$ and the maximum is $\frac{\sqrt{2}}{2}$.