Math 240 (7:30pm section) - Exam 1 Solutions

1. Let $\mathbf{a} = \langle 2, 6, 3\rangle$ and $\mathbf{b} = \langle 2, 1, 2\rangle$.

   (a) Find the scalar projection of $\mathbf{a}$ onto $\mathbf{b}$. (6 pts)

   Solution.
   $$\operatorname{comp}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \frac{2\cdot 2 + 6\cdot 1 + 3\cdot 2}{\sqrt{2^2+1^2+2^2}} = \frac{16}{3}.$$

   (b) Find the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$. (6 pts)

   Solution.
   $$\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{16}{3\sqrt{2^2+6^2+3^2}} = \frac{16}{21}.$$

2. Let $f(x, y) = x^2e^y - y^2e^x$.

   (a) Find an equation for the tangent plane to $z = f(x, y)$ at the point $(1, 1, 0)$. (6 pts)

   Solution. The tangent plane is determined by the partial derivatives of f, so we compute
   $$\begin{aligned} f_x(x,y) &= 2xe^y - y^2e^x \\ f_x(1,1) &= 2e - e = e \\ f_y(x,y) &= x^2e^y - 2ye^x \\ f_y(1,1) &= e - 2e = -e. \end{aligned}$$

   The equation of the tangent plan is then
   $$\begin{aligned} z - f(1,1) &= f_x(1,1)(x-1) + f_y(1,1)(y-1) \\ z &= ex - ey \end{aligned}$$

   (b) Find the distance from this tangent plane to the point $(2, 2, 2)$. (8 pts)

   Solution. The simple way to do this is to use the formula: the distance from a point (x_0, y_0, z_0) to a plane $ax + by + cz + d = 0$ is
   $$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}.$$

   This gives $\frac{|2e-2e-2|}{\sqrt{e^2+e^2+1}} = \frac{2}{\sqrt{2e^2+1}}$.
   Alternatively, you can construct a vector $\mathbf{v} = \langle 1, 1, 2\rangle$ from $(1, 1, 0)$ to $(2, 2, 2)$ (note that you can use any point on the plane instead of $(1, 1, 0)$, so $(0, 0, 0)$ would also work). The distance is then found by computing the absolute value of the scalar projection of $\mathbf{v}$ onto a normal vector $\mathbf{n} = \langle e, -e, -1\rangle$. Note the z component here: you have to put the equation of the plane into the form $ax + by + cz + d = 0$ in order to extract the normal vector.
   The scalar projection is
   $$\operatorname{comp}_{\mathbf{n}}(\mathbf{v}) = \frac{|\mathbf{n}\cdot\mathbf{v}|}{|\mathbf{n}|} = \frac{2}{\sqrt{e^2+1}}.$$

3. Consider the curve parameterized by $\mathbf{r}(t) = \langle t^3 + t^2, t^3 - t^2, t^2 \rangle$. Find the arclength along this curve between $(0, 0, 0)$ and $(2, 0, 1)$. (15 pts)

Solution. Arclength is given by $L = \int_a^b |\mathbf{r}'(t)| dt$, where $\mathbf{r}(a) = (0, 0, 0)$ and $\mathbf{r}(b) = (2, 0, 1)$. We find that $a = 0$ and $b = 1$ and

$$\mathbf{r}'(t) = \langle 3t^2 + 2t, 3t^2 - 2t, 2t \rangle.$$

So, using the substitution $u = 3t^2 + 2$, we have

$$\begin{aligned}
L &= \int_0^1 \sqrt{(3t^2 + 2t)^2 + (3t^2 - 2t)^2 + (2t)^2}\, dt \\
&= \int_0^1 \sqrt{9t^4 + 12t^3 + 4t^2 + 9t^4 - 12t^3 + 4t^2 + 4t^2}\, dt \\
&= \sqrt{6} \int_0^1 t\sqrt{3t^2 + 2}\, dt \\
&= \frac{\sqrt{6}}{6} \int_2^5 \sqrt{u}\, du \\
&= \frac{(5\sqrt{5} - 2\sqrt{2})\sqrt{6}}{9}
\end{aligned}$$

4. Suppose that C is a curve that passes through the origin, where it has curvature $1/2$, unit tangent and unit normal vectors

$$\begin{aligned}
\mathbf{T} &= \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right\rangle, \\
\mathbf{N} &= \langle 0, 0, 1 \rangle.
\end{aligned}$$

(a) Find the binormal vector $\mathbf{B}$ at the origin. (6 pts)

Solution. We have

$$\begin{aligned}
\mathbf{B} &= \mathbf{T} \times \mathbf{N} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \rangle.
\end{aligned}$$

(b) Find the center of the osculating circle to C at the origin. (6 pts)

Solution. The osculating circle is contained within the plane spanned by $\mathbf{T}$ and $\mathbf{N}$, and has the same tangent vector as the curve. Its center is thus in the direction of $\mathbf{N}$. Since the radius is the inverse of the curvature, it has radius 2 and thus center at $(0, 0, 2)$.

5. Suppose $f(u, v)$ is a differentiable function, $z(x, y) = f(xy, x^2 + y^2)$ and that $f_u(2, 5) = 3$. If the level curve for $z(x, y)$ through $(1, 2)$ is horizontal, find $f_v(2, 5)$. Justify your answer. (15 pts)

Solution. If the level curve through $(1, 2)$ is horizontal, that means that the gradient vector points up or down in the xy-plane and thus $z_x(1, 2) = 0$. Alternatively, since the level curve is horizontal, the value of $z(x, y)$ does not change with a change in the x-coordinate, so again we see that $z_x(1, 2) = 0$.

We now use the chain rule to find $z_x(1,2)$. Setting $u = xy$ and $v = x^2 + y^2$ and noting that $u_x = y$ and $v_x = 2x$, we have

$$\begin{aligned} z_x(1,2) &= f_u(2,5)u_x(1,2) + f_v(2,5)v_x(1,2) \\ 0 &= 3 \cdot 2 + f_v(2,5) \cdot 2. \end{aligned}$$

Solving for $f_v(2,5)$ yields $f_v(2,5) = -3$.

6. Suppose that $f(x,y,z)$ is differentiable and that an equation for the tangent plane to the surface $f(x,y,z) = 0$ at the point $(0,1,0)$ is given by

$$x + y - 2z = 1.$$

Suppose further that $D_{\mathbf{u}} f(0,1,0) = 4\sqrt{2}$ when $\mathbf{u} = \langle \sqrt{2}/2, \sqrt{2}/2, 0 \rangle$.

(a) Find ∇f at $(0,1,0)$. (8 pts)

Solution. The gradient vector is perpendicular to the level surface and thus parallel to the normal vector of the tangent plane. Thus

$$\nabla f = \langle 1, 1, -2 \rangle t$$

for some scalar t. We can determine t by the condition that $D_{\mathbf{u}} f(0,1,0) = 4\sqrt{2}$. We compute

$$\begin{aligned} D_{\mathbf{u}} f(0,1,0) &= \mathbf{u} \cdot \nabla f \\ &= \langle \sqrt{2}/2, \sqrt{2}/2, 0 \rangle \cdot \langle 1, 1, -2 \rangle t \\ &= \sqrt{2} t, \end{aligned}$$

so $t = 4$ and $\nabla f = \langle 4, 4, -8 \rangle$.

(b) If $\mathbf{v}$ points in the direction of $\langle 0, 1, 1 \rangle$, find $D_{\mathbf{v}} f(0,1,0)$. (6 pts)

Solution. With ∇f in hand, we again use the formula for the directional derivative. Note that we must scale $\mathbf{v}$ so that it is a unit vector, ie $\mathbf{v} = \langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$. Then

$$\begin{aligned} D_{\mathbf{v}} f(0,1,0) &= \langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle \cdot \langle 4, 4, -8 \rangle \\ &= 2\sqrt{2} - 4\sqrt{2} \\ &= -2\sqrt{2}. \end{aligned}$$

7. Let $f(x,y) = x^3 + y^3 - 3y^2 - 3xy + 3x + 3y$. Find the critical points of f and determine which are local minima, local maxima, or saddle points. (18 pts)

Solution. We compute the partial derivatives of f and set them equal to 0.

$$\frac{\partial f}{\partial x} = 3x^2 - 3y + 3 = 0 \tag{1}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 6y - 3x + 3 = 0. \tag{2}$$

Solving (2) for x gives

$$x = y^2 - 2y + 1. \tag{3}$$

Substituting (3) into (1) yields

$$\begin{aligned} 3(y^2 - 2y + 1) - 3y + 3 &= 0 \\ 3(y-1)(y-2) &= 0 \end{aligned}$$

Thus $y = 1$ or $y = 2$. If $y = 1$, substituting into (3) gives $x = 1^2 - 2 \cdot 1 + 1 = 0$. If $y = 2$, substituting into (3) gives $x = 2^2 - 2 \cdot 2 + 1 = 1$. Thus the two critical points are $(0, 1)$ and $(1, 2)$. Note that you could also solve in the reverse order, solving (1) for y and substituting into (2).

We now need to determine which are local minima, local maxima or saddle points. To do so, we use the second derivative test. We compute

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= 6x \\
\frac{\partial^2 f}{\partial x \partial y} &= -3 \\
\frac{\partial^2 f}{\partial y^2} &= 6y - 6 \\
D &= \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\
&= 36x(y-1) - 9.
\end{aligned}$$

Evaluating D at $(0, 1)$ gives $-9 < 0$ so $(0, 1)$ is a saddle point. Evaluating D at $(1, 2)$ gives $27 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 6 > 0$, so $(1, 2)$ is a local minimum.