Math 220 - Practice Final (Spring 2005) Solutions

3. (a) We find the derivative,
\[ f'(x) = (x - 1)^2 + 2x(x - 1) \]
\[ = (x - 1)(3x - 1). \]

The critical points are at \( x = 1 \) and \( x = 1/3 \). Since \( f'(x) < 0 \) for \( 1/3 < x < 1 \) and \( f'(x) > 0 \) elsewhere, \( x = 1/3 \) is a maximum and \( x = 1 \) is a minimum (by the first derivative test).

(b) The second derivative is
\[ f''(x) = (3x - 1) + 3(x - 1) \]
\[ = 6x - 4. \]

There is one inflection point at \( x = 2/3 \), where \( f''(x) \) changes sign.

4. The equation to obtain the next approximation in Newton’s method is
\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]

Applying this to the function \( f(x) = x^4 - 10100 \) with \( x_1 = -10 \) gives
\[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \]
\[ = -10 - \frac{10000 - 10100}{4 \cdot (-1000)} \]
\[ = -10 + \frac{1}{40} \]
\[ = -10.025. \]

5. (a) The slope is given by \( y' = \frac{1}{4} x^{-3/4} = \frac{1}{4000} \), and the equation by \( y - 10 = \frac{1}{4000}(x - 10000). \)
(b) We find $y$ on the tangent line when $x = 1000$:

$$y = 10 + \frac{1}{4000}(100)$$

$$= 10.025.$$

1. (a) This has indeterminate form $0/0$, so L’Hospital’s rule applies. Differentiating top and bottom, we get $\frac{\frac{1}{x^2}}{2\sqrt{x+4}} = 2\sqrt{x+4}$. Evaluating at $x = 0$ yields 4, so

$$\lim_{x \to 0} \frac{x}{\sqrt{x+4} - 2} = 4.$$

(b) When $x < 3$, $|x - 3| = 3 - x$, so

$$\lim_{x \to 3^-} \frac{|x - 3|}{x - 3} = \lim_{x \to 3^-} \frac{3 - x}{x - 3} = -1.$$

(c) Near $x = 1$, $-\pi/2 < 2x - 5 < \pi/2$ so arctan(tan(2x - 5)) = 2x - 5. Therefore

$$\lim_{x \to 1} \frac{\arctan(tan(2x - 5))}{2x - 5} = \lim_{x \to 1} \frac{2x - 5}{2x - 5} = 1.$$

(d) As $x \to -\infty$, numerator and denominator both tend to 0, so L’Hospital’s rule applies. Differentiating numerator and denominator yields

$$\frac{1/(1 + \frac{3}{x^2}) \cdot \frac{-9}{x^2}}{\cos\left(\frac{4}{x}\right) \cdot \frac{-8}{x^3}} = \frac{9}{8x (1 + \frac{3}{x^2}) \cos\left(\frac{4}{x}\right)}.$$  

As $x \to -\infty$, this ratio tends to 0. Therefore

$$\lim_{x \to -\infty} \frac{\ln(1 + \frac{3}{x^2})}{\sin\left(\frac{4}{x}\right)} = 0.$$

(e) This limit is of the form $0 \cdot (-\infty)$, so we rewrite it as $\lim_{x \to 0} \frac{\ln(x^2)}{x^2}$. L’Hospital’s rule now applies, and we differentiate numerator and denominator, yielding

$$\frac{\frac{2x}{x^2}}{-\frac{2}{x^3}} = -x^2.$$  

Therefore

$$\lim_{x \to 0} x^2 \ln(x^2) = \lim_{x \to 0} -x^2$$

$$= 0.$$

7. (a) Using the substitution $u = 5x/2$,

$$\int \frac{dx}{4 + 25x^2} = \frac{1}{4} \int \frac{dx}{1 + (5x/2)^2}$$

$$= \frac{1}{10} \int \frac{du}{1 + u^2}$$

$$= \frac{1}{10} \tan^{-1}(5x/2) + C.$$
(b) \[ \int \left( 12^x + x^{1/2} \right) \, dx = \frac{12^x}{\ln(12)} + \frac{2}{3} x^{3/2} + C. \]

(c) Let \( g(u) = \int_0^u \frac{du}{\sqrt{1+u^2}} \) and \( u = 2x \). Then \( f(x) = g(u(x)) \) and \( \frac{df}{dx} = \frac{dg}{du} \frac{du}{dx} = \frac{1}{\sqrt{1+u^2}} \cdot 2. \)

8. (a) Differentiating, we have \[ 2yy' + e^y + 2xyy'e^{y^2} = 0. \] Substituting \( (x, y) = (0, 1) \) and solving gives \( 2y' + e = 0 \), so \( \frac{dy}{dx} = \frac{e}{2}. \)

(b) Differentiating, we have
\[
\begin{align*}
y' &= \frac{6 \sin(x) \cos(x)}{1 + 9 \sin^4(x)} \\
y'\left(\pi/4\right) &= \frac{6 \sin(\pi/4) \cos(\pi/4)}{1 + 9 \sin^4(\pi/4)} \\
&= \frac{6/2}{1 + 9/4} \\
&= \frac{12}{13}.
\end{align*}
\]

(c) Writing \( y = e^{2x \ln(x)} \) and differentiating, we get \[ \frac{dy}{dx} = 2(\ln(x) + 1)x^{2x}. \]

You can also use logarithmic differentiation.

9. Implicitly differentiating with respect to time,
\[ 4xx' - x'y - xy' + 6yy' = 0. \]

Substituting \( x = -3, y = 1 \) and \( x' = 5 \), we solve for \( y' \):
\[
\begin{align*}
4(-3)(5) - (5)(1) - (-3)y' + 6(1)y' &= 0 \\
9y' &= 65 \\
y' &= \frac{65}{9}.
\end{align*}
\]