Math 220 - Practice Exam 1

1. Give the limit if it exists

   (a) $\lim_{x\to 2-} \frac{|x^2-2x|}{x-2}$

   Solution. For $0 < x < 2$, $x^2 - 2x < 0$, so $\frac{|x^2-2x|}{x-2} = \frac{2x-x^2}{x-2} = \frac{x(2-x)}{x-2} = -x$. Therefore

   $$\lim_{x\to 2-} \frac{|x^2-2x|}{x-2} = \lim_{x\to 2-}(-x) = -2.$$

   (b) $\lim_{x\to 2+} \frac{|x^2-2x|}{x-2}$

   Solution. For $x > 2$, $x^2 - 2x > 0$, so $\frac{|x^2-2x|}{x-2} = \frac{x^2-2x}{x-2} = \frac{x(x-2)}{x-2} = x$. Therefore

   $$\lim_{x\to 2-} \frac{|x^2-2x|}{x-2} = \lim_{x\to 2-} x = 2.$$

   (c) $\lim_{x\to 2} \frac{\sqrt{x+2}-2}{x-2}$

   Solution. Multiply numerator and denominator by the conjugate, then cancel terms.

   $$\begin{aligned}
   \lim_{x\to 2} \frac{\sqrt{x+2}-2}{x-2} &= \lim_{x\to 2} \frac{(\sqrt{x+2}-2)(\sqrt{x+2}+2)}{(x-2)(\sqrt{x+2}+2} \\
   &= \lim_{x\to 2} \frac{x+2-4}{(x-2)(\sqrt{x+2}+2} \\
   &= \lim_{x\to 2} \frac{1}{\sqrt{x+2}+2} \\
   &= \frac{1}{\sqrt{2+2}+2} \\
   &= \frac{1}{4}
   \end{aligned}$$

2. Determine the derivatives of the following functions:

   (a) $f(x) = x^4 - 2\sqrt{x} + 1 - \frac{2}{x^3} + \pi x^\pi$,

   Solution.

   $$f'(x) = 4x^3 - \frac{1}{\sqrt{x}} + \frac{6}{x^4} + \pi^2 x^{\pi-1}.$$

   (b) $f(x) = (x^2 - 4x + 5)^6 (2x^3 - 1)^3$,

   Solution.

   $$f'(x) = 6(x^2-4x+5)^5(2x-4)(2x^3-1)^3 + 3(x^2-4x+5)^6(2x^3-1)^2(6x^2)$$

   (c) $g(x) = \frac{3}{(x-\sin(x))^2}$,

Solution.

$$g'(x) = \frac{-6}{(x - \sin(x))^3} \cdot (1 - \cos(x))$$

(d) $f(x) = \frac{\sin(x)\tan(3x)}{1+\cos^2(x)}$,

Solution.

$$f'(x) = \frac{(\cos(x)\tan(3x) + 3\sin(x)\sec^2(3x))(1+\cos^2(x)) + 2\sin^2(x)\tan(3x)\cos(x)}{(1+\cos^2(x))^2}$$

(e) $h(x) = \sin^3(x^3)$.

Solution.

$$h'(x) = 9x^2\sin^2(x^3)\cos(x^3)$$

3. Evaluate the limit

$$\lim_{h\to 0} \frac{\tan(\pi/4 + h) - 1}{h}.$$

Solution. We recognize this limit as the derivative of the function $f(x) = \tan(x)$ at $x = \pi/4$. We know that $f'(x) = \sec^2(x)$, so

$$\lim_{h\to 0} \frac{\tan(\pi/4 + h) - 1}{h} = \sec^2(\pi/4) = \frac{1}{\cos^2(\pi/4)} = \frac{1}{(\sqrt{2}/2)^2} = 2.$$

4. If $f(x) = x^3 - 2x + 1$,

(a) What is the differential of $f(x)$ at $x = 2$?

Solution.

$$df = f'(x)dx = (3x^2 - 2)dx.$$

(b) $f(2) = 5$. Use the differential to approximate $f(1.9)$.

Solution. Note that $dx = 1.9 - 2 = -0.1$ and $df = (3 \cdot 2^2 - 2) \cdot dx = 10 \cdot (-0.1) = -1$. Therefore

$$f(1.9) \approx f(2) + df = 5 - 1 = 4.$$

5. Use implicit differentiation to determine the equation of the tangent line to the curve

$$x^3y^2 - x^2y^3 + 2x^2 - y^2 = 1$$

at the point $(1, 1)$.

Solution. Differentiating both sides,

$$3x^2y^2 + 2x^3yy' - 2xy^3 - 3x^2y^2y' + 4x - 2yy' = 0,$$

and solving yields

$$y' = \frac{-3x^2y^2 + 2xy^3 - 4x}{2x^3y - 3x^2y^2 - 2y}.$$

Evaluating at $(1, 1)$ gives the derivative:

$$y' = \frac{-3 + 2 - 4}{2 - 3 - 2} = \frac{5}{3}$$

Thus the equation of the tangent line is

$$y - 1 = \frac{5}{3}(x - 1).$$

6. A $10m$ ladder leans against a wall. As the base of the ladder is pulled away at a rate of $0.6m/s$, the ladder slides down the wall. At what rate is the height of the top of the ladder changing when the base of the ladder is $6m$ from the wall? (Give an exact numerical answer)

   Solution. If x is the distance that the base has been pulled from the wall, the height of the top of the ladder is $H = \sqrt{100 - x^2}$. Therefore

$$\frac{dH}{dt} = \frac{-x}{\sqrt{100 - x^2}} \frac{dx}{dt}.$$

   Evaluating when $x = 6$ and $\frac{dx}{dt} = 0.6$ gives

$$\frac{dH}{dt} = \frac{-6}{\sqrt{100 - 36}} \cdot (0.6) = \frac{-6}{8} \cdot \frac{6}{10} = \frac{-9}{20}.$$

   So the height of the top of the ladder is changing at a rate of $-\frac{9}{20} = -0.45m/s$.

7. Determine where the function $f(x) = x\sqrt{8 - x^2}$ has a horizontal tangent line.

   Solution. Differentiate

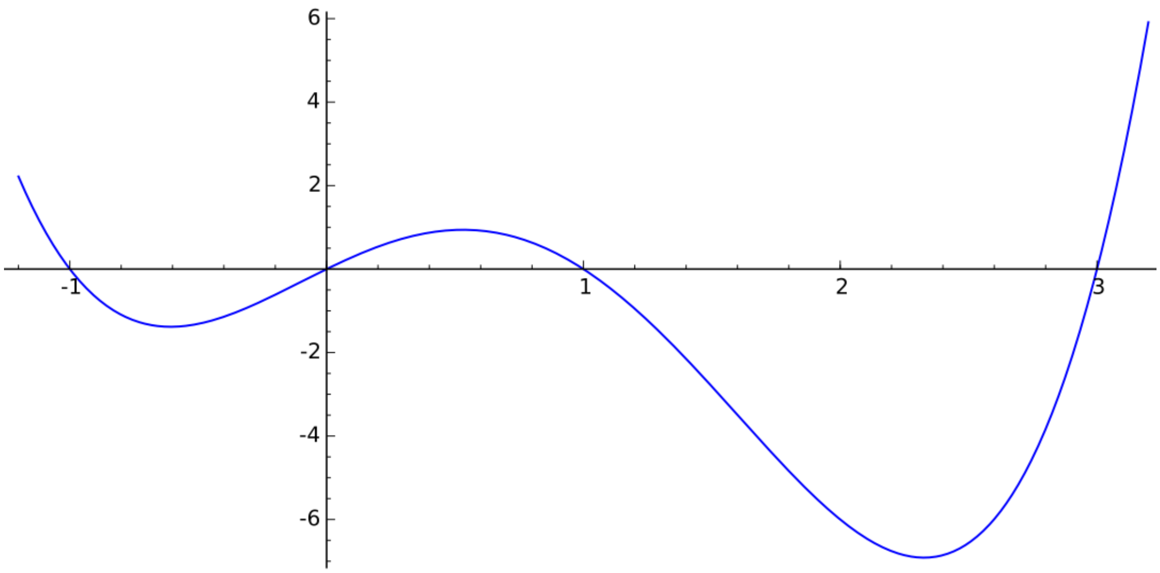
$$f'(x) = \sqrt{8 - x^2} - x\frac{x}{\sqrt{8 - x^2}},$$

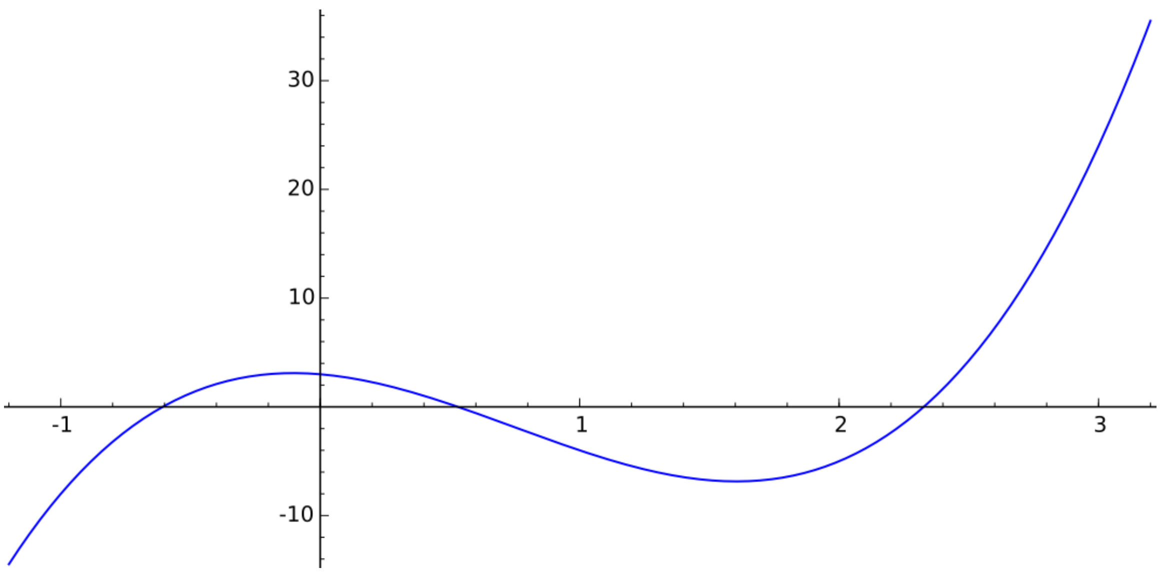
   set equal to zero and solve for x:

$$\begin{aligned} 0 &= \sqrt{8 - x^2} - \frac{x^2}{\sqrt{8 - x^2}} \\ 0 &= (8 - x^2) - x^2 \\ 2x^2 &= 8 \\ x &= \pm 2 \end{aligned}$$

   So $f(x)$ has horizontal tangent lines at $x = 2$ and $x = -2$.

8. Sketch the graph of the derivative of the given function $f(x)$:





I would be looking to see that you have the approximate shape correct and that you have the sign of y as a function of x correct (including the locations of the zeros of $f'(x)$). I would not look to check that you had the correct vertical scale (in fact, you might notice that the y scale on this solution graph is different from the blank graph provided).

9. The chart below gives the values of $f(x)$ and $g(x)$ and their derivatives at $x = 2$, $x = 3$ and $x = 4$.

	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	4	-2	3	9
3	0	29	-1	4
4	3	2	7	11

Use it to find the following derivatives,

(a) If $h(x) = f(g(x))$, determine $h'(4)$.

Solution. By the chain rule,

$$\begin{aligned} h'(4) &= f'(g(4)) \cdot g'(4) \\ &= f'(2) \cdot g'(4) \\ &= 3 \cdot 11 \\ &= 33. \end{aligned}$$

(b) If $k(x) = g(f(x))$, determine $k'(4)$.

Solution. By the chain rule,

$$\begin{aligned} k'(4) &= g'(f(4)) \cdot f'(4) \\ &= g'(3) \cdot f'(4) \\ &= 4 \cdot 7 \\ &= 28. \end{aligned}$$