Math 220 - Practice Final (Spring 2008) Solutions

1. We write $y = \frac{x+1}{2x+1}$ and solve for x:

$$\begin{aligned}(2x+1)y &= x+1\\ 2xy - x + y - 1 &= 0\\ (2y-1)x &= 1-y\\ x &= \frac{1-y}{2y-1}.\end{aligned}$$

So $f^{-1}(x) = \frac{1-x}{2x-1}$.

3. The derivative at $x = 0$ is

$$\begin{aligned}\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} &= \frac{|h|\sqrt[3]{h} - |0|\sqrt[3]{0}}{h}\\ &= \lim_{h\to 0}\frac{|h|}{h}\sqrt[3]{h}.\end{aligned}$$

When $h > 0$, $\frac{|h|}{h} = 1$ and $\sqrt[3]{h}$ tends to 0 as $h \to 0$. When $h < 0$, $\frac{|h|}{h} = -1$ and $-\sqrt[3]{h}$ also tends to 0 as $h \to 0$. Therefore the limit exists, and $f(x)$ is differentiable at 0 with $f'(0) = 0$.

4. (a) $f'(x) = 20\cos(5x) + \frac{2}{3\sqrt[3]{x^5}} + \frac{1}{x}$.

   (b) $g'(x) = \frac{2e^{2x}(1+x^2) - 2xe^{2x}}{(1+x^2)^2} = \frac{2(1-x+x^2)e^{2x}}{(1+x^2)^2}$.

   (c) $y' = 3x^2\tan^{-1}(4x) + \frac{4x^3}{1+16x^2}$.

   (d) $y = e^{\ln(x+1)x}$ so $y' = (\frac{x}{x+1} + \ln(x+1))(x+1)^x$. You can also use logarithmic differentiation.

   (e) $f'(x) = \frac{\sin(x)}{x^3+2}$.

5. We use implicit differentiation to get

$$2x + 4y + 4xy' + 3y^2y' = 0.$$

Solving for y' and substituting $x = 2$ and $y = 1$,

$$\begin{aligned}y'(4x+3y^2) &= -2x-4y\\ y'(8+3) &= -4-4\\ y' &= -\frac{8}{11}.\end{aligned}$$

Using the point-slope equation of the line passing through $(2,1)$ with slope $-\frac{8}{11}$, we get

$$y - 1 = -\frac{8}{11}(x-2).$$

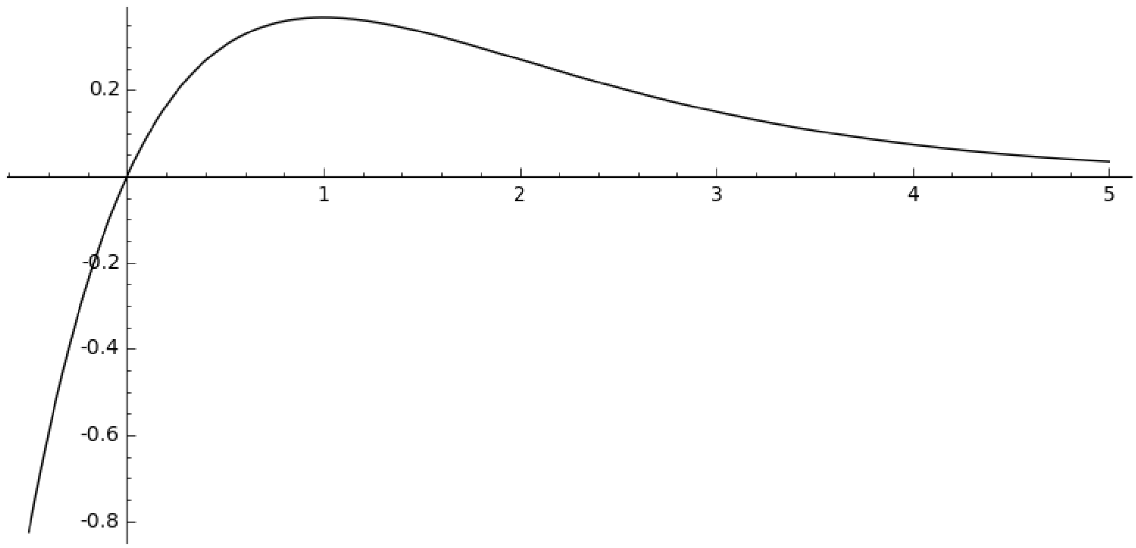
6. (a) As $x \to \infty$, e^{-x} goes to 0 faster than x goes to infinity, so $\lim_{x\to\infty} f(x) = 0$. There are no other vertical or horizontal asymptotes.

   (b) $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, which is negative when $x > 1$ and positive when $x < 1$. So $f(x)$ is increasing for $x < 1$ and decreasing for $x > 1$.

   (c) There is a critical point at $x = 1$, which is a local maximum by the first derivative test.

   (d) Differentiating again, $f''(x) = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$. So $f(x)$ is concave up for $x > 2$ and concave down for $x < 2$, with an inflection point at $x = 2$.

   (e) Using the points $(0, 0)$ and $(1, 1/e)$, we get the following graph:



7. We have $A = LW$. Differentiating with respect to time,

$$A' = L'W + W'L = 8 \cdot 10 + 3 \cdot 20 = 140.$$

   So the area is increasing at a rate of 140 square centimeters per second.

8. Differentiating,

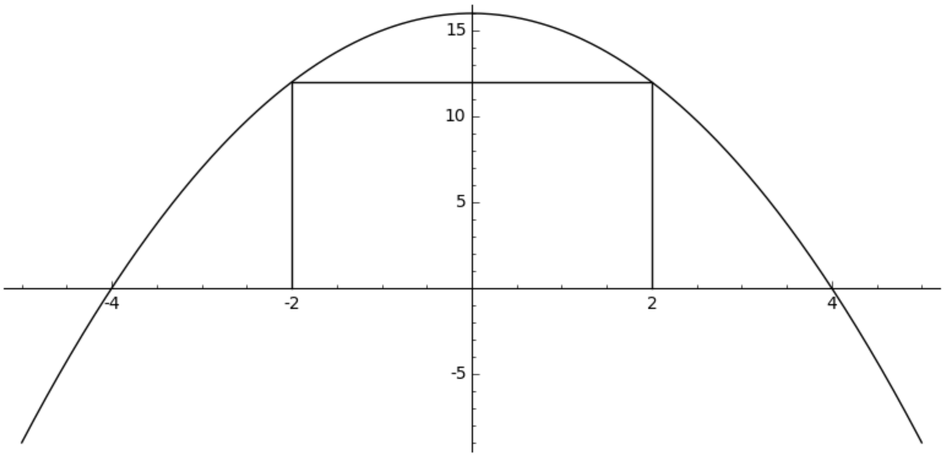
$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$$

   We compute the value of $f(x)$ at the three critical points $0, 1$ and -1, and at the endpoints of the interval, -2 and 3.

$$\begin{aligned} f(-2) &= 16 - 8 + 3 = 11 \\ f(-1) &= 1 - 2 + 3 = 2 \\ f(0) &= 0 - 0 + 3 = 3 \\ f(1) &= 1 - 2 + 3 = 2 \\ f(3) &= 81 - 18 + 3 = 66. \end{aligned}$$

   So the absolute minimum is 2 and the absolute maximum is 66.

9. The picture is



So the area is

$$A = 2xy,$$

and the fact that (x, y) lies on the parabola implies that $y = 16 - x^2$. Thus

$$A = 2x(16 - x^2).$$

Differentiating, we have

$$0 = 32 - 6x^2,$$

so $x = \frac{4}{\sqrt{3}}$ and $y = 16 - x^2 = \frac{32}{3}$. So the largest rectangle has width $2x = \frac{8}{\sqrt{3}}$ and height $\frac{32}{3}$.

10. Draw the tangent line to the curve above x_1, and x_2 will be the intersection of that tangent line with the x-axis. Repeat to get x_3 and x_4.

11. The linear approximation near $a = 1$ is

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= \sqrt[9]{a} + \frac{1}{9}a^{-8/9}(x - a) \\ &= 1 + \frac{1}{9}(x - 1) \end{aligned}$$

Therefore

$$\begin{aligned} \sqrt[9]{1.1} &\approx L(1.1) \\ &= 1 + 0.1/9 \\ &\approx 1.011111 \end{aligned}$$

12. (a)

$$\begin{aligned} \lim_{x \to 2^-} \frac{|x - 2|}{x^2 - 4} &= \lim_{x \to 2^-} \frac{2 - x}{x^2 - 4} \\ &= \lim_{x \to 2^-} \frac{-1}{x + 2} \\ &= -\frac{1}{4} \end{aligned}$$

(b)

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2}$$
$$= \frac{3}{5}$$

(c)

$$\lim_{x\to 0} \frac{\sin(6x)}{\ln(x+1)} = \lim_{x\to 0} \frac{6\cos(6x)}{1/(x+1)}$$
$$= 6$$

by L'Hospital's rule. Note that we need to check that $\frac{\sin(6\cdot 0)}{\ln(0+1)} = \frac{0}{0}$, so this is an indeterminate form where L'Hospital's rule applies.

(d) Since

$$\frac{1}{x} - \frac{1}{\sin(x)} = \frac{\sin(x) - x}{x\sin(x)}$$

and both numerator and denominator evaluate to 0 when $x = 0$, we may use L'Hospital's rule. Differentiating top and bottom, we get

$$\frac{\cos(x) - 1}{\sin(x) + x\cos(x)}.$$

Again, both numerator and denominator evaluate to 0 so we apply L'Hospital's rule again.

$$\frac{-\sin(x)}{2\cos(x) - x\sin(x)}$$

evaluates to $\frac{0}{2}$, so

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin(x)}\right) = 0.$$

(e) This limit is of indeterminate form 1^∞, so we need to take the logarithm and apply L'Hospital's rule. The natural log is

$$\frac{\ln(1+3x)}{x},$$

and differentiating numerator and denominator yields

$$\frac{3/(1+3x)}{1}.$$

Evaluating at $x = 0$ gives 3, so the original limit is e^3.

13. We integrate to find that $f'(t) = 2e^t - 3\cos(t) + C$, and integrate again to get

$$f(t) = 2e^t - 3\sin(t) + Ct + D$$

Evaluating at 0 we get $0 = f(0) = 2 + D$, so $D = -2$. Evaluating at π we get $0 = f(\pi) = 2e^\pi + C\pi - 2$, so $C = \frac{2-e^\pi}{\pi}$. Thus

$$f(t) = 2e^t - 3\sin(t) + \frac{2 - e^\pi}{\pi}t - 2.$$

14. There are four intervals, with midpoints at $1.5, 2.5, 3.5$ and 4.5. The relevant Riemann sum is

$$\frac{1}{(1.5)^3+1}+\frac{1}{(2.5)^3+1}+\frac{1}{(3.5)^3+1}+\frac{1}{(4.5)^3+1}.$$

15. (a)

$$\begin{aligned}\int_1^9 \frac{3x-1}{\sqrt{x}}\,dx &= \int_1^9 3x^{1/2}-x^{-1/2}\,dx\\ &= [2x^{3/2}-2x^{1/2}]_1^9\\ &= (2\cdot 27-2\cdot 3)-(2-2)\\ &= 48.\end{aligned}$$

(b) Using substitution with $u=4+t^2$,

$$\begin{aligned}\int_0^2 t\sqrt{4+t^2}\,dt &= \frac{1}{2}\int_4^8 \sqrt{u}\,du\\ &= [\frac{1}{3}u^{3/2}]_4^8\\ &= \frac{16\sqrt{2}-8}{3}\end{aligned}$$

(c) This is the area of a semicircle of radius 2, which is 2π.

16. (a) Using substitution with $u=\ln(x)$ and $du=\frac{dx}{x}$,

$$\begin{aligned}\int \frac{\ln(x)}{x}\,dx &= \int u\,du\\ &= \frac{u^2}{2}+C\\ &= \frac{\ln(x)^2}{2}+C.\end{aligned}$$

(b) Using integration by parts with $u=\ln(x)$ and $dv=x\,dx$,

$$\begin{aligned}\int x\ln(x)\,dx &= \frac{1}{2}x^2\ln(x)-\frac{1}{2}\int x\,dx\\ &= \frac{1}{2}x^2\ln(x)-\frac{1}{4}x^2+C.\end{aligned}$$

(c) Using the identity $\sin^3(x)=\sin(x)(1-\cos^2(x))$ and the substitution $u=\cos(x)$,

$$\begin{aligned}\int \sin^3(x)\,dx &= \int \sin(x)-\cos^2(x)\sin(x)\,dx\\ &= -\cos(x)+\int u^2\,du\\ &= -\cos(x)+\frac{1}{3}\cos^3(x)+C.\end{aligned}$$