Math 220 Final Exam (part 2) Solutions

1. Evaluate the following integrals. (5 points each)

   (a)
   $$\int 25x^4 \ln(x)\, dx$$

   Solution. We use integration by parts, with $u = \ln(x)$ and $dv = 25x^4 dx$. Then $du = \frac{1}{x} dx$ and $v = 5x^5$. We then have

   $$\begin{aligned} \int 25x^4 \ln(x)\, dx &= uv - \int v\, du \\ &= 5x^5 \ln(x) - \int 5x^5 \cdot \frac{1}{x}\, dx \\ &= 5x^5 \ln(x) - \int 5x^4\, dx \\ &= 5x^5 \ln(x) - x^5 + C. \end{aligned}$$

   (b)
   $$\int_0^\pi 15 \sin^2(x) \cos^3(x)\, dx$$

   Solution. We use $\cos^2(x) = 1 - \sin^2(x)$ and $u = \sin(x)$. Then $du = \cos(x)dx$. So

   $$\begin{aligned} \int_0^\pi 15 \sin^2(x) \cos^3(x)\, dx &= \int_0^\pi 15 \sin^2(x)(1 - \sin^2(x)) \cos(x)\, dx \\ &= \left[\int 15(u^2 - u^4)\, du \right]_{x=0}^{x=\pi} \\ &= \left[5u^3 - 3u^5 \right]_{x=0}^{x=\pi} \\ &= \left[5\sin^3(x) - 3\sin^5(x) \right]_{x=0}^{x=\pi} \\ &= 0. \end{aligned}$$

(c)

$$\int \frac{4x}{1+4x^4}\,dx$$

Solution. We use the substitution $u = 2x^2$. Then $du = 4xdx$, and

$$\begin{aligned}
\int \frac{4x}{1+4x^4}\,dx &= \int \frac{du}{1+u^2} \\
&= \tan^{-1}(u) + C \\
&= \tan^{-1}(2x^2) + C.
\end{aligned}$$

(d)

$$\int_0^1 x\sinh(x)\,dx$$

Solution. We use integration by parts with $u = x$ and $dv = \sinh(x)dx$. Then $du = dx$ and $v = \cosh(x)$. We then have

$$\begin{aligned}
\int_0^1 x\sinh(x)\,dx &= [uv]_0^1 - \int_0^1 v\,du \\
&= [x\cosh(x)]_0^1 - \int_0^1 \cosh(x)\,dx \\
&= \cosh(1) - [\sinh(1)]_0^1 \\
&= \cosh(1) - \sinh(1) \\
&= e^{-1}.
\end{aligned}$$

2. Suppose that a car's velocity is given by $v(t) = 60 + t\sin(t)$. Find its average velocity during the interval between $t = 0$ and $t = 10$. (10 points)

Solution. The average velocity is

$$\frac{1}{10-0}\int_0^{10} 60 + t\sin(t)\,dt = 60 + \frac{1}{10}\int_0^{10} t\sin(t)\,dt.$$

Using integration by parts with $u = t$ and $v = \sin(t)\,dt$, we get

$$\begin{aligned}
&= 60 + \frac{1}{10}\left([-t\cos(t)]_0^{10} + \int_0^{10} \cos(t)\,dt\right) \\
&= 60 + \frac{1}{10}\left(-10\cos(10) + [\sin(t)]_0^{10}\right) \\
&= 60 - \cos(10) + \frac{\sin(10)}{10}.
\end{aligned}$$

3. Consider a window formed from two panes of glass: a semicircular pane above a rectangular pane. The two panes are touching, and the diameter of the semicircle is the same as one edge of the rectangle. If the total perimeter is required to be 10 feet, what is the radius r of the semicircle that maximizes the area? (12 points)

Solution. Let h be the height of the rectangle; it has width $2r$ by the problem statement. Then the perimeter is

$$10 = P = 2r + 2h + \pi r,$$

so

$$h = \frac{10 - 2r - \pi r}{2}.$$

The total area is

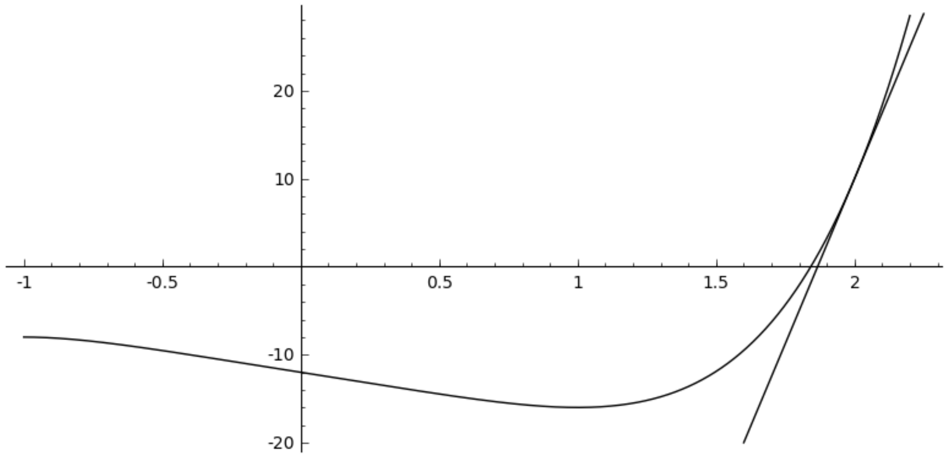
$$\begin{aligned} A &= 2rh + \frac{1}{2}\pi r^2 \\ &= r(10 - 2r - \pi r) + \frac{1}{2}\pi r^2 \\ &= 10r - 2r^2 - \frac{1}{2}\pi r^2 \end{aligned}$$

Differentiating with respect to r and setting $A' = 0$, we get

$$\begin{aligned} A' &= 10 - 4r - \pi r \\ 0 &= 10 - (4 + \pi)r \\ r &= \frac{10}{4 + \pi}. \end{aligned}$$

So the maximum area occurs when the semicircle has radius $\frac{10}{4+\pi}$ feet.

4. Consider the function $f(x) = x^5 - 5x - 12$, whose graph is shown below.



(a) If you want to approximate the smallest positive root of $f(x)$ using Newton's method, what is a good initial estimate x_0? Note that there may be multiple acceptable answers. (1 point)

Solution. 2 is a good choice because it is close to the root and the tangent line there will intersect the x-axis even closer to the root. Other answers between 1.5 and 2.2 will also work.

(b) On the graph above, draw the process used to find x_1, starting from your choice of x_0 in part (a). (3 points)

Solution. We draw the tangent line at $x = 2$. The intersection of this tangent line with the x-axis gives x_1.

(c) Find x_1 numerically (you do not need to simplify). (4 points)

Solution. We compute

$$\begin{aligned} f(2) &= 32 - 10 - 12 = 10 \\ f'(2) &= 5 \cdot 16 - 5 = 75 \\ x_1 &= x_0 - \frac{f(2)}{f'(2)} \\ &= 2 - \frac{10}{75} \\ &= \frac{28}{15}. \end{aligned}$$

5. Let

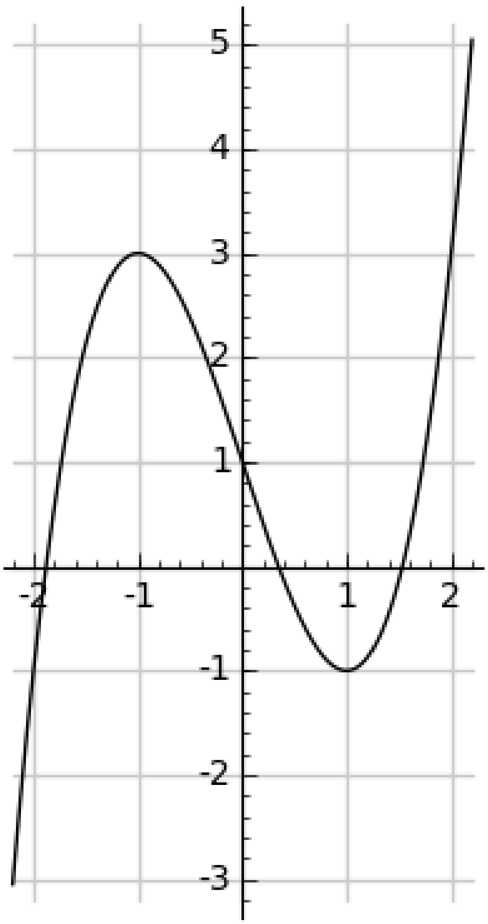
$$f(x) = \int_{-x^2}^{0} e^{t^2}\, dt.$$

Find $f'(x)$. (5 points)

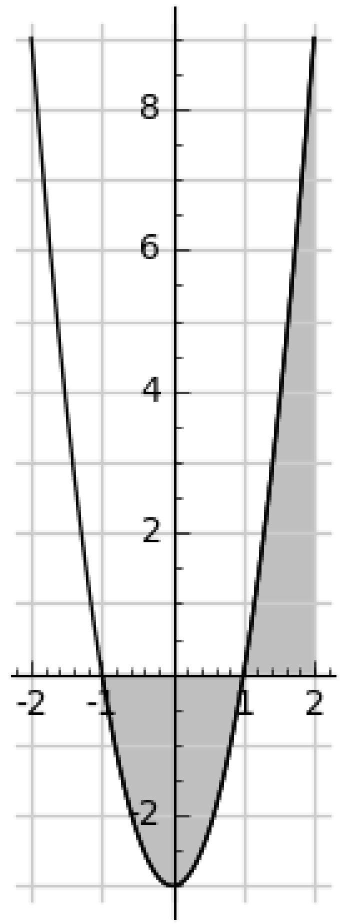
Solution. Let $F(u) = \int_0^u e^{t^2}\, dt$ and $u(x) = x^2$. Note that $f(x) = \int_0^{x^2} e^{t^2}\, dt = F(x^2)$. Now by the Fundamental Theorem of Calculus and the chain rule,

$$f'(x) = F'(u(x))u'(x) = 2xe^{x^4}.$$

6. Shown below are the graphs of a function $f(x)$ and its derivative $f'(x)$.



$f(x)$



$f'(x)$

Which of the two shaded areas is larger? **Justify your answer**. (5 points)

Solution. They are the same size, since

$$\int_{-1}^{2} f'(x)\,dx = f(2) - f(-1) = 3 - 3 = 0.$$

You can also argue that

$$\int_{1}^{-1} f'(x)\,dx = f(-1) - f(1) = 4 = f(2) - f(1) = \int_{1}^{2} f'(x)\,dx.$$