1. (10 pts.) Find all values of the constant c that make the function $f(x)$ continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} c^2x^2 - 3x - 1, & \text{if } x < 1 \\ 3c\,\cos(x-1), & \text{if } x \geq 1 \end{cases}$$

answer: Continuity at $x = 1$ must have $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$. Therefore, $c^2(1) - 3(1) - 1 = 3c \Rightarrow c^2 - 3c - 4 = 0$. Therefore $c = 4, -1$.

2. (10 pts.) Find the linear approximation of the function $f(x) = \sin(2x)$ at $a = \dfrac{\pi}{6}$.

answer: $f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$. $f'(\frac{\pi}{6}) = 1$. Therefore: $L(x) = \dfrac{\sqrt{3}}{2} + 1\left(x - \dfrac{\pi}{6}\right)$.

3. (10 pts.) Find the derivative of the function $f(x) = \sqrt{3x-1}$ using the limit definition of the derivative. NO CREDIT will be given if Limit Definition is not used.

answer: $f'(x) = \lim\limits_{h\to 0} \dfrac{\sqrt{3(x+h)-1} - \sqrt{3x-1}}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)-1} - \sqrt{3x-1}}{h} \cdot \frac{\sqrt{3(x+h)-1} + \sqrt{3x-1}}{\sqrt{3(x+h)-1} + \sqrt{3x-1}}$$

$$= \lim_{h\to 0} \frac{(3(x+h)-1) - (3x-1)}{h(\sqrt{3(x+h)-1} + \sqrt{3x-1}} = \frac{3}{2\sqrt{3x+1}}$$

4. (10 pts.) Find an equation of the tangent line to the curve

$$4\sqrt{3x-y} = x\,y + 9$$

at the point $(1, -1)$.

answer: $\dfrac{2\left(3 - \frac{dy}{dx}\right)}{\sqrt{3x-y}} = y + x\dfrac{dy}{dx}$

$\dfrac{6 - 2\frac{dy}{dx}}{2} = -1 + \dfrac{dy}{dx}$ $\quad 3 - \dfrac{dy}{dx} = -1 + \dfrac{dy}{dx}$ $\quad \dfrac{dy}{dx} = 2$

$y = -1 + 2(x-1)$

5. (25 pts; 5 pts. each) Find the limit, if it exists. If the limit does not exist, explain why. All work must be shown.

(a) $\displaystyle\lim_{x\to 2^+} \frac{|2-x|}{x^2-7x+10} = \lim_{x\to 2^+} \frac{|2-x|}{(x-2)(x-5)} = -\frac{1}{3}$

(b) $\displaystyle\lim_{x\to\infty} \frac{\sin(5x)}{x} = 0$ since $|\sin(5x)| \leq 1$.

(c) $\displaystyle\lim_{x\to\infty} \frac{3-x^3}{(2x^2+1)(x-2)} = \lim_{x\to\infty} \frac{3-x^3}{2x^3-4x^2+x-2} = -\frac{1}{2}$

(d) $\displaystyle\lim_{x\to 0} \frac{e^x-e^{2x}}{\sqrt{5x+1}-1} = \lim_{x\to 0} \frac{e^x-2e^{2x}}{\frac{5}{2\sqrt{5x+1}}} = -\frac{2}{5}$

(e) $\displaystyle\lim_{x\to 0}(1+\arcsin(2x))^{\frac{1}{x}} = P = e^2$

$\displaystyle\ln P = \lim_{x\to 0} \frac{\ln(1+\arcsin(2x))}{x} = \lim_{x\to 0} \frac{\frac{2/\sqrt{1-4x^2}}{1+\arcsin(2x)}}{1} = 2$

6. (25 pts; 5 pts. each) Differentiate the following functions. You do not have to simplify your answers.

(a) $f(x) = x^2\cos(3x+1)$
$f'(x) = 2x\cos(3x+1) - 3x^2\sin(3x+1)$

(b) $\displaystyle f(x) = \frac{\sqrt{x}}{2\tan(x)}$
$\displaystyle f'(x) = \frac{\frac{\tan(x)}{\sqrt{x}} - 2\sqrt{x}\sec^2(x)}{4\tan^2(x)}$

(c) $f(x) = (\tan^{-1}(3x) + 2^x)^2$
$\displaystyle f'(x) = 2(\arctan(3x) + 2^x)\left(\frac{3}{1+9x^2} + 2^x\ln 2\right)$

(d) $f(x) = (\sin x)^{\sqrt[3]{x}}$
$\displaystyle f'(x) = (\sin x)^{\sqrt[3]{x}}\left(\frac{1}{3x^{2/3}}\ln(\sin x) + \sqrt[3]{x}\cot x\right)$

(e) $\displaystyle f(x) = \int_2^{x^2} \sqrt{1+t^3}\,dt$
$f'(x) = 2x\sqrt{1+x^6}$

7. (10 pts.) Use Newton's method with the initial approximation $x_1 = 0$ to find the second approximation x_2 to the root of the equation $x - \cos(2x) = 0$.
answer: $x_2 = 0 - \dfrac{-1}{1} = 1$

8. (10 pts.) Find the absolute maximum and the absolute minimum of $f(x) = \dfrac{x}{x^2+1}$ on the interval $[0, 3]$.
$f'(x) = \dfrac{(1-x)(1+x)}{(x^2+1)^2}$
$f(0) = 0$ absolute minimum $\quad f(1) = \dfrac{1}{2}$ absolute maximum $\quad f(3) = \dfrac{3}{10}$.

9. (10 pts.) A bacteria culture grows with constant relative growth rate. After 2 hours, there are 400 bacteria, and after 3 hours, there are 1600 bacteria. Find the initial population. Simplify your answer as much as possible.
answer: This is an algebra problem: There is no calculus used here. Relative growth rate implies that $P(t) = Ab^t$. Solve simultaneous equations:
$400 = Ab^2$ and $1600 = Ab^3$. $b = 4$, plug in and $A = P(0) = 25$

10. (10 pts.) The base of a triangle is increasing at a rate of 3 cm/s, and the height is decreasing at a rate of 2 cm/s. At what rate is the area of the triangle changing when the height is 10 cm and the area is 80 cm?
$A = \dfrac{1}{2}BH \qquad \dfrac{dA}{dt} = \dfrac{1}{2}\dfrac{dB}{dt}H + \dfrac{1}{2}B\dfrac{dH}{dt}$
$\dfrac{dA}{dt} = \dfrac{1}{2}(10)(3) + \dfrac{1}{2}(16)(-2)$ cm^2/s = -1cm^2/s.

11. (10 pts.) A box with an open top and a rectangular base is to be constructed from 24 ft^2 of cardboard. The length of the base of the box is twice its width. Find the largest possible volume of the box.
answer: Maximize Volume under constraint cardboard=24
$V = 2x^2h \qquad 2x^2 + 6xh = 24$ Therefore: $h = \dfrac{24-2x^2}{6x}$
$V = 2x^2\left(\frac{24-2x^2}{6x}\right)$
$V = 8x - \frac{2}{3}x^3$ and $V' = 8 - 2x^2$
$x^2 = 4$, so $x = 2$ and the base of the box is 2×4, $h = \frac{5}{3}$.

12. (20 pts; 5 pts. each) Given the function $f(x) = x^4e^{-x}$,

   (a) Find the critical numbers of $f(x)$.
   $f'(x) = 4x^3e^{-x} - x^4e^{-x} = x^3e^{-x}(4-x)$: C.V. $x = 0, 4$.

(b) Find the intervals on which $f(x)$ is increasing and decreasing, and the local maximum and minimum values of $f(x)$.
On $0 < x < 4$, $f'(x) > 0$ so $f(x)$ is increasing.
On $x < 0$, and on $x > 4$, $f'(x) < 0$ so $f(x)$ is decreasing.

(c) Find the inflection points of $f(x)$.
$f''(x) = 12x^2e^{-x} - 4x^3e^{-x} - 4x^3e^{-x} + x^4e^{-x} = x^2e^{-x}(2-x)(6-x)$
Inflection points: $x = 2, 6$

(d) Find the intervals of concavity.
On $x < 2$, and on $x > 6$, $f''(x) > 0$ so $f(x)$ is concave up.
On $2 < x < 6$, $f''(x) > 0$ so $f(x)$ is concave down.

13. (25 pts; 5 pts. each) Evaluate the following integrals.

(a) $\int_1^4 |3 - x|\, dx$ (break into triangles)
$=2+.5=2.5$

(b) $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx$
$= \arcsin x\Big|_{x=0}^{x=1/2} = \frac{\pi}{6}$

(c) $\int \sin^3 x \cos^4 x\, dx$ $(du = \sin x dx)$
$= -\frac{1}{5}\cos^5 x - \frac{1}{7}\cos^7 x + c$

(d) $\int \frac{1}{x(1+(\ln x)^2)}\, dx$ $(u = \ln x)$
$= \arctan(\ln x)$

(e) $\int \sqrt{x} \ln x\, dx$ (by parts: $u = \ln x$, $dV = \sqrt{x}dx$)
$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{2/3} + c$

14. (15 pts.) Sketch the graph of the function that satisfies all of the given conditions. Indicate clearly all asymptotes, the intervals on which $f(x)$ is increasing and decreasing, local maxima and minima, the inflection points, and the intervals of concavity.

$f(x)$ is defined for all $x \neq 0$, $x \neq 3$;

$\lim_{x \to -\infty} f(x) = 1, \qquad \lim_{x \to +\infty} f(x) = 0$

$$\lim_{x\to 0^-} f(x) = -\infty, \qquad \lim_{x\to 0^+} f(x) = -\infty,$$

$$\lim_{x\to 3^-} f(x) = +\infty, \qquad \lim_{x\to 3^+} f(x) = -\infty,$$

$f'(x) > 0$ if $0 < x < 3$ or $3 < x < 5$;
$f'(x) = 0$ if $x = 5$
$f'(x) < 0$ if $x < 0$ or $x > 5$;

$f''(x) > 0$ if $1 < x < 3$ or $x > 6$;
$f''(x) = 0$ if $x = 1$ or $x = 6$;
$f''(x) < 0$ if $x < 0$, $0 < x < 1$, or $3 < x < 6$.