Final Practice:

1. Find the inverse function $f^{-1}(x)$ of $f(x) = \dfrac{x-2}{3x+4}$

   Set $x = \dfrac{y-2}{3y+4}$ and multiply both sides by $3y+4$.

$$f^{-1}(x) = -\frac{4x+2}{3x-1}$$

2. Use a linear approximation to estimate $\sqrt[3]{7.7}$.

   $x_0 = 8$, $f(x) = \sqrt[3]{x}$ then $f'(x) = \dfrac{1}{3x^{2/3}}$ and $y = 2 + \dfrac{1}{12}(x-8)$.

$$\sqrt[3]{7.7} \approx 2 + \frac{1}{12} \cdot \frac{-3}{10} = 1\frac{39}{40}$$

3. Use the Limit Definition of the Derivative to find the derivative of $f(x) = \sqrt{2x+3}$ at the point $a = 3$. NO CREDIT will be given if Limit Definition is not used.

$$\lim_{x\to 3} \frac{\sqrt{2x+3}-3}{x-3} = \lim_{x\to 3} \left(\frac{\sqrt{2x+3}-3}{x-3}\right)\left(\frac{\sqrt{2x+3}+3}{\sqrt{2x+3}+3}\right)$$

$$= \lim_{x\to 3} \frac{2(x-3)}{(x-3)(\sqrt{2x+3}+3)} = \frac{1}{3}$$

4. Find the slope of the tangent line to the curve below at the point $(1,2)$

$$x^4y^2 + 6x^5 - y^3 + 2x = 4$$

   $4x^3y^2 + x^4 2y\dfrac{dy}{dx} + 30x^4 - 3y^2\dfrac{dy}{dx} + 2 = 0.$

$$4(1)(2^2) + 1(2)(2)\frac{dy}{dx} + 30 - 3(2^2)\frac{dy}{dx} + 2 = 0. \qquad \frac{dy}{dx} = 6$$

5. Use Newton's Method with initial approximation $x_1 = 0$ to find the second approximation x_2 and the third approximation x_3 to the root of the equation $x^3+x^2+x-1 = 0$.

$$x_2 = 0 - \frac{-1}{1} = 1 \qquad x_3 = 1 - \frac{2}{6} = \frac{2}{3}.$$

6. A particle is moving in a straight line and its acceleration at the time t is given by $a(t) = \cos t - 6t^2 + 1$ (m/s^2). If the initial velocity of the particle is $v(0) = 2$ (m/s) and the initial position is $s(0) = 3$ (m), find the position of the particle at the time $t = \pi$ (s).
$v(t) = \sin t - 2t^3 + t + 2$

$$s(t) = -\cos t - \frac{1}{2}t^4 + \frac{1}{2}t^2 + 2t + 4 \qquad s(\pi) = 5 - \frac{\pi^4}{2} + \frac{\pi^2}{2}.$$

7. A ladder 3 meters long is leaning against a verticle wall. The base of the ladder starts to slide away from the wall at 5 m/min. How fast is the angle between the ladder and the ground changing when the base is 1 meter away from the wall?
$\cos\theta = \dfrac{x}{3}$. Differentiating with respect to t gives: $-\sin\theta\dfrac{d\theta}{dt} = \dfrac{1}{3}\dfrac{dx}{dt}$

$$\frac{d\theta}{dt} = -\frac{5}{\sqrt{8}}. \qquad \text{Speed} = \frac{5}{\sqrt{8}}\text{rad/min}$$

8. A box with a rectangular base without a lid must have a volume of 18 ft^3. The length of the base of the box is three times its width. Find the dimensions of the box that minimize the amount of material used.
Minimize material under constraint volume=18.
$M = 3x^2 + 8xh$ $\qquad$ $3x^2h = 18$
$M(x) = 3x^2 + \dfrac{48}{x}$. Differentiate: $M'(x) = 6x - \dfrac{48}{x^2}$.

$$x = 2. \quad \text{base} = 2ft \times 6ft \quad \text{height} = \frac{3}{2}ft.$$

9. Evaluate each of the following limits. All work must be shown.

(a) $\displaystyle\lim_{x\to+\infty} \frac{3x^2+x+1}{x^3-2} = \lim_{x\to+\infty} \frac{\frac{3x^2}{x^3} + \frac{x}{x^3} + \frac{1}{x^3}}{\frac{x^3}{x^3} - \frac{2}{x^3}} = 0$

(b) $\displaystyle\lim_{x\to0} \frac{e^{2x} + \ln(3x+1) - 1}{\sin(3x)} = \lim_{x\to0} \frac{2e^{2x} + \frac{3}{3x+1}}{3\cos(3x)} = \frac{5}{3}$

(c) $\displaystyle\lim_{x\to+\infty} x\left(\tan\left(\frac{2}{x}\right)\right) = \lim_{x\to+\infty} \frac{\tan\left(\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{(\sec^2(\frac{2}{x}))(\frac{-2}{x^2})}{\frac{-1}{x^2}} = 2$

(d) $\lim_{x\to 0+} (1-\sin x)^{3/x} = P = e^{-3}$

$\ln P = \lim_{x\to 0^+} \dfrac{3\ln(1-\sin x)}{x} = \lim_{x\to 0^+} \dfrac{\frac{-3\cos x}{1-\sin x}}{1} = -3$

(e) $\lim_{x\to 5^+} \dfrac{x^2-6x+5}{|5-x|} = \lim_{x\to 5^+} \dfrac{(x-5)(x-1)}{|5-x|} = 4$

10. Find the derivative of the following functions. You do not have to simplify your answers.

(a) $f(x) = \log_2 x + e^{\sin x} + \dfrac{1}{\sqrt[5]{x}}$

$f'(x) = \dfrac{1}{x\ln 2} + e^{\sin x}\cos x - \dfrac{1}{5x^{6/5}}$

(b) $g(x) = (3x-1)\sin^{-1}(2x)$

$g'(x) = 3\sin^{-1}(2x) + \dfrac{2(3x-1)}{\sqrt{1-4x^2}}$

(c) $h(t) = \dfrac{3^{t^3}}{\cos t}$

$h'(t) = \dfrac{(\cos t)3^{t^3}3t^2\ln 3 + 3^{t^3}\sin t}{\cos^2 t}$

(d) $f(x) = (\tan(5x))^{\sqrt{x}}$

$f'(x) = (\tan(5x))^{\sqrt{x}}\left(\dfrac{\ln(\tan(5x))}{2\sqrt{x}} + \sqrt{x}\dfrac{5\sec^2(5x)}{\tan(5x)}\right)$

(e) $f(x) = \int_3^x 2^{-t^2}\,dt$

$f'(x) = 2^{-x^2}$

11. Evaluate the following integrals

(a) $\int_1^2 \dfrac{(x+1)(x+2)}{x}\,dx = \int_1^2 \left(x+3+\dfrac{2}{x}\right) = \dfrac{1}{2}x^2 + 3x + 2\ln x\Big|_{x=1}^{x=2} = 4.5 + 2\ln 2$

(b) $\int_0^1 \dfrac{1}{t^2+1}\,dt = \arctan(t)|_{t=0}^{t=1} = \dfrac{\pi}{4}$

(c) $\int \sin^2 t\cos^3 t\,dt = \int \sin^2 t(1-\sin^2 t)\cos t\,dt = \int (\sin^2 t - \sin^4 t)\cos t\,dt$

$= \dfrac{1}{3}\sin^3 t - \dfrac{1}{5}\sin^5 t + c$

(d) $\displaystyle\int_1^4 \frac{5^{\sqrt{x}}}{\sqrt{x}}\,dx = \frac{2}{\ln 5}(5^{\sqrt{x}})\Big|_{t=1}^{t=4} = \frac{40}{\ln 5}$

(e) $\displaystyle\int x^3 \ln(3x)\,dx = \frac{1}{4}x^4 \ln(3x) - \int \frac{1}{4}x^3\,dx = \frac{1}{4}\ln(3x) - \frac{1}{16}x^4 + c$

(f) $\displaystyle\int \frac{t}{t^2+1}\,dt = \frac{1}{2}\ln(t^2+1) + c$

12. Find the absolute maximum and the absolute minimum values of $f(x) = \dfrac{x^3}{3} - 3x^2 + 5$ on the interval $[-3, 1]$.
$f'(x) = x^2 - 6x$ and is equal to zero when $x = 0, 6$ but only $x = 0$ in $[-3, 1]$.
$f(-3) = -31$, the minimum. $f(0) = 5$, the maximum. $f(1) = 2\frac{1}{3}$.

13. Given the function

$$f(x) = \frac{(x-1)^2}{(x+2)(x-4)}$$

and its derivatives:

$$f'(x) = \frac{18(1-x)}{(x+2)^2(x-4)^2} \qquad f''(x) = \frac{54(x^2-2x+4)}{(x+2)^3(x-4)^3}$$

(a) Find all horizontal and vertical asymptotes, if any.
Vertical asymptote at $x = -2$ and $x = 4$. Horizontal asymptote at $y = 1$.

(b) Determine on what intervals $f(x)$ is increasing or decreasing, and find all local maximum and minimum values.
$x < -2$, $f'(x) > 0$, $f(x)$ is increasing.
$-2 < x < 1$, $f'(x) > 0$, $f(x)$ is increasing.
$1 < x < 4$, $f'(x) < 0$, $f(x)$ is decreasing.
$x > 4$, $f'(x) < 0$, $f(x)$ is decreasing.
Local maximaum value at $x = 1$.

(c) Determine on what intervals $f(x)$ is concave up and concave down, and find all inflection points, if any.
No inflection points since numerator is never zero.
$x < -2$, concave up. $-2 < x < 4$, concave down. $x > 4$, concave up.

(d) Use a coordinate axes to sketch the graph of $f(x)$.